A Slacks-Based Measure of Efficiency

in

Data Envelopment Analysis

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First Version: August 1, 1997
Second Version: August 15, 1997
Third Version: November 14, 1997

The author will welcome comments on this draft.

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Abstract

In this paper, we will propose a Slacks-Based measure (SBM) of efficiency in DEA. This scalar measure deals directly with the input surplus and the output shortage of the decision making unit (DMU) concerned. It is unit invariant and monotone decreasing with respect to input surplus and output shortage. Furthermore, this measure is decided only by consulting with the reference set of the DMU and is not affected by statistics over the whole data set. The new measure has a close connection with other measures proposed so far, e.g., CCR and BCC. The dual side of this model can be interpreted as profit maximization, in contrast to the ratio maximization of the CCR model. Numerical experiments show its validity as an efficiency measurement tool and its compatibility with other measures of efficiency.

Keywords: DEA, efficiency, slacks, profit, unit invariant, monotone, returns to scale

1 Introduction

Since the innovative work by Charnes, Cooper and Rhodes (1978), studies in Data Envelopment Analysis (DEA) have been extensive: more than one thousand papers by 1996. A main objective of DEA is to measure the efficiency of a Decision Making Unit (DMU) by a scalar measure, ranging between

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zero (the worst) and one (the best). This scalar value is measured through a linear programming model. Specifically, the Charnes-Cooper-Rhodes (CCR) model deals with the ratio of multiple inputs and outputs in an attempt to gauge the relative efficiency of the DMU concerned among all the DMUs. This fractional program is solved by transforming it into an equivalent linear program. The optimal objective value (θ^*) is called the *ratio* (or *radial*) *efficiency* of the DMU. The optimal solution reveals also the existence, if any, of a surplus in inputs and a shortage in outputs (called *slacks*). A DMU with the full ratio efficiency, $\theta^* = 1$, and with no *slacks* in any optimal solution is called *CCR-efficient*. Otherwise, the DMU has a disadvantage against the DMUs in its reference set. Therefore, in discussing total efficiency, it is important to observe both the ratio efficiency and the slacks. Some attempts have been done to unify θ^* and slacks into a scalar measure, see Tone (1993), Pastor (1995) among others.

Meanwhile, Charnes *et al.* (1985) developed the Additive model of DEA that deals directly with input surplus and output shortage. This model has no scalar measure (ratio efficiency) *per se*. Although this model can discriminate efficient and inefficient DMUs by the existence of slacks, it has no means to gauge the depth of inefficiency similar to θ^* in the CCR model. In an attempt to define inefficiency based on the slacks, Pastor (1996), Lovell and Pastor (1995), Cooper and Tone (1997), Thrall (1997) and others have proposed several formulae for finding a scalar measure. The following properties are considered as important in designing the measures.

- **P1) Unit invariant** : The measure should be invariant with respect to the unit of data.
- **P2) Monotone**: The measure should be monotone decreasing in each

slack in input and output.

- **P3) Translation invariant**: The measure should be invariant under parallel translation of the coordinate system applied. (Ali and Seiford (1990) and Pastor (1996).)

In this paper, we further introduce a new property below:

- **P4) reference set dependent** : The measure should be decided only by consulting with the reference set of the DMU concerned and should not be affected by the minimum and/or the maximum of the whole data set. The concrete meanings of this idea will be clarified in Section 2.

Since the minimum and the maximum of data fluctuate largely depending on the selection of DMUs to be compared, the measures using these extreme values are influenced by the selection. One of the reasons why we propose this property is that, in DEA, an inefficient DMU is 'inefficient' with respect to DMUs in its reference set. Therefore, the measure of efficiency should be decided by the reference set dependent values and should not be influenced by the extreme values and by the statistics over the whole data set.

The new measure proposed in this paper satisfies the properties P1), P2) and P4). Furthermore, it is possible to connect this measure with the ratio measure of the CCR as well as that of the Banker, Charnes and Cooper (BCC) model (1984), as special cases.

The rest of the paper is organized as follows. Section 2 proposes a new measure of efficiency (SBM) based on an input surplus and output shortage, along with the computational scheme for solving the fractional program that defines SBM. Then, it is shown that the SBM can be interpreted as the product of input and output inefficiencies. The relationship between the SBM model and the CCR (Charnes-Cooper-Rhodes) model is described in Section 3. Then, in Section 4, we modify the SBM model to cope with input-orientation and output-orientation, and further generalize the basic model so that the connection with the CCR model can be clarified by a scalar parameter. The dual side of the SBM model is presented in Section 5, where it is shown that the SBM model maximizes the virtual profit instead of the virtual ratio of the CCR model. Returns to scale issues are discussed in Section 6. We will relax the positivity assumption of the data set in Section 7. Finally, in Section 8 numerical examples are exhibited to show the validity of the proposed method.

2 A Slacks-Based Measure of Efficiency

The definition of a Slacks-Based measure of efficiency (SBM) will be given, along with its interpretation as the product of input and output inefficiencies.

2.1 Definition and Computational Scheme of SBM

We will deal with n DMUs (Decision Making Units) with the input and output matrices $X = (x_{ij}) \in R^{m \times n}$ and $Y = (y_{ij}) \in R^{s \times n}$, respectively. We assume that the data set is positive, i.e., $X > O$ and $Y > O$ [1] .

The production possibility set P is defined as

(1) $$P = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \geq X\boldsymbol{\lambda},\ \boldsymbol{y} \leq Y\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq 0\},$$

where $\boldsymbol{\lambda}$ is a nonnegative vector in R^n [2].

[1]This assumption will be relaxed in Section 7.

[2]We can impose some constraints on $\boldsymbol{\lambda}$, such as $\sum_{j=1}^{n} \lambda_j = 1$ (the BCC model), if it is needed to modify the production possibility set.

We consider an expression for describing a certain DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ as

$$(2) \qquad \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^-$$

$$(3) \qquad \boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+,$$

with $\boldsymbol{\lambda} \geq \mathbf{0}$, $\boldsymbol{s}^- \geq \mathbf{0}$ and $\boldsymbol{s}^+ \geq \mathbf{0}$. The values $\boldsymbol{s}^- \in R^m$ and $\boldsymbol{s}^+ \in R^s$ indicate the *input surplus* and *output shortage* of this expression, respectively, and are called *slacks*. From the conditions $X >$O and $\boldsymbol{\lambda} \geq \mathbf{0}$, it holds

$$(4) \qquad \boldsymbol{x}_o \geq \boldsymbol{s}^-.$$

Using $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$, we define an index ρ as follows:

$$(5) \qquad \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^m s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{i=1}^s s_i^+/y_{io}}.$$

It can be verified that ρ satisfies the properties P1) (unit invariant) and P2) (monotone). Furthermore, from (4), it holds

$$(6) \qquad 0 < \rho \leq 1.$$

In an effort to estimate the efficiency of $(\boldsymbol{x}_o, \ \boldsymbol{y}_o)$, we formulate the following fractional program in $\boldsymbol{\lambda}$, $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$.

