

An epsilon-based measure of efficiency in DEA revisited
-A third pole of technical efficiency-

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Abstract

In DEA, we have two measures of technical efficiency with different characteristics: radial and non-radial. In this paper we compile them into a composite model called “epsilon-based measure (EBM).” For this purpose we introduce two parameters which connect radial and non-radial models. These two parameters are obtained from the newly defined affinity index between inputs or outputs along with principal component analysis on the affinity matrix. Thus, EBM takes into account diversity of input/output data and their relative importance for measuring technical efficiency.

Keywords: Data envelopment analysis, Radial, Non-radial, CCR, SBM, EBM, Principal component analysis

1. Introduction

DEA (data envelopment analysis) is a data driven tool for measuring efficiency of decision making units (DMU) and shows a sharp contrast to so-called “parametric methods” such as SFA. The latter methods assume specific production function forms to be identified. This assumption is not so reasonable in several instances and aspects. Since DEA can deal with multiple input vs. multiple output relations in a single framework, it has been becoming a method of choice for efficiency evaluation in recent days. However, DEA has several shortcomings to be explored further. In DEA, we have two measures of technical efficiency with different characteristics: radial and non-radial. Historically, the radial measure, represented by the CCR model (Charnes, Cooper and Rhodes [5]), was the first DEA model, whereas the non-radial model, represented by the SBM model (slacks-based measure by Tone [8], see also Cooper et al. [6]) was a latecomer. For instance, in the input-oriented case, the CCR deals mainly with proportionate reduction of input resources. In other words, if the organisational unit under study, also known as a DMU, has two inputs, this model aims at obtaining the maximum rate of reduction with the same proportion, i.e. a radial contraction in the two inputs that can produce the current outputs. In contrast, the non-radial models put aside

the assumption of proportionate contraction in inputs and aim at obtaining maximum rates of reduction in inputs that may discard varying proportions of original input resources.

In this paper, after introducing radial and non-radial models briefly, we propose a composite model which combines both models in a unified framework. This model has two parameters: one scalar and one vector. In order to determine these two parameters, we introduce a new affinity index associated with inputs or outputs. We apply principal component analysis to thus defined affinity matrix.

This paper unfolds as follows. In Section 2, we briefly survey radial and non-radial models in DEA. In Section 3, we propose the epsilon-based measure of efficiency (EBM). EBM needs two parameters. After observing two extreme diversities of dataset, we introduce a new correlation coefficient called “affinity index” in Section 4. We utilize this index for defining affinity matrix among input/output data. From this matrix we derive two parameters for EBM in Section 5. We discuss rationality of the scheme in Section 6. We demonstrate illustrative examples in Section 7. In Section 8 we extend the model to other orientations and variable returns-to-scale environment. We conclude this paper in Section 9.

2. Radial and non-radial measures of efficiency

In this section we introduce the CCR and SBM models as representative radial and non-radial measures of efficiency respectively, and point out their shortcomings. Throughout this paper, we deal with n DMUs ($j=1,\dots,n$) having m inputs ($i=1,\dots,m$) and s outputs ($r=1,\dots,s$). The input and output matrices are denoted by $\mathbf{X}=\{x_{ij}\}\in R^{m\times n}$ and $\mathbf{Y}=\{y_{rj}\}\in R^{s\times n}$, respectively. We assume $\mathbf{X}>\mathbf{0}$ and $\mathbf{Y}>\mathbf{0}$.

2.1 The CCR and SBM Models

We briefly explain the CCR and SBM models, and compare their inefficiency status.

(a) *The CCR Model*

The input-oriented CCR model evaluates the technical efficiency θ^* of DMU $(\mathbf{x}_o, \mathbf{y}_o)$ by solving the following linear program:

[CCR-I]

$$\theta^* = \min_{\theta, \lambda, s} \theta \tag{1}$$

subject to

$$\begin{aligned}
\theta \mathbf{x}_o &= \mathbf{X}\boldsymbol{\lambda} + \mathbf{s}^- \\
\mathbf{y}_o &\leq \mathbf{Y}\boldsymbol{\lambda} \\
\boldsymbol{\lambda} &\geq \mathbf{0}, \mathbf{s}^- \geq \mathbf{0},
\end{aligned} \tag{2}$$

where $\boldsymbol{\lambda}$ represents the intensity vector and \mathbf{s}^- denotes the non-radial slacks.

Usually, we solve [CCR-I] in a two phase process. In the first phase, we solve [CCR-I] and obtain

θ^* (weak efficiency). Then, in the second phase, we maximize $\sum_{i=1}^m \frac{s_i^-}{x_{io}}$ in terms of $\boldsymbol{\lambda}$ and \mathbf{s}^- ,

subject to (2) and $\theta = \theta^*$.

(b) The SBM Model

Here, we continue with input orientation consistent with our exposition of the CCR model in the preceding paragraph. The input-oriented SBM model under the constant returns-to-scale assumption evaluates the efficiency τ^* of DMU $(\mathbf{x}_o, \mathbf{y}_o)$ by solving the following linear program where the abbreviations I and C indicate Input-oriented and Constant-returns-to-scale, respectively.

