

An epsilon-based measure of efficiency in DEA

Dedicated to Professor W. W. Cooper

with

affection, gratitude, and respect

Kaoru Tone

*National Graduate Institute for Policy Studies
7-22-1 Roppongi, Minato-ku, Tokyo 106-8677, Japan*

Miki Tsutsui

*Central Research Institute of Electric Power Industry
2-11-1 Iwado Kita, Komae-shi, Tokyo 201-8511, Japan*

Abstract

Firstly, we discuss differences between Farrell and Pareto-Koopmans efficiency measures in DEA, and propose a composite method for discriminating them. Then, we extend the method to so-called “epsilon based-measure (EBM).” The EBM can examine the robustness and stability of efficiency measure of DMUs regarding parametric change of input multiplier variables. Lastly, we propose a scheme for selecting an appropriate value of epsilon with recourse to actual cost shares of input resources.

Keywords: Farrell efficiency, Pareto-Koopmans efficiency, stability analysis, EBM

1. Introduction

In the seminal paper (Charnes et al. (1978)), Charnes, Cooper and Rhodes have succeeded in implementing Farrell’s efficiency (Farrell (1957)) in the linear programming framework called the CCR model. On the other hand, in the welfare economics, Pareto-Koopmans concept of efficiency (Pareto (1909), Koopmans (1951)) says “A DMU (decision-making unit) is fully efficient if and only if it is not possible to

improve any input or output without worsening some other input or output (Cooper et al. (2007), p.45).” Hereafter we will cite this as Koopmans efficiency. Between the CCR efficiency and Koopmans efficiency there is a gap regarding slacks. Many researches have been done on this subject. See Cooper et al. (2001) and Cooper et al. (2007) among others. In this paper we revisit this subject and propose a practical scheme for discriminating the both. In the latter half of this paper we extend our approach to stability analysis of DEA scores against the lower bound of the shadow price of multipliers. There are several methods for dealing with multiplier restrictions, e.g. the assurance region (AR) models (Thompson et al (1986)) and the cone-ratio models (Charnes et al. (1990)). The AR models impose the upper and lower bounds for the ratio of multipliers for pair of inputs or outputs, or both, while the cone-ratio models impose a set of linear restrictions that define a convex cone. In this paper, we investigate the situation that the lower bounds of shadow prices of input resources vary parametrically and efficiency scores are subject to change accordingly. Suppose that two DMUs A and B utilize two input resources R1 and R2 for producing the same amount of products, and A consumes more R1 than B whereas A uses less R2 than B. We assume that A and B are judged to be efficient when we impose no restrictions on the multipliers (shadow prices) of input resources. However, if the (shadow) price of R1 goes up, the relative efficiency of A would be turn down compared with B, since A consumes more R1 than B. This kind of price change is not unusual in actual business situations, since resource acquisitions are becoming more competitive in recent days. Using this model, we can see how stable DMUs are against the stress of input price increases. This is a new feature of our model which, as far as we know, other previous researches have not yet pointed to. We further propose a method for estimating the degree of input price stress based on an analogy between the virtual (shadow) price and the actual price of input resources.

The rest of this paper unfolds as follows. Section 2 defines Koopmans efficiency and introduces two-stage approach for finding Koopmans efficiency. Utilizing the strong theorem complementarity we introduce the epsilon-based measure model (EBM). In Section 3 we prove the equivalence of Koopmans efficiency and the EBM model, and discuss the subject how to determine the magnitude of epsilon referring to an illustrative example. Motivated by these observations, we extend the EBM model by employing the epsilon as a parameter and apply it for testing stability of EBM scores in Section 4. We demonstrate an example of the extended EBM model in Section 5. We propose a scheme for finding an appropriate level of epsilon based on an analogy between the shadow price and the actual price of input resources in Section 6. We demonstrate another interpretation of the EBM and point that EBM is a composite of

technical efficiency model and cost minimizing model in Section 7. We conclude this paper in Section 8.

2. Koopmans efficiency and epsilon-based model (EBM)

In this section, we define Koopmans efficiency and introduce an epsilon-based model. We deal with n DMUs ($j=1, \dots, n$) each having m inputs $\{x_{ij}\}$ ($i=1, \dots, m$) and s outputs $\{y_{ij}\}$

($i=1, \dots, s$). We denote DMU j by $(\mathbf{x}_j, \mathbf{y}_j)$ ($j=1, \dots, n$) with $\mathbf{x}_j \in R^m$ and $\mathbf{y}_j \in R^s$, and the

input/output data matrices by $\mathbf{X} = (x_{ij}) \in R^{m \times n}$ and $\mathbf{Y} = (y_{ij}) \in R^{s \times n}$, respectively.

Throughout this paper we assume $\mathbf{x} > \mathbf{0}$ and $\mathbf{y} > \mathbf{0}$.

2.1. Two-stage approach

The input-oriented CCR model (Charnes et al. (1978)) for evaluating the efficiency of the DMU $(\mathbf{x}_o, \mathbf{y}_o)$ is described in the multiplier form [D] and the development form [P] as follows:

[D]

$$\begin{aligned} \theta^* &= \max_{\mathbf{v}, \mathbf{u}} \mathbf{u} \mathbf{y}_o \\ \text{subject to } & \mathbf{v} \mathbf{x}_o = 1 \\ & -\mathbf{v} \mathbf{X} + \mathbf{u} \mathbf{Y} \leq \mathbf{0} \\ & \mathbf{v} \geq \mathbf{0}, \mathbf{u} \geq \mathbf{0}. \end{aligned} \quad (1)$$

[P]

$$\begin{aligned} \theta^* &= \min_{\theta, \boldsymbol{\lambda}, \mathbf{s}^-} \theta \\ \text{subject to } & \theta \mathbf{x}_o - \mathbf{X} \boldsymbol{\lambda} - \mathbf{s}^- = \mathbf{0} \\ & \mathbf{Y} \boldsymbol{\lambda} \geq \mathbf{y}_o \\ & \boldsymbol{\lambda} \geq \mathbf{0} \\ & \mathbf{s}^- \geq \mathbf{0}, \end{aligned} \quad (2)$$

where $\mathbf{s}^- \in R^m$ is the input slacks. [P] and [D] are primal-dual each other.

Let an optimal solution of [P] be $(\theta^*, \boldsymbol{\lambda}^*, \mathbf{s}^{-*})$. We have two definitions of efficiency.

[Definition 1] (Weak efficient)

If DMU $(\mathbf{x}_o, \mathbf{y}_o)$ fulfills $\theta^* = 1$, then it is called weak efficient.

[Definition 2] (Strong efficient)

If DMU $(\mathbf{x}_o, \mathbf{y}_o)$ fulfills $\theta^* = 1$ and $\mathbf{s}^{-*} = \mathbf{0}$ for all optimal solution, then it is called strong efficient.

In order to check the strong efficiency, we have two-stage approach as follows.

[Two-stage approach]

First stage: Solve [P] and obtain θ^* .

Second stage: Solve

$$\begin{aligned}
 w^* &= \max_{\lambda, s^-} \epsilon s^- \\
 \text{subject to } & \theta^* \mathbf{x}_o - \mathbf{X}\lambda - \mathbf{s}^- = \mathbf{0} \\
 & \mathbf{Y}\lambda \geq \mathbf{y}_o \\
 & \lambda \geq \mathbf{0} \\
 & \mathbf{s}^- \geq \mathbf{0},
 \end{aligned} \tag{3}$$

where $\epsilon \in R^m$ is the row vector with all elements being unity.

[Definition 3] Koopmans efficiency (*K-eff*)

A DMU $(\mathbf{x}_o, \mathbf{y}_o)$ is Koopmans efficient if it satisfies:

$$\theta^* = 1 \text{ and } w^* = 0. \tag{4}$$

Otherwise, $(\mathbf{x}_o, \mathbf{y}_o)$ is Koopmans inefficient, i.e. $\theta^* < 1$ or $(\theta^* = 1 \text{ and } w^* > 0)$.

Koopmans efficiency is equivalent to the strong efficiency.