$$(7) \qquad \text{[SBM]} \qquad \begin{aligned} \min \rho &= \frac{1 - \frac{1}{m}\sum_{i=1}^m s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{i=1}^s s_i^+/y_{io}} \\ \text{subject to } \boldsymbol{x}_o &= X\boldsymbol{\lambda} + \boldsymbol{s}^- \\ \boldsymbol{y}_o &= Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\ \boldsymbol{\lambda} &\geq \mathbf{0}, \ \boldsymbol{s}^- \geq \mathbf{0}, \ \boldsymbol{s}^+ \geq \mathbf{0}. \end{aligned}$$

[SBM] can be transformed into the program below by introducing a positive scalar variable t. (See Charnes and Cooper (1962).)

$$(8) \qquad \text{[SBMt]} \qquad \min \tau = t - \frac{1}{m}\sum_{i=1}^m t s_i^-/x_{io}$$

$$\begin{aligned} \text{subject to } 1 &= t + \frac{1}{s}\sum_{i=1}^{s} ts_i^+/y_{io} \\ \boldsymbol{x}_o &= X\boldsymbol{\lambda} + \boldsymbol{s}^- \\ \boldsymbol{y}_o &= Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\ \boldsymbol{\lambda} &\geq 0,\ \boldsymbol{s}^- \geq 0,\ \boldsymbol{s}^+ \geq 0,\ t > 0. \end{aligned}$$

Now, let us define

$$\boldsymbol{S}^- = t\boldsymbol{s}^-, \boldsymbol{S}^+ = t\boldsymbol{s}^+, \text{and } \boldsymbol{\Lambda} = t\boldsymbol{\lambda}.$$

Then, [SBMt] becomes to the following linear program in t, $\boldsymbol{S}^-$, $\boldsymbol{S}^+$, and $\boldsymbol{\Lambda}$:

$$\text{(9)} \qquad \text{[LP]} \qquad \begin{aligned} \min \tau &= t - \frac{1}{m}\sum_{i=1}^{m} S_i^-/x_{io} \\ \text{subject to } 1 &= t + \frac{1}{s}\sum_{i=1}^{s} S_i^+/y_{io} \\ t\boldsymbol{x}_o &= X\boldsymbol{\Lambda} + \boldsymbol{S}^- \\ t\boldsymbol{y}_o &= Y\boldsymbol{\Lambda} - \boldsymbol{S}^+ \\ \boldsymbol{\Lambda} &\geq 0,\ \boldsymbol{S}^- \geq 0,\ \boldsymbol{S}^+ \geq 0,\ t > 0. \end{aligned}$$

Let an optimal solution of [LP] be

$$(\tau^*, t^*, \boldsymbol{\Lambda}^*, \boldsymbol{S}^{-*}, \boldsymbol{S}^{+*}).$$

Then, we have an optimal solution of [SBM] as defined by,

$$\text{(10)} \qquad \rho^* = \tau^*,\ \boldsymbol{\lambda}^* = \boldsymbol{\Lambda}^*/t^*,\ \boldsymbol{s}^{-*} = \boldsymbol{S}^{-*}/t^*,\ \boldsymbol{s}^{+*} = \boldsymbol{S}^{+*}/t^*.$$

Based on this optimal solution, we decide a DMU as *SBM-efficient* as follows:

Definition 1 (SBM-efficient) *A DMU* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *is SBM-efficient, if* $\rho^* = 1$.

This condition is equivalent to $s^{-*} = 0$ and $s^{+*} = 0$, i.e., no input surplus and no output shortage in any optimal solution.

For an SBM inefficient DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, we have the expression:

$$\begin{aligned} \boldsymbol{x}_o &= X\boldsymbol{\lambda}^* + \boldsymbol{s}^{-*} \\ \boldsymbol{y}_o &= Y\boldsymbol{\lambda}^* - \boldsymbol{s}^{+*}. \end{aligned}$$

The DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ can be improved and becomes efficient by deleting the input surplus and augmenting the output shortage as follows:

$$\boldsymbol{x}_o \leftarrow \boldsymbol{x}_o - \boldsymbol{s}^{-*} \quad (11)$$

$$\boldsymbol{y}_o \leftarrow \boldsymbol{y}_o + \boldsymbol{s}^{+*}. \quad (12)$$

Based on $\boldsymbol{\lambda}^*$, we define the reference set to $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ as:

Definition 2 (Reference set) *The set of indices corresponding to positive λ_j^*s is called the reference set to $(\boldsymbol{x}_o, \boldsymbol{y}_o)$.*

In the occurrence of multiple optimal solutions, the reference set is not unique. We can choose any one for our purpose.

The reference set R_o is

$$R_o = \{j | \lambda_j^* > 0\} \quad (j \in \{1, \ldots, n\}). \quad (13)$$

Using R_o, we can express $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ by,

$$\boldsymbol{x}_o = \sum_{j \in R_o} \boldsymbol{x}_j \lambda_j^* + \boldsymbol{s}^{-*} \quad (14)$$

$$\boldsymbol{y}_o = \sum_{j \in R_o} \boldsymbol{y}_j \lambda_j^* - \boldsymbol{s}^{+*}. \quad (15)$$

Since the Slacks-Based measure ρ^* depends only on s^{-*} and s^{+*}, i.e., the reference set dependent values, ρ^* is not affected by values attributed to other DMUs not in the reference set. In this sense, ρ^* proposed in this paper is different from other efficiency measures which incorporate statistics over the whole data set.

For instance, Cooper and Pastor (1996) proposed the RAM (Range Adjusted Measure of Inefficiency) as follows:

$$\text{RAM} = \left(\sum_{i=1}^{m} \frac{s_{io}^{-*}}{R_i^-} + \sum_{r=1}^{s} \frac{s_{ro}^{+*}}{R_r^+} \right) / (m+s), \tag{16}$$

where $R_i^- = \bar{x}_i - \underline{x}_i : R_r^+ = \bar{y}_r - \underline{y}_r$, with $\bar{x}_i = \max_j\{x_{ij}\}, \underline{x}_i = \min_j\{x_{ij}\}$ and $\bar{y}_r = \max_j\{y_{rj}\}, \underline{y}_r = \min_j\{y_{rj}\}$, for $i = 1, .., m : r = 1, .., s$.

The measure 1−RAM satisfies properties P1) and P2), and further P3), if the condition $\sum \lambda_j = 1$ is added. Thus, in the latter case, it can deal with negative inputs and outputs. So far, it has good properties [3]. However, RAM is largely affected by the range of the data set, as we see from the definition. The range changes by addition and/or deletion of the extreme data, which often occurs in empirical studies. This dependency on the extreme values is, in a sense, opposite to the principle of DEA in measuring efficiency compared with the efficient frontiers, i.e., DMUs in the reference set of the DMU concerned.