[SBM-I-C]

$$\begin{aligned}
\tau^* &= \min 1 - \frac{1}{m} \sum_{i=1}^m \frac{s_i^-}{x_{io}} \\
\text{subject to} \\
x_{io} &= \sum_{j=1}^n x_{ij} \lambda_j + s_i^- \quad (i=1, \dots, m) \\
y_{io} &\leq \sum_{j=1}^n y_{ij} \lambda_j \quad (i=1, \dots, s) \\
\lambda_j &\geq 0 (\forall j), \quad s_i^- \geq 0 (\forall i),
\end{aligned} \tag{3}$$

where $\boldsymbol{\lambda}$ is the intensity vector, and \mathbf{s}^- represents non-radial input slacks vector.

Let an optimal solution of [SBM-I-C] be $(\boldsymbol{\lambda}^*, \mathbf{s}^{-*})$. Then, the objective function can be rewritten as

$$\tau^* = \frac{1}{m} \sum_{i=1}^m \frac{x_{io} - s_i^{-*}}{x_{io}}. \tag{4}$$

Hence the SBM score τ^* is the average of the component-wise reduction rates which may vary from one input to another. The SBM model is non-radial. On the other hand, as noted in (2), the CCR score θ^* satisfies the relationship $\theta^* \mathbf{x}_o = \mathbf{X}\boldsymbol{\lambda}^* + \mathbf{s}^{-*} = \mathbf{x}^* + \mathbf{s}^{-*}$.

Hence, we have

$$\theta^* = \frac{x_{io}^* + s_i^{-*}}{x_{io}} (\forall i). \tag{5}$$

The component-wise reduction rates are the same for all inputs. This same proportional reduction rate, i.e. radial reduction rate, is the CCR score.

Between the SBM τ^* and the CCR θ^* we have the inequality $\tau^* \leq \theta^*$. See Tone [8] for more details of their comparisons.

2.2 Shortcomings of the radial and non-radial models

In this section we point out shortcomings of the radial and non-radial DEA models.

(a) Shortcomings of the CCR Model

The main shortcoming of the CCR model is the neglect of non-radial slacks s^- in reporting of the efficiency score θ^* . In many cases, we find a lot of remaining non-radial slacks. So, if these slacks have an important role in evaluating managerial efficiency, the radial approaches may mislead the decision when we utilize the efficiency score θ^* as the only index for evaluating performance of DMUs. Furthermore, as to the proportional change $\theta^* \mathbf{x}_o$, if we employ labor, materials and capital as inputs, some of them are substitutional and do not change proportionally. The radial (CCR) model cannot cope with such cases properly.

(b) Shortcomings of the SBM Model

Since models such as SBM capture the non-radial slacks directly, the optimal efficiency value τ^* accounts for the non-radial slacks which are not considered in the radial models. The SBM-projection to the efficient frontier is defined by $\bar{\mathbf{x}}_o = \mathbf{x}_o - \mathbf{s}^-$. Thus, the projected DMU may lose the proportionality in the original \mathbf{x}_o because \mathbf{s}^- is not necessarily proportional to \mathbf{x}_o . This is characteristic of the non-radial models, and if the loss of the original proportionality is inappropriate for the analysis, then this becomes a shortcoming for non-radial models. Yet another equally significant shortcoming of SBM arises from the nature of the linear programming solution, where the optimal slacks tend to exhibit a sharp contrast in taking positive and zero values. See Avkiran et al. [4] for more detailed comparisons of the zero and non-zero patterns in the optimal slacks in the SBM model.

3. An epsilon-based measure of efficiency (EBM)

As pointed out in the preceding section, both radial and non-radial models have merits and demerits regarding the proportionality of the inputs/outputs change. In this section, we propose a compromised model called “epsilon-based measure (EBM)” which has both radial and non-radial features in a unified framework.

We define the primal and dual pair [EBM-I-C] and [Dual] as follows:

[EBM-I-C]

$$\gamma^* = \min_{\theta, \lambda, \mathbf{s}^-} \theta - \varepsilon_x \sum_{i=1}^m \frac{w_i \bar{s}_i^-}{x_{io}} \quad (6)$$

$$\text{subject to } \theta \mathbf{x}_o - \mathbf{X}\lambda - \mathbf{s}^- = \mathbf{0} \quad (7)$$

$$\begin{aligned} \mathbf{Y}\lambda &\geq \mathbf{y}_o \\ \lambda &\geq \mathbf{0} \\ \mathbf{s}^- &\geq \mathbf{0}. \end{aligned} \quad (8)$$

[Dual]

$$\gamma^* = \max_{\mathbf{v}, \mathbf{u}} \mathbf{u} \mathbf{y}_o \quad (9)$$

$$\text{subject to } \mathbf{v} \mathbf{x}_o = 1 \quad (10)$$

$$-\mathbf{v} \mathbf{X} + \mathbf{u} \mathbf{Y} \leq \mathbf{0} \quad (11)$$

$$v_i \geq \frac{\varepsilon_x w_i^-}{x_{io}} \quad (i=1, \dots, m) \quad (12)$$

$$\mathbf{u} \geq \mathbf{0},$$

where w_i^- is the weight (relative importance) of input i and satisfies $\sum_{i=1}^m w_i^- = 1$ ($w_i^- \geq 0 \forall i$), and ε_x is a key parameter which combines the radial θ and the non-radial slacks terms. Parameters ε_x and \mathbf{w}^- must be supplied prior to the efficiency measurements. As can be seen from the term $\frac{w_i^- s_i^-}{x_{io}}$ in the objective function of [EBM-I-C], $\frac{s_i^-}{x_{io}}$ is units-invariant and so w_i^- should be a units-invariant value reflecting the relative importance of resource i . We will discuss this subject in the succeeding sections.