2.2. Strong theorem of complementarity and its applications

[Definition 4] *K-eff* set

We define the set of the Koopmans efficient DMUs as:

$$KE = \left\{ o \mid (\mathbf{x}_o, \mathbf{y}_o) \text{ is } K\text{-eff} \right\}. \tag{5}$$

[Property 1]

If $(\mathbf{x}_o, \mathbf{y}_o)$ is *K-eff*, then by the strong theorem of complementarity there exists \mathbf{v}_o^* for the solution of [D] such that $\mathbf{v}_o^* > \mathbf{0}$.

[Definition 5] Minimum tolerance (ϵ^*)

We define a minimum tolerance ϵ^* by:

$$\epsilon^* = \min_{i,o} \left\{ m v_{io}^* x_{io} \mid i=1, \dots, m : o \in KE, \mathbf{v}_o^* > \mathbf{0} \right\}. \tag{6}$$

2.3 An ϵ^* -based measure model (EBM)

Using the minimum tolerance ϵ^* , we define ϵ^* -based measure model (EBM) as follows.

[D(ϵ^*)]

$$\begin{aligned}
 \tau_o^*(\epsilon^*) &= \max_{\mathbf{v}, \boldsymbol{\mu}} \boldsymbol{\mu} \mathbf{y}_o \\
 \text{subject to } & \mathbf{v} \mathbf{x}_o = 1 \\
 & -\mathbf{v} \mathbf{X} + \boldsymbol{\mu} \mathbf{Y} \leq \mathbf{0} \\
 & v_i \geq \frac{\epsilon^*}{m x_{io}} \quad (i=1, \dots, m) \\
 & \boldsymbol{\mu} \geq \mathbf{0}.
 \end{aligned} \tag{7}$$

[P(ϵ^*)]

$$\begin{aligned}
\tau^*(\mathcal{E}^*) &= \min_{\tau, \xi, \pi} \tau - \frac{\mathcal{E}^*}{m} \sum_{i=1}^m \frac{\pi_i}{x_{i0}} \\
\text{subject to } & \tau \mathbf{x}_0 - \mathbf{X}\xi - \boldsymbol{\pi} = \mathbf{0} \\
& \mathbf{Y}\xi \geq \mathbf{y}_0 \\
& \xi \geq \mathbf{0} \\
& \boldsymbol{\pi} \geq \mathbf{0}.
\end{aligned} \tag{8}$$

[P(\mathcal{E}^*)] and [D(\mathcal{E}^*)] are mutually primal-dual. We notice that the EBM is a “mixed” type model, since it has both the radial factor τ and the slacks π in the objective function of [P(\mathcal{E}^*)]. This feature is quite different from the CCR that has only the radial factor in the objective function and the SBM (slack-based measure (Tone (2001))) that has only the slacks in the objective function. We also notice that the radial factor τ is not upper bounded by 1, although the optimal objective value $\tau^*(\mathcal{E}^*)$ is bounded by 1.

[Lemma 1]

Between [P] and [P(\mathcal{E}^*)], we have the following inequality.

$$0 \leq \tau^*(\mathcal{E}^*) \leq \theta^* \leq 1. \tag{9}$$

Let an optimal solution of [P(\mathcal{E}^*)] be $(\tau^*, \xi^*, \boldsymbol{\pi}^*)$. We define EBM- efficiency as follows.

[Definition 6] EBM-*eff*

DUM($\mathbf{x}_0, \mathbf{y}_0$) is EBM-efficient if it holds that

$$\tau^*(\mathcal{E}^*) = 1. \tag{10}$$

Otherwise if $\tau^*(\mathcal{E}^*) < 1$, DUM($\mathbf{x}_0, \mathbf{y}_0$) is EBM-inefficient.

If DUM($\mathbf{x}_0, \mathbf{y}_0$) is EBM-efficient, we have an optimal solution represented by $\tau^* = 1, \boldsymbol{\pi}^* = \mathbf{0}$. However, multiple optima, i.e. $\tau^* > 1, \boldsymbol{\pi}^* \neq \mathbf{0}$, may exist. Even in such case, we designate the solution $\tau^* = 1, \boldsymbol{\pi}^* = \mathbf{0}$ as the representative one for the efficient($\mathbf{x}_0, \mathbf{y}_0$).

3. Equivalence of Koopmans efficiency and the EBM efficiency

In this section we first demonstrate equivalence of Koopmans efficiency and EBM efficiency, and discuss how to determine \mathcal{E}^* for practical use.

3.1. Equivalence of two efficiencies

[Theorem 1]

If $(\mathbf{x}_0, \mathbf{y}_0)$ is *K-eff*, then $(\mathbf{x}_0, \mathbf{y}_0)$ is EBM-*eff*, i.e. $\tau^*(\mathcal{E}^*) = 1$ and $\boldsymbol{\pi}^* = \mathbf{0}$.

Proof: Since $(\mathbf{x}_0, \mathbf{y}_0)$ is *K-eff*, there exists $(\mathbf{v}^*, \mathbf{u}^*)$ such that

$$\begin{aligned}
\theta^* &= \mathbf{u}^* \mathbf{y}_o = 1 \\
\mathbf{v}^* \mathbf{x}_o &= 1 \\
-\mathbf{v}^* \mathbf{X} + \mathbf{u}^* \mathbf{Y} &\leq \mathbf{0} \\
v_i^* &\geq \frac{\varepsilon^*}{mx_{io}} \quad (i=1, \dots, m) \\
\mathbf{u}^* &\geq \mathbf{0}.
\end{aligned} \tag{11}$$

Let us define $\mathbf{v}^* = \mathbf{v}^*$ and $\boldsymbol{\mu}^* = \mathbf{u}^*$ in $[D(\varepsilon^*)]$. Then (x_o, y_o) is $[D(\varepsilon^*)]$ -efficient. Hence, $\tau^*(\varepsilon^*) = \theta^* = 1$. Therefore on the $[P(\varepsilon^*)]$ side, we have

$$\tau^*(\varepsilon^*) = 1 \leq 1 - \frac{\varepsilon^*}{m} \sum_{i=1}^m \frac{\pi_i}{x_{io}}. \tag{12}$$

This leads to $\boldsymbol{\pi}^* = \mathbf{0}$.

Q.E.D.

[Theorem 2]

If (x_o, y_o) is *EBM-eff*, then (x_o, y_o) is *K-eff*.

Proof: Since (x_o, y_o) is *EBM-eff*, there exists $(\mathbf{v}^*, \boldsymbol{\mu}^*)$ such that

$$\begin{aligned}
\boldsymbol{\mu}^* \mathbf{y}_o &= 1 \\
\mathbf{v}^* \mathbf{x}_o &= 1 \\
-\mathbf{v}^* \mathbf{X} + \boldsymbol{\mu}^* \mathbf{Y} &\leq \mathbf{0} \\
v_i^* &\geq \frac{\varepsilon^*}{mx_{io}} \quad (i=1, \dots, m) \\
\boldsymbol{\mu}^* &\geq \mathbf{0}.
\end{aligned} \tag{13}$$

Let us define $\mathbf{v}^* = \mathbf{v}^*$ and $\mathbf{u}^* = \boldsymbol{\mu}^*$. Then $(\mathbf{v}^*, \mathbf{u}^*)$ is optimal for $[D]$ and we have $\theta^* = 1$. Let an optimal solution for $[P]$ be $(\boldsymbol{\lambda}^*, \mathbf{s}_o^{*-})$. By the complementarity between \mathbf{v}^* and \mathbf{s}_o^{*-} we have $v_i^* s_{io}^{*-} = 0 \quad (i=1, \dots, m)$. Since $v_i^* > 0$, we have $s_{io}^{*-} = 0 \quad (i=1, \dots, m)$. Thus, (x_o, y_o) is *K-eff*. Q.E.D.

[Corollary 1]

If (x_o, y_o) is *K-ineff*, then (x_o, y_o) is *EBM-ineff* and vice-versa.