In another example, Lovell and Pastor (1995) use the standard deviation for each data. Specifically, they employ the measure

$$\sum_{i=1}^{m} \frac{s_{io}^{-*}}{\sigma_i^-} + \sum_{r=1}^{s} \frac{s_{ro}^{+*}}{\sigma_r^+}, \tag{17}$$

where σ_i^- represents the standard deviation of the data recorded for input $i = 1, \dots, m$ and σ_r^+ represents the standard deviation for output $r = 1, \dots, s$. This measure is also affected by statistics over the whole data set and is not purely dependent on the reference set (frontiers) of the DMU concerned.

[3] Thrall (1996) pointed out that optimal dual solutions for the Additive and the BCC models are not invariant under translation.

2.2 Interpretation of SBM as Product of Input and Output Inefficiencies

The formula for ρ in (5) can be transformed into

$$\rho = \left(\frac{1}{m}\sum_{i=1}^{m}\frac{x_{io} - s_i^-}{x_{io}}\right)\left(\frac{1}{s}\sum_{i=1}^{s}\frac{y_{io} + s_i^+}{y_{io}}\right)^{-1}.$$

The ratio $(x_{io} - s_i^-)/x_{io}$ evaluates the relative reduction rate of input i and therefore the first term corresponds to the mean reduction rate of inputs or *input inefficiency*. Similarly, in the second term, the ratio $(y_{io} + s_i^+)/y_{io}$ evaluates the relative expansion rate of output i and $(1/s)\sum(y_{io} + s_i^+)/y_{io}$ is the mean expansion rate of outputs. Its inverse, the second term, measures *output inefficiency*. Thus, SBM ρ can be interpreted as the product of input and output inefficiencies. Further, we have the theorem:

Theorem 1 *If DMU A dominates DMU B, i.e.,* $\boldsymbol{x}_A \leq \boldsymbol{x}_B$ *and* $\boldsymbol{y}_A \geq \boldsymbol{y}_B$*, then it holds that* $\rho_A^* \geq \rho_B^*$.

Proof. Let an optimal solution of [SBM] for A be $(\boldsymbol{\lambda}_A^*, \boldsymbol{s}_A^{-*}, \boldsymbol{s}_A^{+*})$. Then, $(\boldsymbol{\lambda}_A^*, \boldsymbol{x}_B - \boldsymbol{x}_A + \boldsymbol{s}_A^{-*}, \boldsymbol{y}_A - \boldsymbol{y}_B + \boldsymbol{s}_A^{+*})$ is feasible for B and its objective function satisfies the following inequality.

$$\frac{1 - \frac{1}{m}\sum_{i=1}^{m}(x_{iB} - x_{iA} + s_{iA}^{-*})/x_{iB}}{1 + \frac{1}{s}\sum_{i=1}^{s}(y_{iA} - y_{iB} + s_{iA}^{+*}))/y_{iB}} \leq \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_{iA}^{-*}/x_{iA}}{1 + \frac{1}{s}\sum_{i=1}^{s} s_{iA}^{+*}/y_{iA}} = \rho_A^*.$$

Thus, we have the theorem. □

The reverse of this theorem is not always true.

With regard to the ordering of DMUs, Dr. Thrall asked the author, in a private communication, that, if $1 > \rho_A^* > \rho_B^*$, is A more efficient than B? The answer is yes, if decision makers consent to employ the above definition of input/output inefficiency. There may be other possibility for measuring the means, e.g., the weighted means that reflect the intention of decision makers. We will discuss this issue in Section 4.2.

3 Relationship with the CCR Model

In this section, we will prove that the SBM ρ^* is not greater than the CCR (Charnes-Cooper-Rhodes) efficiency measure (θ^*) and that a DMU is SBM-efficient if and only if it is CCR-efficient.

3.1 SBM and the CCR Measure

The CCR model can be formulated as follows:

$$
\begin{array}{llrcl}
 & \text{[CCR]} & \min & & \theta \\
(18) & \text{subject to} & \theta \boldsymbol{x}_o & = & X\boldsymbol{\mu} + \boldsymbol{t}^- \\
(19) & & \boldsymbol{y}_o & = & Y\boldsymbol{\mu} - \boldsymbol{t}^+ \\
 & & \boldsymbol{\mu} & \geq & 0,\ \boldsymbol{t}^- \geq 0,\ \boldsymbol{t}^+ \geq 0.
\end{array}
$$

Definition 3 (CCR-efficient) *A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-efficient, if the optimal objective value θ^* is equal to one and the optimal slacks $\mathbf{t}^{-*}$ and $\mathbf{t}^{+*}$ are zero for every optimal solution of* [CCR].

Let an optimal solution of [CCR] be $(\theta^*, \boldsymbol{\mu}^*, \boldsymbol{t}^{-*}, \boldsymbol{t}^{+*})$. From (18), it holds

$$
\begin{array}{lrcl}
(20) & \boldsymbol{x}_o & = & X\boldsymbol{\mu}^* + \boldsymbol{t}^{-*} + (1-\theta^*)\boldsymbol{x}_o \\
(21) & \boldsymbol{y}_o & = & Y\boldsymbol{\mu}^* - \boldsymbol{t}^{+*}.
\end{array}
$$

Let us define

$$
\begin{array}{lrcl}
(22) & \boldsymbol{\lambda} & = & \boldsymbol{\mu}^* \\
(23) & \boldsymbol{s}^- & = & \boldsymbol{t}^{-*} + (1-\theta^*)\boldsymbol{x}_o \\
(24) & \boldsymbol{s}^+ & = & \boldsymbol{t}^{+*}.
\end{array}
$$

Then, $(\boldsymbol{\lambda}, s^-, s^+)$ is feasible for [SBM] and its objective value is:

$$(25) \quad \rho = \frac{1 - \frac{1}{m}\{\sum_{i=1}^m t_i^{-*}/x_{io} + m(1-\theta^*)\}}{1 + \frac{1}{s}\sum_{i=1}^s t_i^{+*}/y_{io}} = \frac{\theta^* - \frac{1}{m}\sum_{i=1}^m t_i^{-*}/x_{io}}{1 + \frac{1}{s}\sum_{i=1}^s t_i^{+*}/y_{io}}.$$

Evidently, the last term is not greater than θ^*. Thus, we have:

Theorem 2 *The optimal SBM ρ^* is not greater than the optimal CCR θ^*.*

Notice that the coefficient $1/(m\ x_{io})$ of the input surplus s_i^- in ρ plays a crucial role in validating Theorem 2.