[Proposition 1]

γ^* in [EBM-I-C] satisfies $1 \geq \gamma^* \geq 0$ and is units-invariant, i.e. γ^* is independent of the units in which the inputs and outputs are measured.

[Proposition 2]

If we set $\varepsilon_x = 0$ in [EBM-I-C], then it reduces to the input-oriented CCR model.

[Proposition 3]

If we set $\theta = 1$ and $\varepsilon = 1$ in [EBM-I-C], then it reduces to the input-oriented SBM model.

Thus, [EBM-I-C] includes the radial CCR and the non-radial SBM models as special cases, but it is basically non-radial.

The constraints (10) and (12) lead to $1 = \mathbf{v} \mathbf{x}_o = \sum_{i=1}^m v_i x_{io} \geq \varepsilon_x$. Thus, ε_x must be not greater than unity.

[Proposition 4]

[EBM-I-C] and [Dual] have a finite optima for $\varepsilon_x \in [0, 1]$.

[Proposition 5]

For $\varepsilon_x > 1$, [Dual] has no feasible solution and [EBM-I-C] has unbounded solution.

[Proposition 6]

γ^* is non-increasing in ε_x .

[Definition 1] (EBM input-efficiency)

DMU $_o$ is called EBM input-efficient if $\gamma^* = 1$.

[Definition 2] (EBM projection)

Let an optimal solution to (6)-(8) be $(\theta^*, \lambda^*, s^{-*})$. We define the projection of DMU (x_o, y_o) as follows.

$$\begin{aligned} \mathbf{x}_o^* &= \mathbf{X}\lambda^* = \theta^* \mathbf{x}_o - \mathbf{s}^{-*} \\ \mathbf{y}_o^* &= \mathbf{Y}\lambda^* \end{aligned} \quad (13)$$

[Proposition 7]

The projected DMU $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ is EBM input-efficient. (See Appendix A for a proof.)

[EBM-I-C] can be manipulated in another form by introducing a variable $\mathbf{x} = \theta \mathbf{x}_o - \mathbf{s}^-$ as follows.

$$\begin{aligned} \gamma^* &= \min_{\theta, \mathbf{x}, \lambda, \mathbf{s}^-} (1 - \varepsilon_x) \theta + \varepsilon_x \sum_{i=1}^m \frac{w_i^- x_i}{x_{io}} \\ \text{subject to } & \mathbf{x} - \mathbf{X}\lambda = \mathbf{0} \\ & \mathbf{x} = \theta \mathbf{x}_o - \mathbf{s}^- \\ & \mathbf{Y}\lambda \geq \mathbf{y}_o \\ & \lambda \geq \mathbf{0}, \mathbf{s}^- \geq \mathbf{0}. \end{aligned} \quad (14)$$

This formulation indicates that γ^* is obtained as the optimal internally dividing value between the radial θ and the non-radial term $\sum_{i=1}^m w_i^- x_i / x_{io}$. Since θ is not restricted, its optimal value θ^* can be greater than 1, and hence the optimal \mathbf{x}^* is not necessarily less than or equal to \mathbf{x}_o . We notice that the composite single stage approach like the EBM was commented in Ali and Seiford [3] and further developed by Johnson and Ruggiero [7]. However, our objective is quite different from the preceding ones as can be seen in the following sections.

4. How to determine epsilon and weights

In EBM, the values of ε_x and \mathbf{w}^- play the central role for evaluating efficiency of DMUs. We would like to determine them from the data set (\mathbf{X}, \mathbf{Y}) , since DEA is a data driven method. In this section firstly we observe two extreme cases. Then we introduce an affinity index between two vectors which replaces the Pearson's correlation coefficient.

4.1 Two extreme cases

(1) *Narrow range case*

Figure 1 plots an example of data concerning inputs x_1 and x_2 concentrating in a narrow range. If all inputs and outputs go along with the similar behavior, the assumption of proportional (radial) model can be effected. Thus, in such case, we have $\varepsilon_x \doteq 0$ and the CCR model is a valid choice.

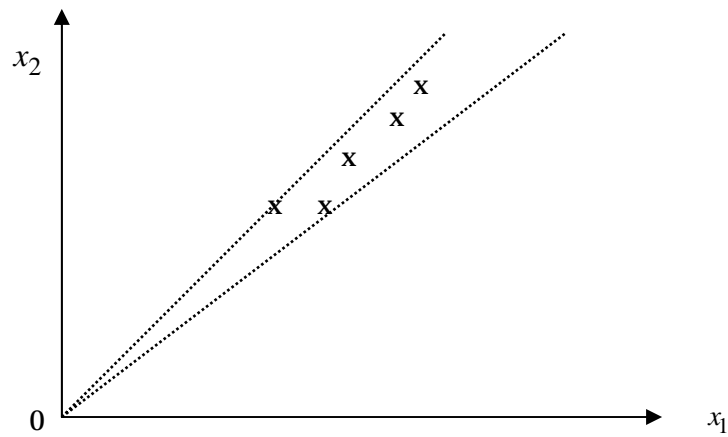


Figure 1: Narrow range case

(2) *Widely scattered case*

In the other extreme case, if the observed data scatters widely as exemplified in Figure 2, the non-radial model can be applied. Thus, we have $\varepsilon_x \doteq 1$ and the SBM model with $\theta = 1$ is a choice although we do not shut off the assumption of radial models depending on the characteristics of problems.