3.2. How to determine ε^*

We have defined ε^* by (6). However, $\mathbf{v}_o^* > \mathbf{0} \quad (o \in KE)$ is usually not determined uniquely. So, ε^* may be non-unique but positive. Determination of ε^* has been a controversial

issue for a long time. Originally, it was introduced as a non-Archimedean number (Charnes et al. (1978)) but not for computational purpose. The composite single-stage approach like the EBM was commented in Ali (1994) (see also Ali and Seiford (1993)) and he reported that, in certain settings of ε^* and the tolerance of LP convergence criteria, the single-stage approach produced unbounded solution and hence miss-identification of efficiency occurred. See also Johnson and Ruggiero (2009) for further reference to this subject. In our model, theoretically, unbounded solution never occurs for ε^* in the range $[0, 1]$. (See Section 4.2 for detail.) We do not intend the EBM as a computational tool.¹ Our final target is to check stability of scores against push-up of the dual variables v (shadow prices) corresponding to inputs in $[D(\varepsilon^*)]$ by means of ε^* . This will be discussed in Section 4.

However, if we employ EBM as a computational purpose, we suggest to set ε^* to 10^{-4} . The reason is that, from the objective function form of $[P(\varepsilon^*)]$, the order of the coefficient of ε^* is around unity. Hence contribution of the ε^* term to the efficiency score is around ε^* . In real world application of DEA, the expected reliability of the efficiency score is usually $10^{-3} \sim 10^{-4}$. For example we cannot differentiate the scores 0.9999 and 1 practically. We illustrate this issue using examples. Table 1 exhibits five DMUs (A, B, C, D and E) with two inputs (x_1, x_2) and single output (y) with the value unity ($y=1$). C, D and E have slacks against A and B in x_1 or x_2 . Figure 1 depicts the situation. So, A and B are efficient in the Koopmans definition.

Table 1: Example 1

DMU	x_1	x_2	y
A	1	2	1
B	2	1	1
C	4	1	1
D	1	4	1
E	1	8	1

¹ If identification of Koopmans efficiency is the main concern, we can do it by using the Additive model (Charnes et al. (1985)) or if evaluation of scores based on remaining slacks is the target, we can use the SBM model (Tone (2001)) and its variants (Tone (2009)).

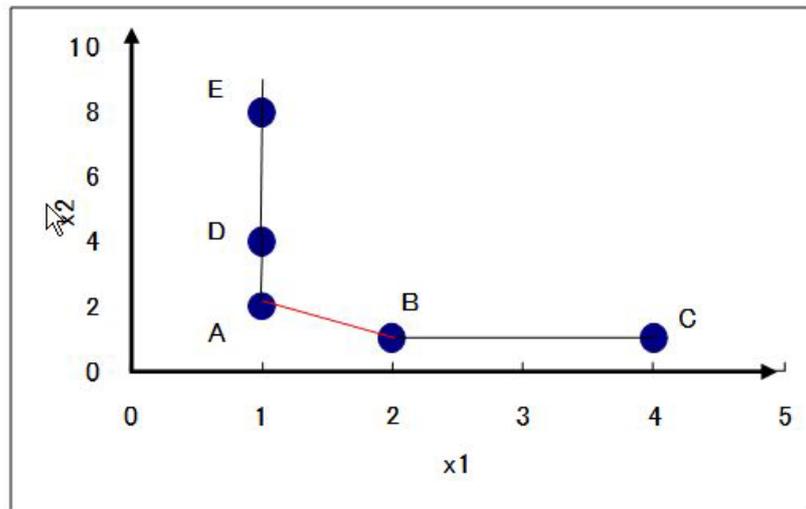


Figure 1: Example 1

Table 2 reports EBM scores for several ε^* values. The last column is the SBM which corresponds to $\tau=1$ and $\varepsilon^*=1$.²

Table 2: EBM score of Example 1

DMU	$\varepsilon^*=0$ (CCR)	$\varepsilon^*=0.0001$	$\varepsilon^*=0.01$	$\tau=1, \varepsilon^*=1$ (SBM)
A	1	1	1	1
B	1	1	1	1
C	1	0.999975	0.9975	0.75
D	1	0.999975	0.9975	0.75
E	1	0.9999625	0.99625	0.625

We observe its dual problem. Corresponding to the constraint $v_1x_1 + v_2x_2 = 1$, we have, for $\varepsilon^*=0$ case, five line segments in Figure 2. On each line segment, we maximize u subject to $-v_1x_{1j} - v_2x_{2j} + u \leq 0$ ($j=A,B,C,D,E$). DMU A is efficient on the line segment $\overline{P_1P_2}$, while B is efficient on $\overline{P_2P_3}$. DMUs D and E is efficient only at the point P_1 ($v_1=1, v_2=0$) whereas C is efficient only at the point P_3 ($v_1=0, v_2=1$). C, D and E are inefficient in the Pareto-Koopmans definition.

² See Proposition 3 in Section 4.2.

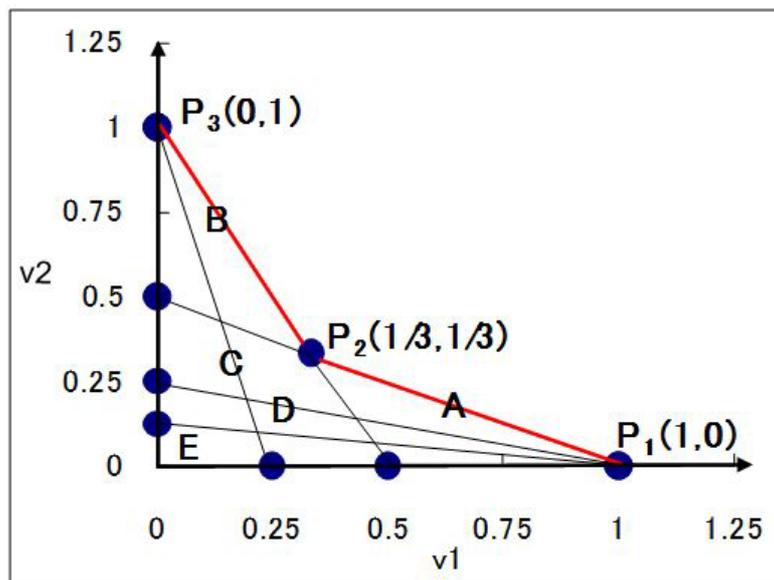


Figure 2: Dual side of Example 1

Now we introduce the constraints $v_1 \geq 0.05$ and $v_2 \geq 0.05$. These constraints are consistent with the setting $\varepsilon^* = 0.1$. Figure 3 depicts this addition. Points P_1 ($v_1 = 1, v_2 = 0$) and P_3 ($v_1 = 0, v_2 = 1$) are cut off from the feasible region and hence C, D and E are no more efficient as exhibited in Table 2. Depending on the increases in ε^* , their efficiency scores decrease.