Conversely, for an optimal solution $(\rho^*, \boldsymbol{\lambda}^*, s^{-*}, s^{+*})$, let us transform the constraints as

$$(26) \quad \theta \boldsymbol{x}_o = X\boldsymbol{\lambda}^* + (\theta^* - 1)\boldsymbol{x}_o + s^{-*}$$

$$(27) \quad \boldsymbol{y}_o = Y\boldsymbol{\lambda}^* - s^{+*}.$$

Further, we add the constraint

$$(28) \quad (\theta - 1)\boldsymbol{x}_o + s^{-*} \geq \mathbf{0}.$$

Then, $(\theta, \boldsymbol{\lambda}^*, \boldsymbol{t}^- = (\theta - 1)\boldsymbol{x}_o + s^{-*}, \boldsymbol{t}^+ = s^{+*})$ is feasible for [CCR].

3.2 SBM-Efficiency and CCR-Efficiency

The relationship between CCR-efficiency and SBM-efficiency is shown by the following theorem.

Theorem 3 *A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-efficient, if and only if it is SBM-efficient.*

Proof. Suppose that $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-inefficient. Then, we have either $\theta^* < 1$ or ($\theta^* = 1$ and $(\boldsymbol{t}^{-*}, \boldsymbol{t}^{+*}) \neq (\mathbf{0}, \mathbf{0})$). From (25), in both cases, we have $\rho < 1$ for a feasible solution of [SBM]. Therefore, $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-inefficient.

On the other hand, suppose that $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-inefficient. Then, it holds $(\boldsymbol{s}^{-*}, \boldsymbol{s}^{+*}) \neq (\mathbf{0}, \mathbf{0})$. By the statements (26) and (27), $(\theta, \boldsymbol{\lambda}^*, \boldsymbol{t}^- = (\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*}, \boldsymbol{t}^+ = \boldsymbol{s}^{+*})$ is feasible for [CCR], provided $(\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*} \geq \mathbf{0}$. There are two cases.

(Case 1) $\theta = 1$ and $(\boldsymbol{t}^- = \boldsymbol{s}^{-*}, \boldsymbol{t}^+ = \boldsymbol{s}^{+*}) \neq (\mathbf{0}, \mathbf{0})$. In this case, an optimal solution for [CCR] is CCR-inefficient.

(Case 2) $\theta < 1$. In this case, $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-inefficient.

Therefore, CCR-inefficiency is equivalent to SBM-inefficiency. Since the definitions of *efficient* and *inefficient* are mutually exclusive, we have proved the theorem. □

4 Modifications of the SBM Model

In this section, two modifications and a generalization of the SBM model will be developed. Section 4.1 will present the input-oriented and the output-oriented SBM models that correspond to those in the CCR model. In section 4.2, we will show a generalization of the SBM model that connects the SBM model to the CCR model via a scalar parameter.

4.1 Input-Oriented and Output-Oriented SBM Models

We will modify the SBM model by introducing a small positive number $\varepsilon(<< 1)$ in the following ways:

1. **Input-Oriented SBM Model**

   In this case we modify the denominator of the measure ρ by ε as:

(29)
$$\rho_{in} = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{io}}{1 + \frac{\varepsilon}{s}\sum_{i=1}^{s} s_i^+ / y_{io}}.$$

This modification puts more emphasis on the input slacks than the output ones and corresponds to the input-oriented CCR model.

2. **Output-Oriented SBM Model**

   We modify the numerator of ρ by ε as:

   (30) $$\rho_{out} = \frac{1 - \frac{\varepsilon}{m}\sum_{i=1}^{m} s_i^- / x_{io}}{1 + \frac{1}{s}\sum_{i=1}^{s} s_i^+ / y_{io}}.$$

   This modification puts more emphasis on the output slacks than the input ones and corresponds to the output-oriented CCR model.

The former serves to find input surpluses rather than output shortages and the latter serves the reverse function. Decision makers or analysts can choose one depending on the purpose of their analysis.

Regarding the above two models, we have developed the following theorems.

Theorem 4 *The optimal objective value ρ_{in}^* of the input-oriented SBM model satisfies*

$$\rho^* \le \rho_{in}^* \le \theta^*.$$

Proof. This can be proved in a similar way as in the case of Theorem 2. □

Theorem 5 *The optimal value ρ_{out}^* of the output-oriented SBM model satisfies*

$$\rho^* \le \rho_{out}^* \le \theta^*.$$

Proof. The output-oriented CCR model is described as:

$$\begin{aligned}
&(31) && \max && \frac{1}{\theta} \\
&(32) && \text{subject to} \quad x_o &=& X\mu + t^- \\
&(33) && \frac{1}{\theta} y_o &=& Y\mu - t^+ \\
&(34) && \mu &\ge& 0,\ t^- \ge 0,\ t^+ \ge 0.
\end{aligned}$$

Let an optimal solution of this model be $(\theta^*, \mu^*, t^{-*}, t^{+*})$. (Notice that the θ^* is the same with that of the input-oriented case.) Then, we can rewrite y_o as

$$y_o = Y\mu^* - (\frac{1}{\theta^*} - 1)y_o - t^{+*}.$$

Noting $\theta^* \leq 1$, let

$$\begin{aligned} \lambda &= \mu^* \\ s^- &= t^{-*} \\ s^+ &= (\frac{1}{\theta^*} - 1)y_o + t^{+*}. \end{aligned}$$

Then, (λ, s^-, s^+) is feasible for [SBM] and its objective value for the output-oriented model is

$$\rho_{out} = \frac{1 - \frac{\varepsilon}{m}\sum_{i=1}^{m} t_i^{-*}/x_{io}}{1/\theta^* + \frac{1}{s}\sum_{i=1}^{s} t_i^{+*}/y_{io}}.$$

Evidently, the last term is less than or equal to θ^*. □

4.2 A Modified Model with Weighted Slacks

We can modify the SBM measure ρ by incorporating weights w^- and w^+ into the input surplus s^- and the output shortage s^+, respectively, as follows:

(35) $$\rho_w = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} w_i^- s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{i=1}^{s} w_i^+ s_i^+/y_{io}},$$

where $w^- \geq \mathbf{0}$ and $w^+ \geq \mathbf{0}$, and

(36) $$\sum_{i=1}^{m} w_i^- = m \quad \text{and} \quad \sum_{i=1}^{s} w_i^+ = s.$$

The weight w_i^- (w_i^+) reflects the relative importance of the input (output) item i. In the basic model, $w_i^- = 1$ $(\forall i)$ and $w_i^+ = 1$ $(\forall i)$ are assumed. Under this modification, Theorems 2 and 3 hold, too. Since the ratio s_i^-/x_{io} (s_i^+/y_{io})

is unit-free, the weight w_i^- (w_i^+) should represent the unit-free importance of the slack or the input (output) i.

Also, the combination of the weighted slacks model with the input or output-oriented SBM models deserves consideration.

The weighted model has a close connection with the goal vectors in Thrall (1997).