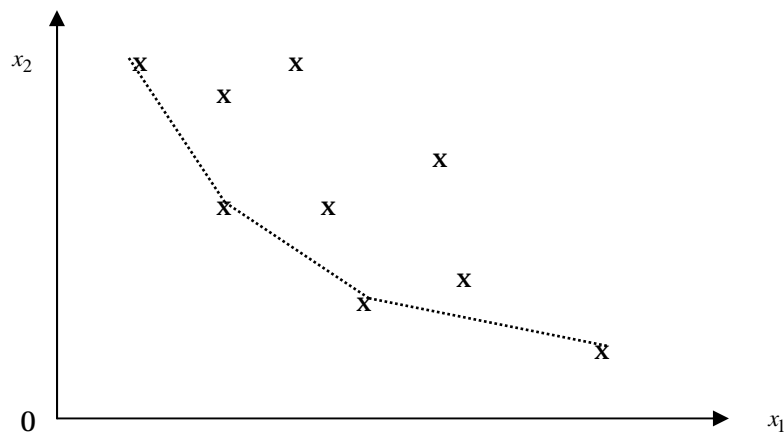


Figure 2: Widely scattered case

These extreme cases suggest that ε_x can be determined in the context of the degree of correlations among inputs (outputs).

Several authors, e.g. Ueda and Hoshiai [9] and Adler and Golany [1, 2] among others, utilized correlation matrix of inputs (outputs) and applied principal component analysis (PCA) to DEA. Their main objectives were integration of inputs (outputs) to other representative indicators.

In this paper, we employ similar but different correlation matrix as described in the next section in order to gauge affinity among inputs which will be utilized to estimate parameters ε_x and \mathbf{w}^- in the EBM.

4.2 Diversity index and affinity index

Let $\mathbf{a} \in R_+^n$ and $\mathbf{b} \in R_+^n$ be two positive vectors with dimension n . They represent observed values for certain input items over n DMUs. We define an affinity index $S(\mathbf{a}, \mathbf{b})$ between \mathbf{a} and \mathbf{b} with the following properties.

- (P1) $S(\mathbf{a}, \mathbf{a}) = 1$ ($\forall \mathbf{a}$) Identical
- (P2) $S(\mathbf{a}, \mathbf{b}) = S(\mathbf{b}, \mathbf{a})$ Symmetric
- (P3) $S(t\mathbf{a}, \mathbf{b}) = S(\mathbf{a}, \mathbf{b})$ ($\forall t > 0$) Units-invariant and
- (P4) $1 \geq S(\mathbf{a}, \mathbf{b}) \geq 0$ ($\forall \mathbf{a}, \mathbf{b}$).

The usual Pearson's correlation coefficient introduces the translation of origin in calculating correlations. In our model, we wish to evaluate affinity of two vectors without translation of origin. Therefore, we introduce another correlation coefficient called "affinity index."

Let us define

$$\begin{aligned} c_j &= \ln \frac{b_j}{a_j} \quad (j=1, \dots, n) \\ \bar{c} &= \frac{1}{n} \sum_{j=1}^n c_j \\ c_{\max} &= \max_j \{c_j\} \text{ and } c_{\min} = \min_j \{c_j\}. \end{aligned} \quad (15)$$

[Definition 3] (diversity index)

We define the "diversity index" of vectors \mathbf{a} and \mathbf{b} as the deviation of $\{c_j\}$ from the average \bar{c} in the following way.

$$\begin{aligned} D(\mathbf{a}, \mathbf{b}) &= \frac{\sum_{j=1}^n |c_j - \bar{c}|}{n(c_{\max} - c_{\min})} \\ &= 0 \quad \text{if } c_{\max} = c_{\min}. \end{aligned} \quad (16)$$

[Proposition 8]

$$0 \leq D(\mathbf{a}, \mathbf{b}) = D(\mathbf{b}, \mathbf{a}) \leq \frac{1}{2}. \quad (17)$$

See Appendix B for a proof. $D(\mathbf{a}, \mathbf{b}) = 0$ occurs if and only if two vectors \mathbf{a} and \mathbf{b} are proportional.

[Definition 4] (affinity index)

We define the "affinity index" $S(\mathbf{a}, \mathbf{b})$ between two vectors \mathbf{a} and \mathbf{b} by

$$S(\mathbf{a}, \mathbf{b}) = 1 - 2D(\mathbf{a}, \mathbf{b}). \quad (18)$$

[Proposition 9]

It holds $1 \geq S(\mathbf{a}, \mathbf{b}) \geq 0$. $S(\mathbf{a}, \mathbf{b})$ satisfies properties (P1), (P2), (P3) and (P4).

The reason why we employ the affinity index (15) instead of the Pearson's correlation coefficient is the following:

1. Pearson's correlation coefficient is defined by

$$r(\mathbf{a}, \mathbf{b}) = \frac{\sum_{j=1}^n (a_j - \bar{a})(b_j - \bar{b})}{\sqrt{\sum_{j=1}^n (a_j - \bar{a})^2} \sqrt{\sum_{j=1}^n (b_j - \bar{b})^2}},$$

where \bar{a} and \bar{b} are respectively averages of $\{a_j\}$ and $\{b_j\}$. In this formula, the absolute magnitude of

a_j and b_j effects $r(\mathbf{a}, \mathbf{b})$ strongly. In contrast, in DEA, the relative measure, e.g. b_j/a_j , is a main

concern.