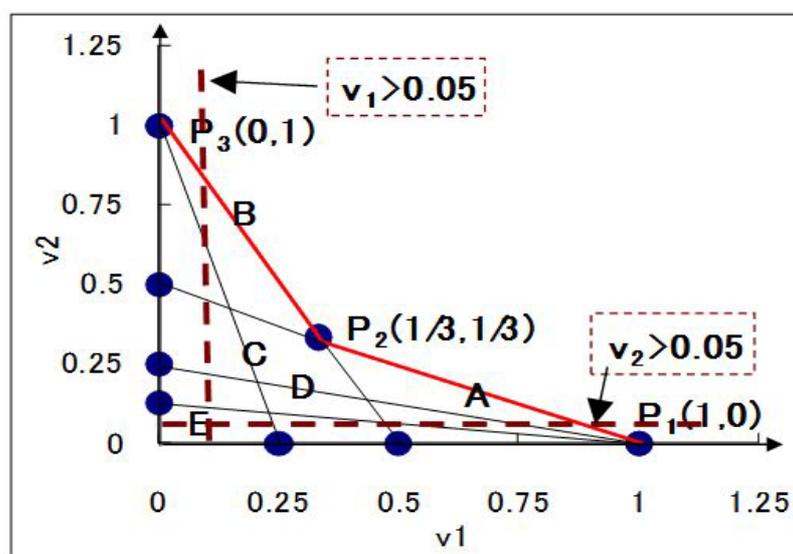


Figure 3: Introduction of positive dual variables

One of the drawbacks of the positive setting $\varepsilon^* = 10^{-4}$ is that, in some exceptional cases, efficient DMUs may be cut off and judged inefficient. Table 3 exhibits such a case, where DMUs D and E have $x_1 = 0.99999$. D is no more inferior to A, although E is dependent on D. The efficient line segment for D is a very short range from $(v_1 = 0.99999, v_2 = 0.000005)$ to $(v_1 = 1.00001, v_2 = 0)$. If we set $\varepsilon^* = 0.0001$, we restrict $v_2 \geq 0.0001 / (2 * 4) = 0.0000125$. Thus, this efficient line segment for D is cut off and D is judged inefficient as Table 4 demonstrates. However, if we set $\varepsilon = 0.000001$ (this corresponds to $v_2 \geq 0.000001 / (2 * 4) = 0.000000125$), a portion of the efficient line segment for D still remains valid and hence D can be identified as efficient. Here we notice that in most DEA applications the degree of accuracy of efficiency score is within 0.001 and hence we need not use a very small ε^* . As we see from Table 4, practically no difference exists among the efficiency scores of A, B, C, D and E. They can be regarded as unity.

In Table 4, we also described the results for $\varepsilon = 0.1, \varepsilon = 1$ and the slacks-based measure (SBM) ($\tau = 1, \varepsilon = 1$) cases. All EBM scores are not increasing in ε contrary to the SBM score. For example, D has EBM score 0.750005 for $\varepsilon = 1$ while its SBM score is 1 verifying its efficient status. The difference is caused by the constraint $\tau = 1$ in the SBM whereas τ is free in the EBM. Actually, the EBM optimal solution of D for $\varepsilon = 1$ is $\tau = 1.00001, \pi_1^- = 0, \pi_2^- = 2.00004, \xi_A = 1, \xi_j = 0 (j \neq A)$. Hence, D $(x_1 = 0.99999, x_2 = 4)$ is projected (referred) to A $(x_1 = 1, x_2 = 2)$ so that the projected D is efficient for $\varepsilon = 1$. This increase in x_1 never occurs in the traditional radial (CCR) and non-radial (SBM) models.

Table 3: Example 2

DMU	x_1	x_2	y
A	1	2	1
B	2	1	1
C	4	1	1
D	0.99999	4	1
E	0.99999	8	1

Table 4: EBM score of Example 2

DMU	$\varepsilon = 0$ (CCR)	$\varepsilon = 0.0001$	$\varepsilon = 0.000001$	$\varepsilon = 0.1$	$\varepsilon = 1$	$\tau = 1, \varepsilon = 1$ (SBM)
A	1	1	1	1	1	1
B	1	1	1	1	1	1
C	1	0.999975	0.9999997	0.975	0.75	0.75
D	1	0.999985	1	0.9750095	0.750005	1
E	1	0.9999725	0.9999997	0.9625095	0.625	0.625

4. Extension of the EBM

So far ε^* is defined by (6), or, as we noticed in Section 3.2, we set $\varepsilon^* = 10^{-4}$. In this section, we extend the role of ε^* as a parameter ranging the interval $[0, 1]$.

4.1 Extended EBM

We define the primal and dual pair $[P(\varepsilon)]$ and $[D(\varepsilon)]$ as follows:

$[P(\varepsilon)]$

$$\theta^*(\varepsilon) = \min_{\theta, \lambda, s^-} \theta - \varepsilon \sum_{i=1}^m \frac{w_i s_i^-}{x_{io}} \quad (14a)$$

$$\text{subject to } \theta \mathbf{x}_o - \mathbf{X}\lambda - \mathbf{s}^- = \mathbf{0} \quad (14b)$$

$$\mathbf{Y}\lambda \geq \mathbf{y}_o$$

$$\lambda \geq \mathbf{0} \quad (14c)$$

$$\mathbf{s}^- \geq \mathbf{0}.$$

$[D(\varepsilon)]$

$$\theta^*(\varepsilon) = \max_{\mathbf{v}, \mathbf{u}} \mathbf{u}\mathbf{y}_o \quad (15a)$$

$$\text{subject to } \mathbf{v}\mathbf{x}_o = 1 \quad (15b)$$

$$-\mathbf{v}\mathbf{X} + \mathbf{u}\mathbf{Y} \leq \mathbf{0} \quad (15c)$$

$$v_i \geq \frac{\varepsilon w_i}{x_{io}} \quad (i=1, \dots, m) \quad (15d)$$

$$\mathbf{u} \geq \mathbf{0},$$

where w_i is the weight (relative importance) of input i which is supplied exogenously and satisfies $\sum_{i=1}^m w_i = 1$ ($w_i \geq 0 \forall i$).³ As can be seen from the term $\frac{w_i s_i^-}{x_{io}}$ in the objective

³ We can impose weights individually to DMU o such that $\sum_{i=1}^m w_{io} = 1$ ($w_{io} \geq 0 \forall i$).

function of $[P(\varepsilon)]$, $\frac{s_i^-}{x_{io}}$ is units-invariant and so w_i should be a units-invariant value reflecting the relative importance of resource i . We will discuss this subject later in Section 4.3. This model is also an extension of the weighted SBM by Tsutsui and Goto (2009).

$[P(\varepsilon)]$ has a feasible solution $\theta = 1, \lambda_o = 1, \lambda_j = 0 (j \neq o)$ and $s^- = \mathbf{0}$. Hence we have $\theta^*(\varepsilon) \leq 1$.

The constraint $v_i \geq \frac{\varepsilon w_i}{x_{io}} (i = 1, \dots, m)$ restricts the lower bound of the shadow price v_i of input resource i . This bound becomes proportionally tight in ε and w_i .

Let an optimal solution of $[P(\varepsilon)]$ be $(\theta^*, \lambda^*, s^{-*})$. Similarly to Definition 6 in Section 2, we have:

[Definition 7] (EBM-efficient)

DMU o is called EBM-efficient if $\theta^*(\varepsilon) = 1$.

If DMU o is EBM-efficient, then $\theta = 1, \lambda_o = 1, \lambda_j = 0 (j \neq o), s^- = \mathbf{0}$ gives an optimal solution.

However, there might be other optimal solutions that have $\theta > 1$ and $s^- \neq \mathbf{0}$.

If DMU o is EBM-inefficient, we define its projection (x_o^*, y_o^*) as follows.

[Definition 8] (EBM-projection)

$$\begin{aligned} x_o^* &= X\lambda^* = \theta^* x_o - s^{-*} \\ y_o^* &= Y\lambda^*. \end{aligned} \tag{16}$$

[Theorem 4]

The projected (x_o^*, y_o^*) is EBM-efficient.

See Appendix A for a proof.

4.2. Several properties of the EBM model

[Proposition 1]

$\theta^*(\varepsilon)$ in $[P(\varepsilon)]$ is units-invariant, i.e. $\theta^*(\varepsilon)$ is independent of the units in which the inputs and outputs are measured.

[Proposition 2]

If we set $\varepsilon = 0$ in $[P(\varepsilon)]$, then $[P(0)]$ reduces to the input-oriented CCR model.

[Proposition 3]

If we set $\theta = 1$ and $\varepsilon = 1$ in $[P(\varepsilon)]$, then $[P(1)]$ reduces to the input-oriented SBM model.

Thus, $[P(\varepsilon)]$ includes the radial CCR and the non-radial SBM models as special cases,

but it is basically non-radial.

The constraints (15b) and (15d) lead to $1 = \mathbf{v}\mathbf{x}_o = \sum_{i=1}^m v_i x_{io} \geq \varepsilon$. Thus, ε must be not greater than unity.

[Proposition 4]

[P(ε)] and [D(ε)] have a finite optima for $\varepsilon \in [0,1]$.

[Proposition 5]

For $\varepsilon > 1$, [D(ε)] has no feasible solution and [P(ε)] has unbounded solution.