4.3 A Generalization of the SBM Model

We will generalize the SBM model by introducing a scalar parameter α ($0 \leq \alpha \leq 1$) in the following way:

$$\text{(37)} \quad \text{[MSBM]} \quad \min \rho_\alpha = \frac{(1-\alpha)\pi + \alpha - \frac{\alpha}{m}\sum_{i=1}^{m} s_i^-/x_{io}}{1 + \frac{\alpha}{s}\sum_{i=1}^{s} s_i^+/y_{io}}$$

$$\text{subject to } x_o = X\lambda + s^-$$

$$y_o = Y\lambda - s^+$$

$$\text{(38)} \quad s^- + (\pi - 1)x_o \geq 0$$

$$\lambda \geq 0, \; s^- \geq 0, \; s^+ \geq 0.$$

This model has nonnegative vectors λ, s^-, s^+ and a free scalar π as variables. However, from the constraints (38) and (4), π is forced to be nonnegative.

We will observe two extreme cases of α:

- **Case 1** ($\alpha = 1$)

  In this case, the objective function becomes

  $$\rho_1 = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{i=1}^{s} s_i^+/y_{io}}.$$

  Thus, this case corresponds to the SBM model.

- Case 2 ($\alpha = 0$)

  We have the objective function as

$$\rho_0 = \pi.$$

  The constraints can be transformed into:

$$\begin{aligned} \pi \boldsymbol{x}_o &= X\boldsymbol{\lambda} + (\pi - 1)\boldsymbol{x}_o + \boldsymbol{s}^- \\ \boldsymbol{y}_o &= Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\ \boldsymbol{s}^- + (\pi - 1)\boldsymbol{x}_o &\geq \boldsymbol{0}. \end{aligned}$$

  Therefore, this case corresponds to the CCR model.

Let the optimal ρ_α be ρ_α^*. Then we have, by Theorem 2 ,

$$\rho_0^* \geq \rho_1^*.$$

Furthermore, the following theorem holds.

Theorem 6 *ρ_α^* is decreasing in α.*

Proof. First, we can rewrite ρ_α into:

$$\rho_\alpha = \frac{\pi + (1 - \pi - \frac{1}{m}\sum_{i=1}^m \frac{s_i^-}{x_{io}})\alpha}{1 + \frac{\alpha}{s}\sum_{i=1}^s \frac{s_i^+}{y_{io}}}.$$

Now, we observe the coefficient of α in the numerator. From the condition (38), we have the relation:

$$\frac{s_i^-}{x_{io}} + \pi - 1 \geq 0. \text{ for } i = 1, \ldots, m$$

Hence, it holds,

$$\pi - 1 + \frac{1}{m}\sum_{i=1}^m \frac{s_i^-}{x_{io}} \geq 0.$$

Thus, the coefficient of α in the numerator is nonnegative and the numerator is not increasing in α. The denominator is increasing in α. Therefore, ρ_α is decreasing in α.

Now, let an optimal solution for α_0 be

$$(\boldsymbol{\lambda}_0^*, \boldsymbol{s}_0^{-*}, \boldsymbol{s}_0^{+*}, \pi_0^*, \rho_{\alpha_0}^*).$$

Since $(\boldsymbol{\lambda} = \boldsymbol{\lambda}_0^*, \boldsymbol{s}^- = \boldsymbol{s}_0^{-*}, \boldsymbol{s}^+ = \boldsymbol{s}_0^{+*}, \pi = \pi_0^*)$ is feasible to [MSBM] for $\alpha \geq \alpha_0$ and ρ_α is decreasing in α, the relation $\rho_\alpha^* \leq \rho_{\alpha_0}^*$ holds for $\alpha \geq \alpha_0$. □

5 Observations on the Dual Problem

An important characteristic of DEA is its dual side, as represented by the dual program of the original linear program. This links the efficiency evaluation with the economic interpretation.

5.1 The Dual Program of the SBM Model as Profit Maximization

The dual program of the problem [LP] in Section 2 can be expressed as follows, with the dual variables $\xi \in R, \boldsymbol{v} \in R^m$ and $\boldsymbol{u} \in R^s$:

$$
\begin{aligned}
&(39) \qquad & [\text{DP}] \qquad & \max\ \xi \\
&(40) \qquad & \text{subject to}\ \ \xi + \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}\boldsymbol{y}_o &= 1 \\
&(41) \qquad & -\boldsymbol{v}X + \boldsymbol{u}Y &\leq 0 \\
&(42) \qquad & \boldsymbol{v} &\geq \frac{1}{m}\,[1/\boldsymbol{x}_o] \\
&(43) \qquad & \boldsymbol{u} &\geq \frac{\xi}{s}\,[1/\boldsymbol{y}_o],
\end{aligned}
$$

where the notation $[1/\boldsymbol{x}_o]$ designates the row vector $(1/x_{1o}, 1/x_{2o}, \ldots, 1/x_{mo})$.

By the equation (40), we can eliminate ξ. Then, this problem is equivalent to the following:

$$\begin{array}{llrcl}
(44) & \text{[DP']} & \max\ \boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o & & \\
(45) & & \text{subject to}\ \ \boldsymbol{u}Y - \boldsymbol{v}X & \leq & 0 \\
(46) & & \boldsymbol{v} & \geq & \frac{1}{m}\,[1/\boldsymbol{x}_o] \\
(47) & & \boldsymbol{u} & \geq & \frac{1 - \boldsymbol{v}\boldsymbol{x}_o + \boldsymbol{u}\boldsymbol{y}_o}{s}\,[1/\boldsymbol{y}_o].
\end{array}$$

The dual variables $\boldsymbol{v} \in R^m$ and $\boldsymbol{u} \in R^s$ can be interpreted as the virtual costs and prices of input and output items, respectively. The dual problem aims to find the optimal virtual costs and prices for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ so that the profit $\boldsymbol{u}\boldsymbol{y}_j - \boldsymbol{v}\boldsymbol{x}_j$ does not exceed zero for every DMU (including $(\boldsymbol{x}_o, \boldsymbol{y}_o)$) and maximizes the profit $\boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o$ for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ concerned. Apparently, the optimal profit is at best zero, and hence $\xi^* = 1$ for the SBM efficient DMUs.

The constraints (42) and (43) restrict the feasible $\boldsymbol{v}$ and $\boldsymbol{u}$ to the positive orthant. In this framework, we can incorporate other important developments related to the virtual dual variables into the SBM model, e.g., the assurance region methods (Thompson *et al.* (1986) (1997), Thompson and Thrall (1994)), the cone-ratio models (Charnes *et al.* (1990), Tone (1997)) among others. These modifications will contribute to enhance the potential usage of the model.

We will now observe the role of the dual variables $\boldsymbol{v}$ and $\boldsymbol{u}$ as coefficients of the supporting hyperplane to the production possibility set P defined by (1) in Section 2. A supporting hyperplane to P satisfies the inequality $-\boldsymbol{v}\boldsymbol{x} + \boldsymbol{u}\boldsymbol{y} \leq 0$ for every $(\boldsymbol{x}, \boldsymbol{y}) \in P$ and touches P at least at one point in P, i.e., there is a point $(\boldsymbol{x}, \boldsymbol{y})$ that satisfies the equality $-\boldsymbol{v}\boldsymbol{x} + \boldsymbol{u}\boldsymbol{y} = 0$. In

Figure 1, the DMU $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ is on the efficient frontier of P and the straight line H passing through $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ is a supporting hyperplane. As is easily seen, such a hyperplane may not be decided uniquely.