2. Pearson's correlation coefficient results in the range $-1 \leq r(\mathbf{a}, \mathbf{b}) \leq 1$. Hence, in the principal component analysis we will utilize in the next section, it is not guaranteed that the principal vector consists of non-negative components. Although it is possible to adjust $r(\mathbf{a}, \mathbf{b})$ into $[0, 1]$, this might bring a skew distribution, since most of $r(\mathbf{a}, \mathbf{b})$ are non-negative in DEA applications.

3. We employ the logarithmic function $\ln b_j/a_j$ instead of b_j/a_j , because the latter violates the property (P2).

5. Use of affinity matrix in EBM

In this section, we measure the diversity of production possibility set by means of the affinity matrix derived from the observed inputs and outputs. Although we describe the method in the input-oriented model under the constant returns-to-scale (CRS) assumption, we can modify it to the output-oriented and non-oriented models under constant or variable returns-to-scale (VRS) assumptions. We discuss this subject in Section 8.

Step 1. Creation of projected VRS-efficient DMUs

In most DEA models, the production possibility set is spanned by the efficient DMUs which usually consist of a small portion of the entire DMUs. In order to increase the accuracy of our estimation, we first project all DMUs to the VRS (variable returns-to-scale)-efficient frontiers using the Additive model or non-oriented SBM model below¹.

¹ We can employ the observed data (X, Y) instead of the projected DMUs in this step. However, we utilized the projected DMUs, because our main concerns are the shape of frontiers.

[ADD]

$$\begin{aligned}
& \max \sum_{i=1}^m \frac{s_i^-}{x_{io}} + \sum_{i=1}^s \frac{s_i^+}{y_{io}} \\
& \text{subject to} \\
& x_{io} = \sum_{j=1}^n x_{ij} \lambda_j + s_i^- \quad (i=1, \dots, m) \\
& y_{io} = \sum_{j=1}^n y_{ij} \lambda_j - s_i^+ \quad (i=1, \dots, s) \\
& \sum_{j=1}^n \lambda_j = 1 \\
& \lambda_j \geq 0 (\forall j), \quad s_i^- \geq 0 (\forall i), \quad s_i^+ \geq 0 (\forall i).
\end{aligned} \tag{19}$$

[SBM]

$$\begin{aligned}
& \min \frac{1 - \frac{1}{m} \sum_{i=1}^m \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s} \sum_{i=1}^s \frac{s_i^+}{y_{io}}} \\
& \text{subject to} \\
& x_{io} = \sum_{j=1}^n x_{ij} \lambda_j + s_i^- \quad (i=1, \dots, m) \\
& y_{io} = \sum_{j=1}^n y_{ij} \lambda_j - s_i^+ \quad (i=1, \dots, s) \\
& \sum_{j=1}^n \lambda_j = 1 \\
& \lambda_j \geq 0 (\forall j), \quad s_i^- \geq 0 (\forall i), \quad s_i^+ \geq 0 (\forall i).
\end{aligned} \tag{20}$$

Using the optimal slacks s_i^{-*} and s_i^{+*} we define the projected input and output for DMU_o by

$$\begin{aligned}
\bar{x}_{io} &= x_{io} - s_i^{-*} \quad (i=1, \dots, m) \\
\bar{y}_{io} &= y_{io} + s_i^{+*} \quad (i=1, \dots, s).
\end{aligned} \tag{21}$$

We notice that [ADD] and [SBM] may produce different projections but they are on the efficient frontiers of the production possibility set.

Thus, we have n VRS-efficient DMUs denoted by

$$\begin{bmatrix} \bar{\mathbf{X}} \\ \bar{\mathbf{Y}} \end{bmatrix} = \begin{bmatrix} \bar{x}_{11} \cdots \bar{x}_{1n} \\ \cdots \cdots \cdots \\ \bar{x}_{m1} \cdots \bar{x}_{mn} \\ \bar{y}_{11} \cdots \bar{y}_{1n} \\ \cdots \cdots \cdots \\ \bar{y}_{s1} \cdots \bar{y}_{sn} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \cdots \\ \bar{\mathbf{x}}_m \\ \bar{\mathbf{y}}_1 \\ \cdots \\ \bar{\mathbf{y}}_s \end{bmatrix} \tag{22}$$

All CRS (constant-returns-to-scale) efficient DMUs are included in this set along with VRS-efficient DMUs.

Step 2. Formation of affinity matrix

In the input-oriented case, we calculate the affinity matrix $\mathbf{S} = [s_{ij}] \in R^{m \times m}$ with the elements

$$s_{ij} = S(\bar{\mathbf{x}}_i, \bar{\mathbf{x}}_j) \quad (i, j = 1, \dots, m) \quad (23)$$

All elements of the matrix \mathbf{S} satisfy the bounds:

$$1 \geq s_{ij} \geq 0 \quad (\forall (ij)). \quad (24)$$

Step 3. Solving the largest eigenvalue and eigenvector of the affinity matrix

By the formation rule, \mathbf{S} is symmetric and non-negative with the diagonal elements equal to unity. It has m pairs of eigenvalue and eigenvector. By the Perron-Frobenius theorem for non-negative matrices, \mathbf{S} has the largest eigenvalue ρ_x with its associated non-negative eigenvector $\mathbf{w}_x (\geq \mathbf{0})$. The non-negative \mathbf{w}_x corresponds to the weight of input factors. Since \mathbf{S} is non-negative definite, we have $m \geq \rho_x \geq 1$.