[Proposition 6]

$\theta_o^*(\varepsilon)$ is non-increasing in ε .

4.3 On the weight $\{w_i\}$

We have several options for choosing the weight w_i , e.g. using an objective gauge based on actual data or employing a subjective judgment of decision makers.

(a) Objective way

From (15d), we have $v_i x_{io} \geq \varepsilon w_i$ ($i=1, \dots, m$). This suggests that w_i has the same dimension with $v_i x_{io}$, i.e. the virtual cost of input i . Thus, if the unit price c_{io} of x_{io} is available, the following scheme might be a choice. The input cost C_o is calculated as $C_o = \sum_{i=1}^m c_{io} x_{io}$ and the cost share r_{io} of input i is defined by $r_{io} = c_{io} x_{io} / C_o$. We can define a cost share based weight as the average of r_{io} over the entire DMUs as follows.

$$w_i = \frac{1}{n} \sum_{j=1}^n r_{ij} \quad (i=1, \dots, m). \quad (17)$$

Then, the objective function (14a) can be rewritten as

$$\theta^*(\varepsilon) = \min_{\theta, \lambda, s^-} \theta - \varepsilon \sum_{i=1}^m \frac{c_{io} s_i^-}{C_o}, \quad (18)$$

in which the second term means the value of input slacks in the total cost.

(b) Subjective way

If decision makers have some special preference on the weight of input resources, they can determine w_i subjectively. In actual situation, there usually exist multiple criteria for deciding the weights. For example, in the hospital case, if we choose doctor and nurse as main resources, their importance should be estimated by considering several factors such as salary, cost of education, principle of scarcity and so forth. For this purpose, the AHP (analytic hierarchy process (Saaty (1980))) would be of help. We remind that we are estimating (value/unit) \times (amount) of resource i , i.e. the value of resource i , but not the value/unit itself.

4.4 Comparison of CCR, SBM and EBM models

We compare the differences among CCR, SBM and EBM by illustration. Figure 4 depicts an example with two inputs and single output case. We assume the output value equal to unity. Efficient frontiers are line segments \overline{AB} and \overline{BC} . We denote an inefficient DMU by x_o . The CCR model contracts x_o radially to P, while the SBM projects x_o to some point on the line segments \overline{QB} and \overline{BR} . Since the EBM imposes no restriction on the expansion or reduction rate θ , the point θx_o may move up and x_o may be projected to some point on the line segments \overline{AB} and \overline{BC} . Hence, it may occur that some inputs increase at the expense of decrease of other inputs. This conforms with Koopmans concept of efficiency. Thus, the potential reference sets of the three models satisfy the relation $CCR \subset SBM \subset EBM$. The optimal θ is determined as the solution of $[P(\varepsilon)]$ and it may be less than, greater than, or equal to unity.

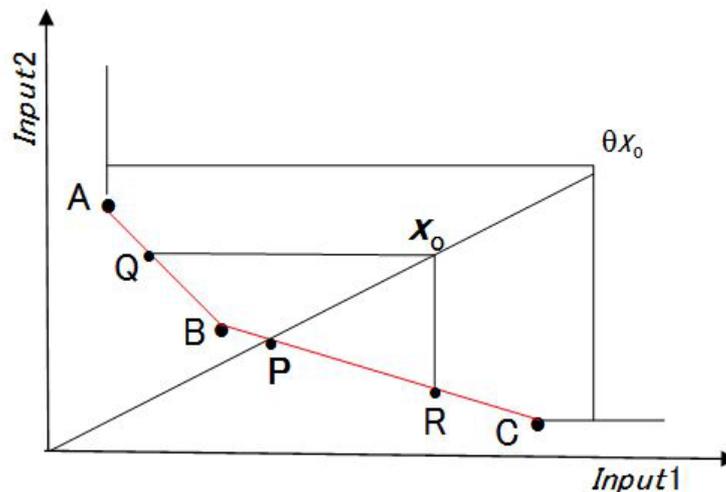


Figure 4: Comparison of CCR, SBM and EBM

5. An example of the extended EBM model

We applied EBM in Section 4 to hospitals A to L exhibited in Table 5. They have two inputs (doctor and nurse) and two outputs (outpatient and inpatient). We analyzed this in two ways: one objective and the other subjective weighting.

Table 5: Hospital

DMU	Inputs		Outputs	
	Doctor	Nurse	Outpatient	Inpatient
A	20	151	100	90

B	19	131	150	50
C	25	160	160	55
D	27	168	180	72
E	22	158	94	66
F	55	255	230	90
G	33	235	220	88
H	31	206	152	80
I	30	244	190	100
J	50	268	250	100
K	53	306	260	147
L	38	273	250	133

5.1 Case 1: Objective weighting

We employed weights to doctor vs. nurse as $w_1 = 0.44$, $w_2 = 0.56$. This is an average ratio of cost shares between doctor and nurse as demonstrated in Table 8.

Table 6 and Figure 5 report EBM score of hospitals corresponding to $\varepsilon = 0, 0.1, 0.2, \dots, 1$ where “Stability” is defined by $\theta^*(1) - \theta^*(0)$

Table 6: EBM score for Case 1

ε	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	Stability
A	1	1	1	1	1	1	1	1	1	1	1	0
B	1	1	1	1	1	1	1	1	1	1	1	0
C	0.883	0.881	0.878	0.876	0.873	0.871	0.868	0.865	0.862	0.859	0.856	-0.0266
D	1	1	1	1	0.996	0.991	0.986	0.981	0.976	0.971	0.966	-0.0342
E	0.763	0.763	0.763	0.763	0.762	0.761	0.761	0.760	0.759	0.758	0.757	-0.00663
F	0.835	0.825	0.815	0.806	0.796	0.784	0.771	0.759	0.746	0.733	0.721	-0.11388
G	0.902	0.901	0.900	0.900	0.899	0.898	0.897	0.896	0.896	0.895	0.894	-0.00798
H	0.796	0.796	0.796	0.796	0.794	0.791	0.788	0.786	0.783	0.780	0.778	-0.01872
I	0.960	0.954	0.948	0.942	0.935	0.929	0.923	0.917	0.910	0.904	0.898	-0.06244
J	0.871	0.865	0.860	0.855	0.848	0.839	0.830	0.820	0.811	0.802	0.793	-0.07765
K	0.955	0.949	0.943	0.937	0.929	0.921	0.912	0.904	0.895	0.886	0.878	-0.07727
L	0.958	0.956	0.953	0.951	0.948	0.946	0.943	0.941	0.938	0.936	0.933	-0.02485

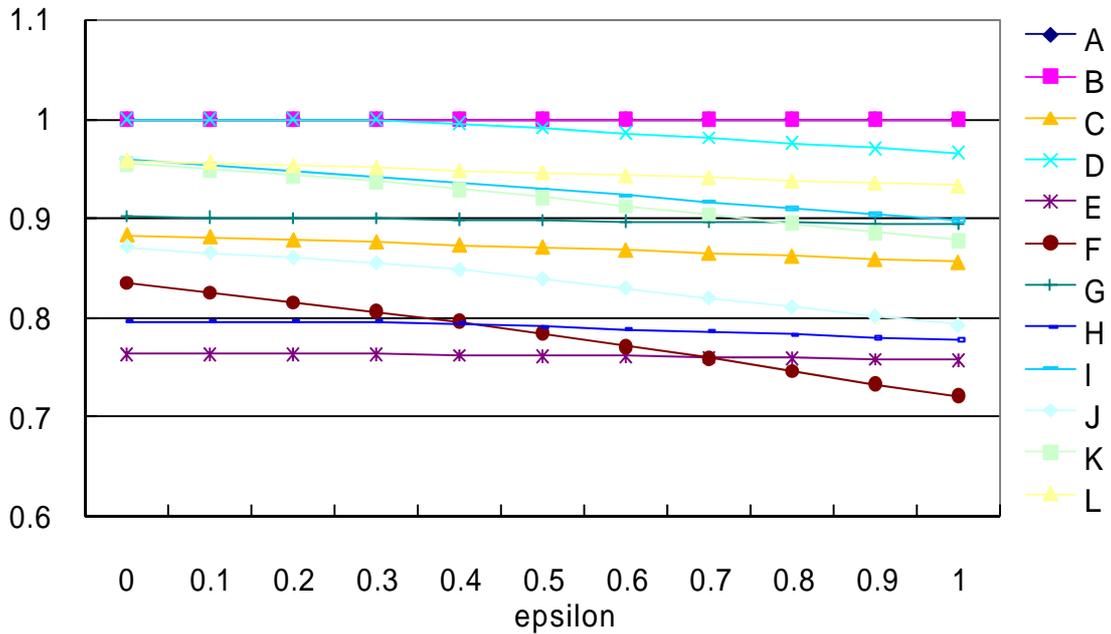


Figure 5: Variation of EBM scores in terms of ϵ for Case 1

Observations:

- Hospitals A and B keep the status of full efficiency throughout the interval $[0,1]$ of ϵ . They are relatively the most robust against the stress of the lower bounds of input shadow prices.
- Hospital D is efficient for $\epsilon = 0-0.3$. However, it drops to inefficient at $\epsilon = 0.4$. (We can calculate the exact turning point by applying the parametric programming technique in ϵ .) This hospital is not robust as A and B against the stress of input shadow prices. Relatively large number of doctors forces this hospital to inefficiency as the stress v_1 to doctor increases. At $\epsilon = 1$, D has the optimal solution $\theta^*(1) = 0.966, \theta = 1.016, \lambda_A = 0.212, \lambda_B = 1.059, s_1^- = 3.078, s_2^- = 0$. Projected numbers of doctor and nurse are respectively 24.35 and 170.68 as contrasted to the observed values 27 and 168. Thus, this hospital is recommended to reduce doctors and increase nurses under the stress of input prices.
- Hospital F is the most sensitive to the input stress. It drops from 0.8348 ($\epsilon = 0$) to 0.721 ($\epsilon = 1$).
- Hospitals E, G, I and L are stable throughout the interval $[0, 1]$ of ϵ . Their references are mostly A and B.
- Rank reversals of efficiency scores are observed in many instances reflecting the

comparative weakness of respective hospitals against input price impacts.

5.2 Case 2: Subjective weighting

Since the role of doctor is increasing in medical services, we assume a scenario that the weight of doctor is five times that of nurse, i.e. $w_1 = 0.833, w_2 = 0.167$. Table 7 and Figure 6 show the results of EBM scores under this weight setting.

Table 7: EBM score for Case 2

ε	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	Stability
A	1	1	1	1	1	1	1	1	1	1	1	0.000
B	1	1	1	1	1	1	1	1	1	1	1	0.000
C	0.883	0.879	0.874	0.869	0.863	0.858	0.852	0.847	0.841	0.836	0.830	-0.052
D	1	1	0.997	0.987	0.978	0.968	0.959	0.949	0.940	0.930	0.921	-0.079
E	0.763	0.763	0.763	0.761	0.759	0.757	0.756	0.754	0.752	0.750	0.749	-0.015
F	0.835	0.816	0.798	0.775	0.751	0.728	0.704	0.680	0.656	0.632	0.609	-0.226
G	0.902	0.902	0.901	0.901	0.901	0.901	0.901	0.900	0.900	0.900	0.900	-0.002
H	0.796	0.796	0.794	0.789	0.784	0.779	0.774	0.769	0.764	0.759	0.754	-0.043
I	0.960	0.959	0.957	0.955	0.953	0.951	0.949	0.947	0.946	0.944	0.942	-0.019
J	0.871	0.861	0.850	0.833	0.815	0.798	0.780	0.763	0.746	0.728	0.711	-0.160
K	0.955	0.944	0.931	0.915	0.899	0.882	0.866	0.850	0.834	0.817	0.801	-0.154
L	0.958	0.957	0.957	0.956	0.955	0.955	0.954	0.953	0.952	0.952	0.951	-0.007

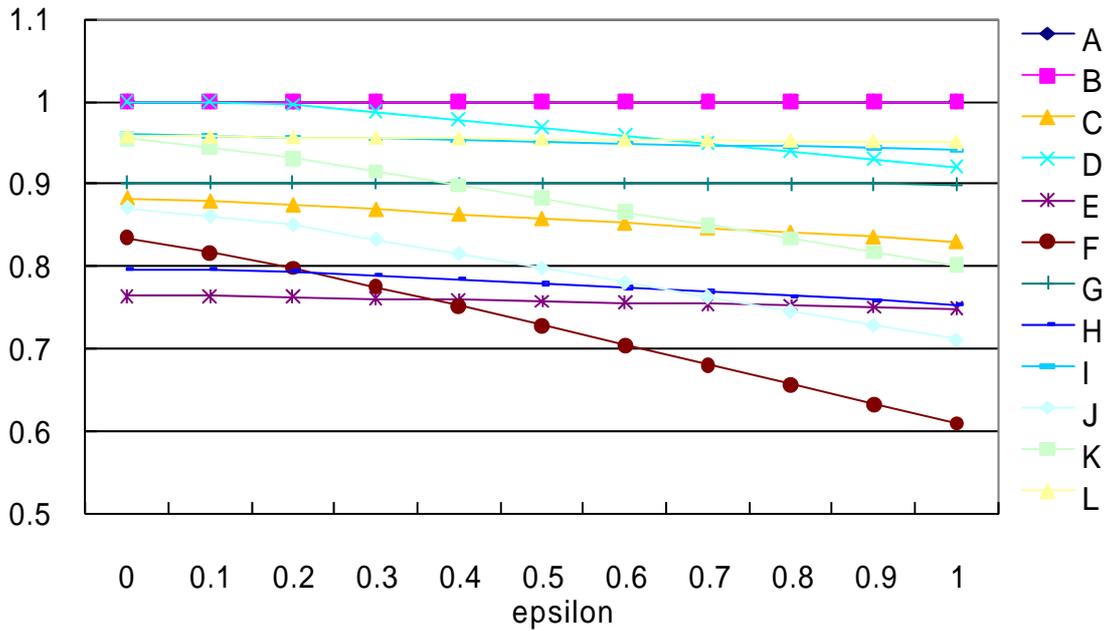


Figure 6: Variation of EBM scores in terms of ϵ for Case 2

Observations

Generally, EBM scores go down in Case 2 compared with Case 1. For example, Hospital D loses its efficiency status at $\epsilon = 0.2$ in Case 2 whereas it becomes inefficient at $\epsilon = 0.4$ in Case 1. These are caused by the large weight to Doctor. Stability becomes weak on average in Case 2 compared with Case 1, exception being Hospitals G, I and L. These three hospitals employ a relatively large number of nurses.

6. How to find an appropriate ϵ

Looking at the above examples, we are solicited to gauge an appropriate level of ϵ at the current stage. The following is our attempt for this purpose based on an analogy between the shadow cost and the actual cost. Recalling the constraint $v_i \geq \frac{\epsilon w_i}{x_{io}}$ ($i = 1, \dots, m$),

we obtain a bound of ϵ by

$$\epsilon \leq \frac{v_{io} x_{io}}{w_i} \quad (i = 1, \dots, m), \quad (19)$$

where w_i denotes the weight (relative importance) of input i , which can be determined objectively or subjectively based on the average cost share of input i (See Section 4.3). And thus, ϵ is bounded by the ratio of shadow cost ($v_{io} x_{io}$) and utilized weight (w_i).

Let the unit price (or its proxy) of input resource i for DMU o be c_{io} . Then the total

input cost C_o of DMU o is obtained as

$$C_o = \sum_{i=1}^m c_{io} x_{io} . \quad (20)$$

This leads to

$$1 = \sum_{i=1}^m \frac{c_{io}}{C_o} x_{io} . \quad (21)$$

Recalling the constraint $\sum_{i=1}^m v_i x_{io} = 1$ of (15b), it looks like that cost share of input i

expressed as $r_{io} = c_{io} x_{io} / C_o$ can play the role of $v_i x_{io}$ as proxy. Then, we can assume the

bounds on shadow cost as

$$\min_j (r_{ij}) \leq v_i x_{io} \leq \max_j (r_{ij}) . \quad (22)$$

Then, inserting (22) into (19), we can obtain

$$\varepsilon \leq \frac{\min_j (r_{ij})}{w_i} = \frac{r_i}{w_i} \quad (i = 1, \dots, m) . \quad (23)$$

where $r_i = \min_j (r_{ij})$ is the minimum cost share of input i among the entire DMUs.