For an inefficient DMU, e.g., $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ in Figure 1, the dual program tries to find a supporting hyperplane that maximizes $-\boldsymbol{v}\boldsymbol{x}_o + \boldsymbol{u}\boldsymbol{y}_o$. Anyhow, the coefficients $\boldsymbol{v}$ and $\boldsymbol{u}$ are restricted to be positive by (42) and (43).

The complementary slackness conditions of the primal and the dual programs are described as follows:

(48) $$s_i^{-*}\,(v_i^* - \frac{1}{m}\frac{1}{x_{io}}) \;=\; 0 \quad (i = 1, \ldots, m)$$

(49) $$s_i^{+*}\,(u_i^* - \frac{\xi^*}{s}\frac{1}{y_{io}}) \;=\; 0. \quad (i = 1, \ldots, s)$$

Hence, it holds that if $s_i^{-*} > 0$, then $v_i^* = \frac{1}{m}\frac{1}{x_{io}}$ and if $s_i^{+*} > 0$, then $u_i^* = \frac{\xi^*}{s}\frac{1}{y_{io}}$. Thus, these $\boldsymbol{v}_i^*$ and $\boldsymbol{u}_i^*$ are uniquely decided. However, if $s_i^* = 0$, there exists, mostly, infinitely many $\boldsymbol{v}_i^*$ such that $v_i^* > \frac{1}{m}\frac{1}{x_{io}}$, by the strong complementary slackness theorem. This corresponds to the existence of multiple supporting hyperplanes at $(\boldsymbol{x}_e, \boldsymbol{y}_e)$ in Figure 1.

insert Figure 1.

5.2 Comparisons of Dual Programs in CCR and SBM Models

The dual program of the CCR model can be expressed as:

(50) $$[\text{DCCR}] \quad \max\ \eta \boldsymbol{y}_o$$

(51) $$\text{subject to} \quad \xi \boldsymbol{x}_o \;=\; 1$$

(52) $$-\xi X + \eta Y \;\leq\; 0$$

(53) $$\xi \geq 0, \quad \eta \;\geq\; 0.$$

Originally, this program comes from the ratio form CCR model (Charnes *et al.* (1978)) below:

$$\text{(54)} \qquad \max \ \frac{\eta y_o}{\xi x_o}$$

$$\text{(55)} \qquad \text{subject to} \ \frac{\eta y_j}{\xi x_j} \leq 1 \quad (\forall j)$$

$$\text{(56)} \qquad \xi \geq 0, \quad \eta \geq 0.$$

Thus, the CCR model tries to find the virtual costs ξ and prices η so that the ratio $\eta y_o / \xi x_o$ is maximized, subject to the ratio constraint $\eta y_j / \xi x_j \leq 1$ for every DMU j.

The SBM model proposed in this paper deals with the *profit* instead of the *ratio* in the CCR model.

It should be noted that, in the SBM model, the optimal dual variable v^* satisfies, by (42), $v^* x_o \geq 1$. Furthermore, if $s^{-*} > 0$, then by the complementary slackness condition, it holds $v^* x_o = 1$. In this case, the SBM model maximizes the ratio form of the CCR model in the more restricted range ((42) and (43)) of v and u. However, this case is exceptional.

6 Returns to Scale Issues

So far, we have dealt with the constant returns-to-scale situation as characterized by the production possibility set P in (1). The variable returns-to-scale scenario will be introduced by imposing the convex constraint on λ as:

$$e\lambda = 1,$$

where e is the row vector with all elements equal to one.

We call the thus extended model the *VSBM* model. Since the production possibility set of the VSBM is the same as that of the BCC model (Banker *et* *al.* (1984)), and VSBM-efficiency is equivalent to BCC-efficiency, the returns to scale characteristics of the VSBM-efficient DMUs can be decided in the same way as in the BCC model.

More concretely, let us begin to consider the dual program of the VSBM model that is:

$$
\begin{array}{llrcl}
(57) & & \max\ \boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o - u_0 & & \\
(58) & \text{subject to} & \boldsymbol{u}Y - \boldsymbol{v}X - \boldsymbol{e}u_0 & \leq & \boldsymbol{0} \\
(59) & & \boldsymbol{v} & \geq & \frac{1}{m}\,[1/\boldsymbol{x}_o] \\
(60) & & \boldsymbol{u} & \geq & \frac{1 - \boldsymbol{v}\boldsymbol{x}_o + \boldsymbol{u}\boldsymbol{y}_o - u_0}{s}\,[1/\boldsymbol{y}_o] \\
(61) & & u_0 & & : \text{unrestricted in sign.}
\end{array}
$$

If the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is VSBM-efficient, its dual variables satisfy the following relations:

$$
\begin{array}{lrcl}
(62) & \boldsymbol{u}^*\boldsymbol{y}_o - \boldsymbol{v}^*\boldsymbol{x}_o - u_0^* & = & 0 \\
(63) & \boldsymbol{u}^*Y - \boldsymbol{v}^*X - \boldsymbol{e}u_0^* & \leq & \boldsymbol{0} \\
(64) & \boldsymbol{v}^* & \geq & \frac{1}{m}\,[1/\boldsymbol{x}_o] \\
(65) & \boldsymbol{u}^* & \geq & \frac{1}{s}\,[1/\boldsymbol{y}_o].
\end{array}
$$

Usually the optimal solution is not unique, and let the upper and lower bounds of u_0 be $\bar{u}_0^*$ and $\underline{u}_0^*$, respectively. Then, following Banker and Thrall (1992), we can decide the returns-to-scale of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ as

1. *Increasing*, if $\underline{u}_0^* < \bar{u}_0^* \leq 0$ or $\underline{u}_0^* = \bar{u}_0^* < 0$.

2. *Constant*, if $\underline{u}_0^* < 0 < \bar{u}_0^*$ or $\underline{u}_0^* = \bar{u}_0^* = 0$.

3. *Decreasing*, if $0 \leq \underline{u}_0^* < \bar{u}_0^*$ or $0 < \underline{u}_0^* = \bar{u}_0^*$.

Furthermore, if a DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is VSBM-inefficient, we can project it onto the efficient frontier by deleting the input surplus and augmenting the output shortage by

$$(\boldsymbol{x}_o - \boldsymbol{s}^{-*}, \boldsymbol{y}_o + \boldsymbol{s}^{+*}).$$

We can decide returns-to-scale characteristics of this projected DMU by a simple rule as follows. (See Tone(1996) for details.)

1. If the DMUs in the reference set of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ have the same returns-to-scale characteristics, then the projected DMU has the same one.