Step 4. Calculation of ε_x and \mathbf{w}^- for the EBM

We define ε_x and \mathbf{w}^- in the EBM as follows.

$$\begin{aligned} \varepsilon_x &= \frac{m - \rho_x}{m - 1} \quad (\text{if } m > 1) \\ &= 0 \quad (\text{if } m = 1). \end{aligned} \quad (25)$$

$$\mathbf{w}^- = \frac{\mathbf{w}_x}{\sum_{i=1}^m w_{xi}}. \quad (26)$$

The thus defined ε_x and \mathbf{w}^- satisfy the relationship $0 \leq \varepsilon_x \leq 1$ and $\mathbf{e}\mathbf{w}^- = 1$.

Step 5. Use of ε_x and \mathbf{w}^- in the EBM

These parameters are utilized in the EBM model [EBM-I-C].

6. Rationale of the proposed EBM

In this section we demonstrate the rationale of the scheme proposed in the preceding section.

Before going into theoretical discussions, we show some real world data concerning input/output items. Figure 4 depicts 814 samples of no. of doctors (as input) vs. no. of beds (as input) of Japanese municipally-owned hospitals. Figure 5 shows no. of doctors (as input) vs. revenue/day (as output) in the same 814 sample hospitals. Since the municipal hospitals are, to some extent, standardized under the control of respective administrative offices, many inputs and outputs have positive relationship and hence the affinity matrix is expected to have high affinity values. Consequently, its principal eigenvalue will be large and hence ε will be small.

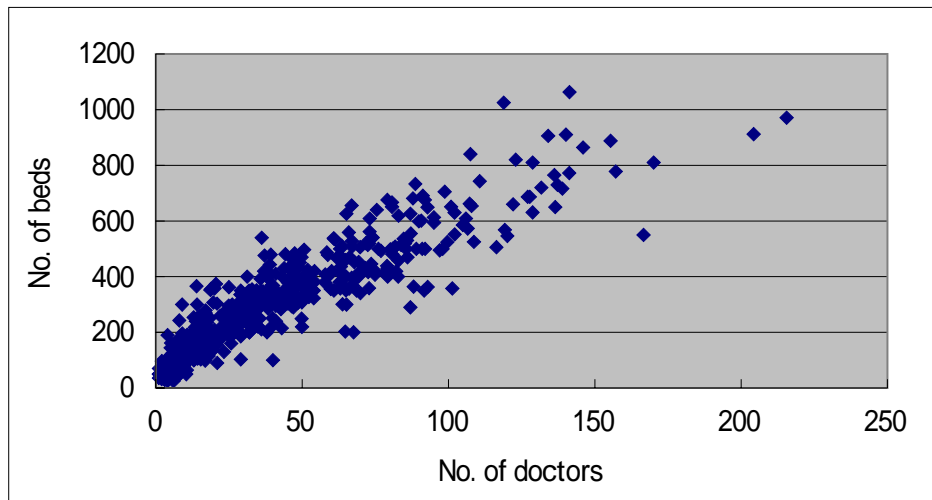


Figure 4: Empirical data 1

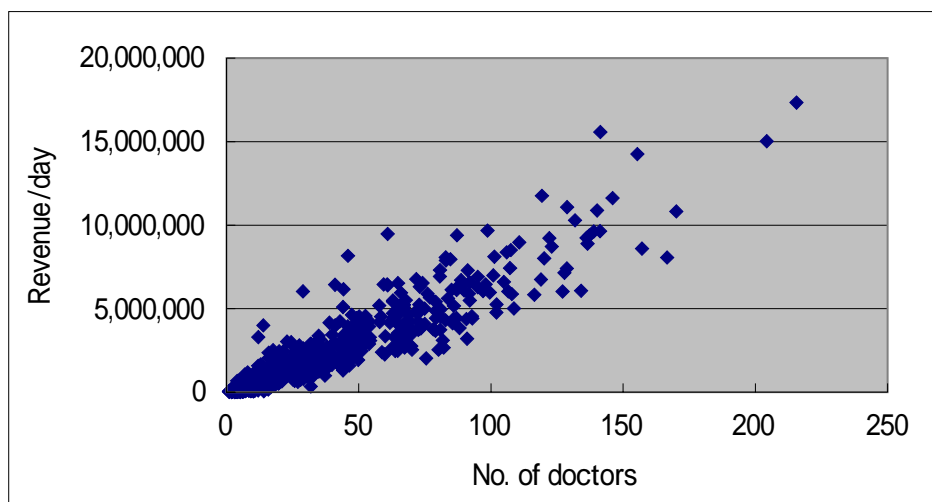


Figure 5: Empirical data 2

Figures 6 and 7 exhibit two plotted data concerning no. of employees vs. no. of visitors and area vs. no. of visitors for Japanese regional museums. Since museum business is not standardized compared with regional hospitals, they are distributed widely. In this case, the affinity matrix is expected to consist of low values with relatively small principal eigenvalue and hence large ε .