Hence we have an approximation of ε by

$$\varepsilon = \min_i \left\{ \frac{r_i}{w_i} \right\} . \quad (24a)$$

which indicates that ε can be determined as the minimum ratio between the minimum cost share r_i as a proxy of a bound on shadow cost and the average cost share w_i .

Cost share r_{io} and its minimum value among DMUs r_i may be obtained directly from financial data even when we have no unit price information. We also notice that, when we have no data on the cost share r_{io} but its proxy r_i is available, we can estimate the appropriate level by

$$\varepsilon = \min_i \left\{ \frac{r_i}{w_i} \right\} . \quad (24b)$$

[Definition 8] (Cost critical resource)

Let resource i^* be the resource that gives the minimum of (24a) or (24b). We call i^* the cost critical resource for the DMU o .

We applied the above scheme to the hospital case in Table 5 and obtained the results

exhibited in Table 8 where c_1 and c_2 indicate unit prices of doctor and nurse, and r_1 and r_2 are cost share of doctor and nurse, respectively. We set the weights to doctor and nurse as $w_1 = 0.44$ and $w_2 = 0.56$ in Case 1 which are their average cost shares and, $w_1 = 0.833$ and $w_2 = 0.167$ in Case 2 by a subjective judgment. From the results we can see that, in Case 1, ε_o are determined case by case by the cost share of doctor or nurse (the shaded number in Table 8) and hence the cost critical resource for hospitals is different from hospital to hospital. The minimum is attained at hospital F with $\varepsilon = 0.807$. Hence we may safely select $\varepsilon = 0.8$ as appropriate in Case 1. The results are quite different in Case 2. All ε_o are determined by doctor and hence doctor is the cost critical resource for all hospitals. The minimum is attained at hospital E with $\varepsilon = 0.448$. Hence we may safely select $\varepsilon = 0.4$ as appropriate in Case 2.

Table 8: Cost data and Appropriate choice of ε for Case 1 and Case 2

	Cost		Cost share		Case 1		Case 2						
	Doctor	Nurse	Doctor	Nurse	Input	Doctor	Nurse	Doctor	Nurse				
	x_1	c_1	x_2	c_2	c_1x_1	c_2x_2	C	r_1	r_2	ε_1	ε_2	ε_1	ε_2
A	20	500	151	100	10000	15100	25100	0.398	0.602	0.905	1.074	0.478	3.610
B	19	350	131	80	6650	10480	17130	0.388	0.612	0.882	1.092	0.466	3.671
C	25	450	160	90	11250	14400	25650	0.439	0.561	0.997	1.003	0.526	3.368
D	27	600	168	120	16200	20160	36360	0.446	0.554	1.013	0.990	0.535	3.327
E	22	300	158	70	6600	11060	17660	0.374	0.626	0.849	1.118	0.448	3.758
F	55	450	255	80	24750	20400	45150	0.548	0.452	1.246	0.807	0.658	2.711
G	33	500	235	100	16500	23500	40000	0.413	0.588	0.938	1.049	0.495	3.525
H	31	450	206	85	13950	17510	31460	0.443	0.557	1.008	0.994	0.532	3.340
I	30	380	244	76	11400	18544	29944	0.381	0.619	0.865	1.106	0.457	3.716
J	50	410	268	75	20500	20100	40600	0.505	0.495	1.148	0.884	0.606	2.970
K	53	440	306	80	23320	24480	47800	0.488	0.512	1.109	0.915	0.585	3.073
L	38	400	284	70	15200	19880	35080	0.433	0.567	0.985	1.012	0.520	3.400

Table 9 compares the EBM scores corresponding to three ε values. The case $\varepsilon = 0$ corresponds to no restriction on dual shadow prices, while $\varepsilon = 1$ corresponds to the most stringent case. The columns $\varepsilon = 0.8$ and $\varepsilon = 0.4$ exhibit scores for Case 1 and Case 2.

Table 9: Comparisons of EBM scores corresponding to Case 1 and Case 2.

ε	Case 1			Case 2		
	0	0.8	1	0	0.4	1
A	1	1	1	1	1	1
B	1	1	1	1	1	1
C	0.883	0.862	0.856	0.883	0.863	0.830
D	1	0.976	0.966	1	0.978	0.921
E	0.763	0.759	0.757	0.763	0.759	0.749
F	0.835	0.746	0.721	0.835	0.751	0.609
G	0.902	0.896	0.894	0.902	0.901	0.900
H	0.796	0.783	0.778	0.796	0.784	0.754
I	0.960	0.910	0.898	0.960	0.953	0.942
J	0.871	0.811	0.793	0.871	0.815	0.711
K	0.955	0.895	0.878	0.955	0.899	0.801
L	0.958	0.938	0.933	0.958	0.955	0.951

7. Another interpretation of the EBM: from CCR to the cost efficiency model

In this section, we demonstrate another interpretation of the EBM and point that it is a composite of the technical (radial) and the non-radial value-dependent factors, and that, if we employ the cost share of each DMU as weight w , EBM reduces to the projection to the cost minimizing point for $\varepsilon=1$.

In the [P(ε)] ((14a)-(14c)), let us define \mathbf{x} by

$$\mathbf{x} = \theta \mathbf{x}_o - \mathbf{s}^- . \quad (25)$$

Then, we can rewrite [P(ε)] as:

$$\theta^*(\varepsilon) = \min_{\theta, \lambda, \mathbf{x}, \mathbf{s}^-} \theta - \varepsilon \sum_{i=1}^m \frac{w_i(\theta x_{io} - x_i)}{x_{io}} = \min_{\theta, \lambda, \mathbf{x}, \mathbf{s}^-} (1-\varepsilon)\theta + \varepsilon \sum_{i=1}^m \frac{w_i x_i}{x_{io}} \quad (26)$$

$$\text{subject to } \mathbf{x} - \mathbf{X}\lambda = \mathbf{0}$$

$$\mathbf{x} - \theta \mathbf{x}_o + \mathbf{s}^- = \mathbf{0}$$

$$\mathbf{Y}\lambda \geq \mathbf{y}_o \quad (27)$$

$$\lambda \geq \mathbf{0}, \mathbf{s}^- \geq \mathbf{0}.$$

The last expression of the objective function (26) specifies the EBM score as a convex combination of the technical efficiency term θ and the weighted average of the normalized inputs where weights associate with the value of input resources.

In the case $\varepsilon=0$, it reduces to the CCR (radial) model, whereas in accordance with the increase in ε , more emphasis is put on the non-radial (value dependent) part. At the another end $\varepsilon=1$, if we employ the cost share of each DMU as weight w_o , we can show

that [P(1)] produces the same optimal solution with the cost efficient model which can be described as follows.

$$\begin{aligned}
 \text{[CM]} \quad & \min_{\mathbf{x}, \boldsymbol{\lambda}} \sum_{i=1}^m c_{io} x_{io} \\
 \text{subject to} \quad & x_i = \sum_{j=1}^n x_{ij} \lambda_j \quad (i=1, \dots, m) \\
 & y_{io} \leq \sum_{j=1}^n y_{ij} \lambda_j \quad (i=1, \dots, s) \\
 & \lambda_j \geq 0 \quad (\forall j).
 \end{aligned} \tag{28}$$