2. If the DMUs in the reference set of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ belong to different classes of characteristics, i.e., (*increasing* and *constant*) or (*constant* and *decreasing*), then the projected DMU shows *increasing* or *decreasing* characteristics, respectively.

The combination of *increasing* and *decreasing* in the reference set never occurs.

7 How to Deal with Zeros in Data

So far, we have assumed that the data set is positive, i.e., $X > 0$ and $Y > 0$. In this section, we relax this assumption and show how to deal with zeros in the input/output data and even negative output data. This will considerably expand the applicability of SBM to real world problems, which essentially involve systematic zeros in the input-output data matrix.

7.1 Zeros in Input Data

If $\boldsymbol{x}_o$ has zero elements, we can neglect the slacks corresponding to these zeros. Suppose, for example, that $x_{1o} = 0$. Then, the first constraint leads to:

$$\sum_{j=1}^{n} x_{1j}\lambda_j + s_1^- = x_{1o} = 0.$$

Hence, we have $s_1^- = 0$ for every feasible solution. Thus, we can delete s_1^- from the set of variables to be determined by the model. Correspondingly, in the objective function, the term s_1^-/x_{1o} is removed and m should be reduced by one. ($m \rightarrow m-1$.) Notice that the above constraint should be kept in the set of constraints.

7.2 Zeros in Output Data

Suppose that $\boldsymbol{y}_o$ has $y_{1o} = 0$. Then, the first output-constraint leads to:

$$\sum_{j=1}^{n} y_{1j}\lambda_j - s_1^+ = y_{1o} = 0.$$

There are two important cases to be considered:

1. **(Case 1)** The target DMU possesses no function to produce the first output.

   In this case, we can delete the term s_1^+/y_{1o} from the objective function, since s_1^+ has no role in evaluating the efficiency of the DMU. The number of terms (s) in the objective function should be reduced by one. ($s \rightarrow s-1$.)

2. **(Case 2)** The target DMU has a function with the potential of producing the first output but does not utilize it.

   In this case, we may replace y_{1o} in the objective function by a small positive number or by

$$y_{1o} \leftarrow \frac{1}{10}\min\{y_{1j} \mid y_{1j} > 0,\ j = 1,\ldots,n\}.$$

It should be remembered that the term s_1^+/y_{1o} in the objective function has the role of a penalty in this case, and that $1/y_{1o}$ should be sufficiently large.

Finally, negative output data can be dealt using the same approach adopted for handling zeros in output data (Case 2).

8 A Numerical Example

Table 1 exhibits the data of 19 public libraries in the Tokyo Metropolitan Area in 1986. As the measurement of efficiency, we use two input and two output items as follows:

- **Input**: number of books (unit=100) and number of staff
- **Output**: number of registered residents (unit 1000) and number of borrowed books (unit=1000)

insert Table 1

Under the constant returns-to-scale assumptions, Table 2 compares the CCR (input-oriented) and the SBM (basic, input-oriented and output-oriented) scores and ranks. $\varepsilon = 10^{-6}$ was used in the input and output-oriented SBM cases. Also, Table 3 shows similar comparisons under the variable returns-to-scale assumption.

insert Table 2

insert Table 3

insert Table 4

The amount of slacks in the input and output-oriented SBM models is listed in Table 4.

In Table 2, it is observed that all SBM scores are less than the CCR score. Since the CCR model in the table is input-oriented, comparisons between CCR and input-oriented SBM are reasonable. Both score and rank show considerable similarity, except for L2. L2 has a large input surplus (books=217.61, staff=6.53) which is reflected in sharp drop in the SBM score (0.56994) from the CCR score (0.76935). Similar drops occur at L4 (0.74504 → 0.678736), L6 (0.94491 → 0.87861), L7 (0.82267 → 0.71233) and L19 (0.85507 → 0.78275). These changes are caused by input surpluses, which are not fully accounted for in the CCR model. Under the variable returns-to-scale assumption, in Table 3, similar drops in score are observed at L6 (0.99761 → 0.90169), L7 (0.91105 → 0.83346) and L12 (0.90920 → 0.80287). It should be noted that the output-oriented SBM score of L6 (0.99791) is better than the (input-oriented) BCC score (0.99761). This is not unusual, since in the BCC model, the input-oriented scores are usually not equal to the output-oriented ones. Actually, L6 has 0.99854 as the output-oriented BCC score.

As expected, in the SBM model, the slacks in input/output are positively accounted for in the score.

9 Conclusion

This article has proposed a new scalar slacks-based measure of efficiency (SBM) in DEA. In contrast to the CCR and BCC measures, which are based on the proportional reduction (enlargement) of input (output) vectors and do not take account of slacks, the new measure deals directly with input surplus and output shortage. Although the Additive model has the (weighted) sum of slacks as its objective and can discriminate *efficient* and *inefficient* DMUs, it has no means to gauge the depth of inefficiency *per se.* In this regard, SBM clearly differs from CCR, BCC and other measures proposed so far.

This measure satisfies such properties as unit invariance and monotone with respect to slacks. Furthermore, it is reference set dependent, i.e., the measure is decided only by its reference set and is not affected by statistics over the whole data set. Also, this model can be modified to cope with input or output-orientation. A generalization of this method showed that SBM has a close relationship with the CCR (BCC) model. The dual program revealed that SBM tries to find the maximum virtual profit instead of the maximum ratio of the CCR model.

The numerical example showed the compatibility of SBM with other measures and its potential applicability for practical purposes.

Although this study concentrated on the basic characteristics of the proposed model, further theoretical research and applications should be developed in diverse areas, including studies in the combinations of this method with other recent developments in DEA, e.g., the assurance region method and the cone-ratio models.

Acknowledgement

I am grateful to Professor Thrall for his comments on the earlier version of this paper.

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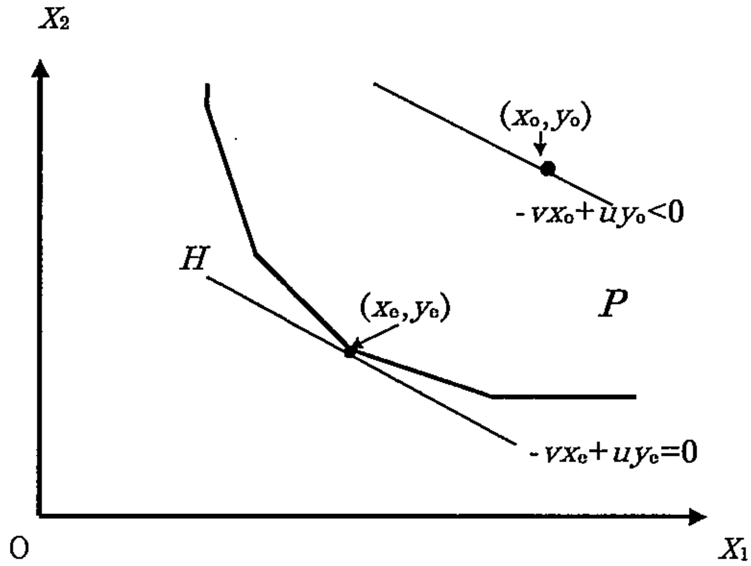