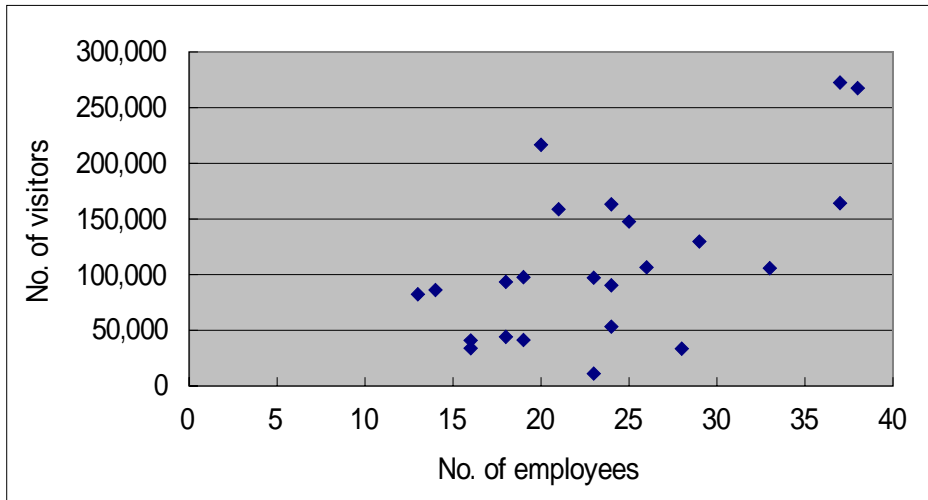


Figure 6: Empirical data 3

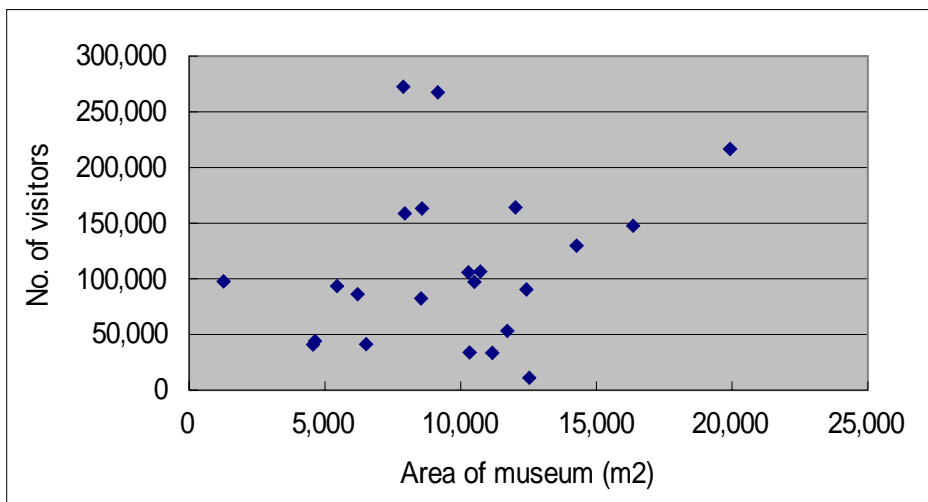


Figure 7: Empirical data 4

Figures 8 and 9 plot data of 273 electric power plants in the U.S. concerning the generating power capacity (GW) (input) vs. no. of employees (input) and the consumed fuel (million BTU) (input) vs. no. of employees (input). They are positively correlated but considerably diversified.