This model aims to obtain the cost minimizing input \mathbf{x} for producing the given output \mathbf{y}_o . If we employ the cost share $c_{io} x_{io} / C_o$ (C_o = total input cost of DMU $_o$) as the weight w_{io} , the EBM [P(1)] reduces to:

$$\begin{aligned}
 \min_{\theta, \boldsymbol{\lambda}, \mathbf{x}, \mathbf{s}^-} \quad & \frac{1}{C_o} \sum_{i=1}^m c_{io} x_i \\
 \text{subject to} \quad & \mathbf{x} - \mathbf{X}\boldsymbol{\lambda} = \mathbf{0} \\
 & \mathbf{x} - \theta \mathbf{x}_o + \mathbf{s}^- = \mathbf{0} \\
 & \mathbf{Y}\boldsymbol{\lambda} \geq \mathbf{y}_o \\
 & \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{s}^- \geq \mathbf{0}.
 \end{aligned} \tag{29}$$

Let an optimal solution to (29) be $(\theta^*, \mathbf{x}^*, \boldsymbol{\lambda}^*, \mathbf{s}^{-*})$. Then, $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is feasible for [CM]. Conversely, let an optimal solution to [CM] be $(\bar{\mathbf{x}}^*, \bar{\boldsymbol{\lambda}}^*)$. We define

$$\bar{\theta}^* = \max \left\{ \frac{x_i^*}{x_{io}^*} \right\} (i=1, \dots, m) \quad \text{and} \quad \bar{\mathbf{s}}^{-*} = \bar{\theta}^* \mathbf{x}_o - \bar{\mathbf{x}}^*. \tag{30}$$

Then, $(\bar{\theta}^*, \bar{\mathbf{x}}^*, \bar{\boldsymbol{\lambda}}^*, \bar{\mathbf{s}}^{-*})$ is feasible for (29). Thus, we have the equality between the two optimal solutions:

$$\mathbf{c}_o \mathbf{x}^* = \mathbf{c}_o \bar{\mathbf{x}}^*. \tag{31}$$

Hence, $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is optimal for [CM] and the projected solution $\bar{\mathbf{x}}^* = \mathbf{X}\boldsymbol{\lambda}^*$ is the cost minimizing point.

Thus, under the assumption that the cost share of each DMU is employed as weight w_o , the projected point of the EBM model moves from the CCR projected point to the cost minimizing point in line with the increase of ε from 0 to 1. Thus, the EBM model includes the cost model as a special case.⁴

⁴ However, we notice that we are not discussing cost efficiency but cost minimizing point, since this subject needs care in handling. See Tone (2002).

8. Concluding remarks

In this paper we have proposed an epsilon-based measure (EBM) model which can discriminate Farrell and Pareto-Koopmans efficiencies practically. Then, we extended EBM to the stability analysis of efficiency score employing the lower bounds of shadow prices of input resources as a parameter. Furthermore, we proposed a tentative scheme for determining an appropriate level of the parameter referring to the actual cost information. As a result, we can evaluate technical efficiency of DMUs based on the cost share of input resources. This subject needs further clarification. The EBM scores are affected by the weight $\{w_i\}$ which can be determined objectively or subjectively. The selection of $\{w_i\}$ is an important subject of management decision to be explored further. We also demonstrated that EBM is a composite of the radial (technical) efficiency and the non-radial (value-dependent) efficiency, and that it includes the CCR and the cost minimizing model as special cases. This might be a new finding for connecting technical and cost models in a unified framework.

Although we have dealt with the constant returns-to-scale situation, we can apply this model to other returns-to-scale cases by imposing constraints on the primal intensity vector.

Future research subjects include extensions to output-oriented (revenue) and non-oriented (profit) models.

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Appendix A. Proof of Theorem 4

Since (x_o, y_o) is EBM-inefficient, it holds that

$$\theta^*(\varepsilon) = \theta^* - \varepsilon \sum_{i=1}^m \frac{w_i s_i^-}{x_{i0}} < 1. \quad (\text{A1})$$

Let an optimal solution for (x_o^*, y_o^*) be $(\theta^{**}(\varepsilon), \theta^{**}, \lambda^{**}, s^{-**})$. The EBM objective function value is:

$$\theta^{**}(\varepsilon) = \theta^{**} - \varepsilon \sum_{i=1}^m \frac{w_i s_i^{-**}}{x_{i0}^*}. \quad (\text{A2})$$

The corresponding constraints for $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ are:

$$\theta^{**} \mathbf{x}_o^* = \mathbf{X}\boldsymbol{\lambda}^{**} + \mathbf{s}^{-**}, \quad \mathbf{y}_o^* \leq \mathbf{Y}\boldsymbol{\lambda}^{**}. \quad (\text{A3})$$

This reduces to:

$$\theta^{**} \theta^* \mathbf{x}^* = \mathbf{X}\boldsymbol{\lambda}^{**} + \mathbf{s}^{-**} + \theta^* \mathbf{s}^{-*}, \quad \mathbf{y}_o \leq \mathbf{y}_o^* \leq \mathbf{Y}\boldsymbol{\lambda}^{**}. \quad (\text{A4})$$

This is another expression for $(\mathbf{x}_o, \mathbf{y}_o)$ and its objective function value is:

$$f = \theta^{**} \theta^* - \varepsilon \sum_{i=1}^m \frac{w_i (s_i^{-**} + \theta^* s_i^{-*})}{x_{i0}^*} = \theta^{**} \theta^*(\varepsilon) - \varepsilon \sum_{i=1}^m \frac{w_i s_i^{-**}}{x_{i0}^*}. \quad (\text{A5})$$

We have three possibilities as follows:

- i) The case $\theta^{**} < 1$. In this case, it holds that $f < \theta^*(\varepsilon)$. This contradicts the optimality of $\theta^*(\varepsilon)$ for $(\mathbf{x}_o, \mathbf{y}_o)$. Thus, this case never occurs.
- ii) The case $\theta^{**} = 1$. In this case, by the optimality of $\theta^*(\varepsilon)$ for $(\mathbf{x}_o, \mathbf{y}_o)$, we have $s_i^{-**} = 0 (\forall i)$. Thus, $\theta^{**}(\varepsilon) = 1$ and $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ is EBM-efficient.
- iii) The case $\theta^{**} > 1$. From the optimality of $\theta^*(\varepsilon)$ for $(\mathbf{x}_o, \mathbf{y}_o)$, it holds that

$$\theta^{**} \theta^*(\varepsilon) - \varepsilon \sum_{i=1}^m \frac{w_i s_i^{-**}}{x_{i0}^*} \geq \theta^*(\varepsilon).$$

Hence we have

$$\theta^{**} \geq 1 + \frac{\varepsilon}{\theta^*(\varepsilon)} \sum_{i=1}^m \frac{w_i s_i^{-**}}{x_{i0}^*}. \quad (\text{A6})$$

Suppose that $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ is EBM-inefficient, i.e. $\theta^{**}(\varepsilon) = \theta^{**} - \varepsilon \sum_{i=1}^m \frac{w_i s_i^{-**}}{x_{i0}^*} < 1$. Then

we have:

$$\theta^{**} < 1 + \varepsilon \sum_{i=1}^m \frac{w_i s_i^{-**}}{x_{i0}^*}. \quad (\text{A7})$$

We compare the terms $\theta^*(\varepsilon)x_{i0}^*$ in (A6) and x_{i0}^* in (A7). Since $x_{i0}^* = \theta^* x_{i0} - s_i^{-*}$, we have

$$\theta^*(\varepsilon)x_{i0}^* - x_{i0}^* = (\theta^*(\varepsilon) - \theta^*)x_{i0} - s_i^{-*} = -\varepsilon x_{i0} \left(\sum_{k=1}^m \frac{w_k s_k^{-*}}{x_{k0}^*} \right) - s_i^{-*} \leq 0.$$

Thus, it holds that

$$\theta^*(\varepsilon)x_{i0}^* \leq x_{i0}^*. \quad (\text{A8})$$

Comparing (A6) and (A7), we have

$$\theta^{**} \geq 1 + \varepsilon \sum_{i=1}^m \frac{w_i s_i^{-**}}{\theta^*(\varepsilon)x_{i0}^*} \geq 1 + \varepsilon \sum_{i=1}^m \frac{w_i s_i^{-**}}{x_{i0}^*} > \theta^{**}. \quad (\text{A9})$$

This cannot occur. Thus, in this case, $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ is EBM-efficient.

Q.E.D.