Figure 1.   Supporting Hyperplane

Table 1: Data of Public Libraries

No.	Libraries	Input		Output	
		Books	Staff	Reg. Res.	Bor. Book
L1	Chiyoda	163.523	26	5.561	105.321
L2	Chuo	338.671	30	18.106	314.682
L3	Taito	281.655	51	16.498	542.349
L4	Arakawa	400.993	78	30.81	847.872
L5	Minato	363.116	69	57.279	758.704
L6	Bunkyo	541.658	114	66.137	1438.746
L7	Sumida	508.141	61	35.295	839.597
L8	Shibuya	338.804	74	33.188	540.821
L9	Toshima	393.815	68	41.197	978.117
L10	Shinjuku	509.682	96	47.032	930.437
L11	Nakano	527.457	92	56.064	1345.185
L12	Shinagawa	601.594	127	69.536	1164.801
L13	Kita	528.799	96	37.467	1348.588
L14	Koto	394.158	77	57.727	1100.779
L15	Katusika	515.624	101	46.16	1070.488
L16	Edogawa	467.617	74	47.236	1223.026
L17	Nerima	669.996	107	69.576	1901.465
L18	Adachi	844.949	120	89.401	1909.698
L19	Ota	1258.981	242	97.941	3055.193

Table 2: Comparisons of Efficiencies under Constant RTS

No.	Libraries	Constant Returns-to-Scale							
		CCR		SBM					
		CCR	Rank	SBM	Rank	Input Or.	Rank	Output Or.	Rank
L1	Chiyoda	0.27945	19	0.25787	19	0.25826	19	0.26928	19
L2	Chuo	0.76935	12	0.55793	17	0.56994	18	0.72162	11
L3	Taito	0.67849	16	0.52621	18	0.63845	16	0.54189	18
L4	Arakawa	0.74504	14	0.62112	15	0.67836	14	0.62214	16
L5	Minato	1	1	1	1	1	1	1	1
L6	Bunkyo	0.94491	6	0.85664	7	0.87861	8	0.88854	7
L7	Sumida	0.82267	11	0.69163	10	0.71233	12	0.81839	9
L8	Shibuya	0.64605	18	0.58385	16	0.60834	17	0.6164	17
L9	Toshima	0.8801	9	0.84579	8	0.87907	7	0.87839	8
L10	Shinjuku	0.66094	17	0.65357	14	0.65727	15	0.65981	15
L11	Nakano	0.9033	7	0.86302	6	0.89514	6	0.88968	6
L12	Shinagawa	0.76674	13	0.71061	9	0.73445	11	0.73815	10
L13	Kita	0.89861	8	0.65654	13	0.84456	9	0.67579	12
L14	Koto	1	1	1	1	1	1	1	1
L15	Katusika	0.73642	15	0.66998	11	0.69628	13	0.67088	13
L16	Edogawa	0.95398	5	0.92855	5	0.94125	5	0.95394	5
L17	Nerima	1	1	1	1	1	1	1	1
L18	Adachi	1	1	1	1	1	1	1	1
L19	Ota	0.85507	10	0.66469	12	0.78275	10	0.66893	14
	Average	0.81795		0.74147		0.77763		0.76915	

Table 3: Comparisons of Efficiencies under Variable RTS

		Variable Returns-to-Scale							
		BCC		SBM					
No.	Libraries	BCC	Rank	SBM	Rank	Input Or.	Rank	Output Or.	Rank
L1	Chiyoda	1	1	1	1	1	1	1	1
L2	Chuo	1	1	1	1	1	1	1	1
L3	Taito	0.94418	11	0.65892	16	0.94045	10	0.65903	18
L4	Arakawa	0.83683	16	0.62112	19	0.82689	15	0.62405	19
L5	Minato	1	1	1	1	1	1	1	1
L6	Bunkyo	0.99761	9	0.90169	10	0.90169	12	0.99791	9
L7	Sumida	0.91105	13	0.77691	12	0.83346	14	0.89078	12
L8	Shibuya	0.81958	17	0.6344	18	0.75392	17	0.67664	17
L9	Toshima	0.97953	10	0.91823	9	0.97786	9	0.91823	10
L10	Shinjuku	0.70321	19	0.65357	17	0.70313	19	0.69531	16
L11	Nakano	0.91931	12	0.88025	11	0.91677	11	0.89376	11
L12	Shinagawa	0.9092	14	0.75086	13	0.80287	16	0.8471	13
L13	Kita	0.90683	15	0.68431	14	0.89479	13	0.71399	15
L14	Koto	1	1	1	1	1	1	1	1
L15	Katusika	0.75082	18	0.66998	15	0.74768	18	0.73495	14
L16	Edogawa	1	1	1	1	1	1	1	1
L17	Nerima	1	1	1	1	1	1	1	1
L18	Adachi	1	1	1	1	1	1	1	1
L19	Ota	1	1	1	1	1	1	1	1
	Average	0.9304		0.85001		0.9105		0.87641	

Table 4: Slacks of Input Oriented and Output Oriented SBM Models under Constant RTS

		Input Oriented				Output Oriented			
		Input		Output		Input		Output	
No.	Libraries	books	staff	reg. res.	bor. b	books	staff	reg. res.	bor. book
L1	Chiyoda	125.61	18.6	0	0	0.72	0	11.35	356.72
L2	Chuo	217.61	6.53	0	0	150.82	0	1.4	218.44
L3	Taito	90.55	20.48	3.35	0	0	0	21.69	204.07
L4	Arakawa	102.24	30.29	0.21	0	0	0	27.66	268.65
L5	Minato	0	0	0	0	0	0	0	0
L6	Bunkyo	29.84	21.4	0	0	0	8.19	13.19	73.96
L7	Sumida	210.65	9.81	0	0	78.63	0	10.15	131.17
L8	Shibuya	119.92	31.77	0	0	0	7.81	16.43	405.37
L9	Toshima	47.22	8.29	0	0	0	0	5.8	133.05
L10	Shinjuku	177.15	32.44	0	0	0	0	23.35	497.5
L11	Nakano	51	10.4	0	0	0	0	7.99	141.89
L12	Shinagawa	140.33	37.83	0	0	0	9.48	18.57	515.29
L13	Kita	53.61	20.11	11.88	0	0	0	34.41	55.28
L14	Koto	0	0	0	0	0	0	0	0
L15	Katusika	135.91	34.73	0	0	0	0.27	29.36	369.51
L16	Edogawa	35.78	3.03	0	0	0	0	1.4	81.87
L17	Nerima	0	0	0	0	0	0	0	0
L18	Adachi	0	0	0	0	0	0	0	0
L19	Ota	182.46	70.08	13.85	0	0	0	83.44	421.4