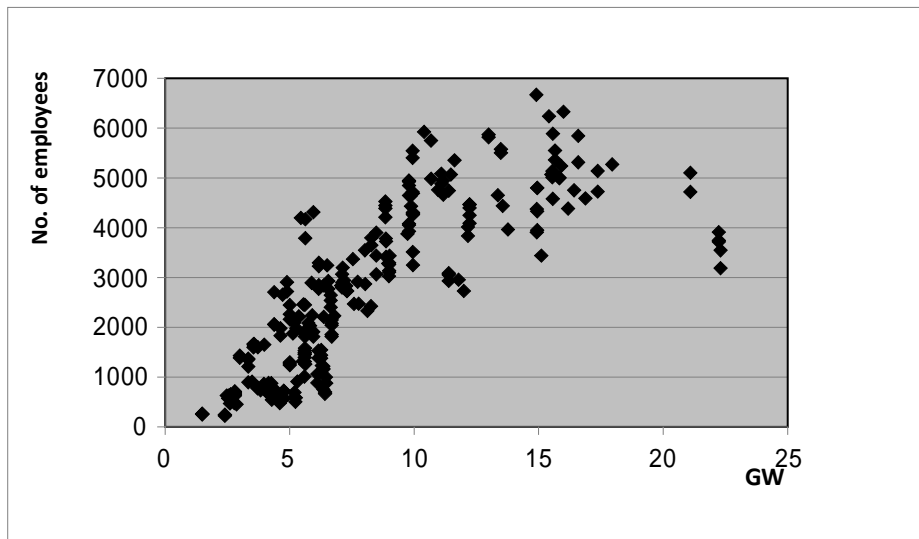


Figure 8: Empirical data 5

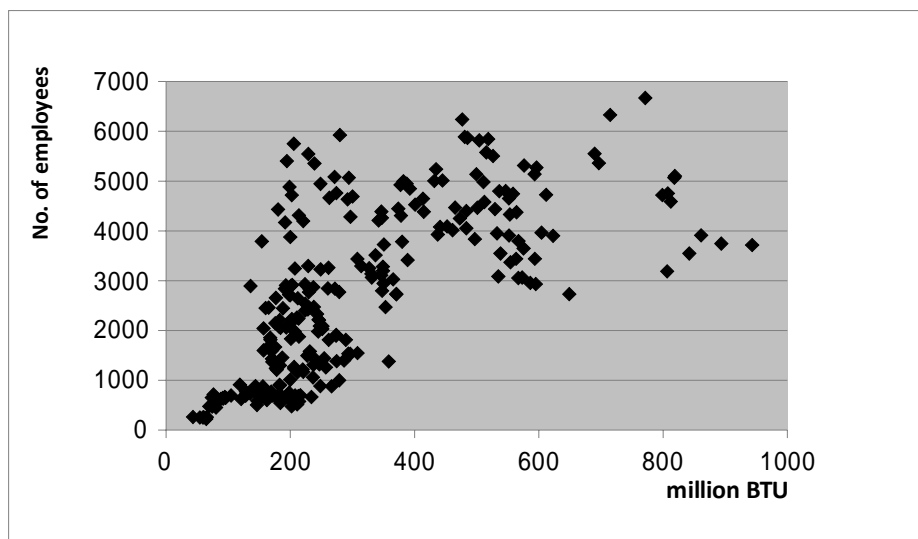


Figure 9: Empirical data 6

The ellipsoid $w^T S w = 1$ has the principal axis in the first (positive) quadrant as exemplified in Figure 10. As the degree of affinity becomes higher and higher, the shape of the ellipsoid comes to be flat and the largest eigenvalue ρ_x tends to m . Thus, ε_x in (25) tends to 0. This comes close to the CCR model, i.e. all inputs and outputs follow proportional changes.

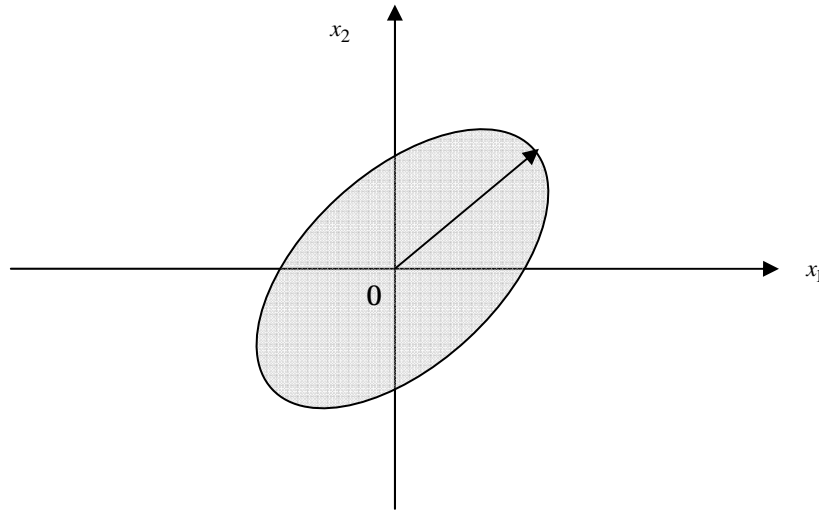


Figure 10: Ellipsoid of affinity matrix

Depending on the degree of affinity among inputs, the principal eigenvalue ρ_x increases up to m . Conversely, the more the data scatters widely, the more ρ_x tends to 1 and the more ε_x grows up. Hence, the model behaves SBM-like. Therefore, it can safely be said that ε_x condenses the affinity matrix in a single value reflecting the scattering of the data set.

We now turn to the positive eigenvector \mathbf{w}_x corresponding to the eigenvalue ρ_x . First of all, we notice that \mathbf{w}_x is units-invariant, since the affinity matrix is units-invariant. Suppose that, in the affinity matrix $\mathbf{S} = (s_{ij})$, $s_{1j} > s_{2j} \geq 0$ ($j = 3, \dots, m$), then it holds that $w_1^- > w_2^-$. This indicates that the item which has higher affinity with others has a large portion in the eigenvector, whereas item i with unrelated to others, i.e., $s_{ij} \doteq 0$ ($\forall j \neq i$) has $w_i^- \doteq 0$. Thus, the magnitude of elements of \mathbf{w}^- indicates importance of the item among the whole items. We can strengthen the discrimination power on efficiency by imposing weight to slacks in proportion to \mathbf{w}^- . Thus, this scheme is an application of the principal component analysis (PCA) to DEA.

We note here that, in the input-oriented model, we estimate ε_x depending only on the input data $\bar{\mathbf{X}}$, but not on the output data $\bar{\mathbf{Y}}$. This means that the objective function in [EBM-I-C] relates to the radial factor θ and the diversity indicator ε_x . The former represents the radial feature of inputs and the later implies the non-radial characteristics of inputs. The interactions between input \mathbf{X} and output \mathbf{Y} are described in the constraints of [EBM-I-C] through the intermediary of the intensity vector $\boldsymbol{\lambda}$.

7. Illustrative examples

In this section, we explain the EBM using three examples and compare the results with the radial

(CCR) and non-radial (SBM) scores.

7.1 Example 1

Table 1 reports comparisons of CCR-I, SBM-I-C and EBM-I-C scores for six DMUs A, B, C, D, E and F with two inputs (x_1 , x_2) and a single output ($y = 1$). Figure 11 plots them graphically. This figure indicates that the data are concentrated in a narrow gauge. See also Figure 12.

Table 1:

	x_1	x_2	y	CCR-I	SBM-I-C	EBM-I-C
A	1	1	1	1	1	1
B	2	3	1	0.500	0.417	0.500
C	3	2	1	0.500	0.417	0.500
D	4	3	1	0.333	0.292	0.333
E	5	6	1	0.200	0.183	0.200
F	7	6	1	0.167	0.155	0.167

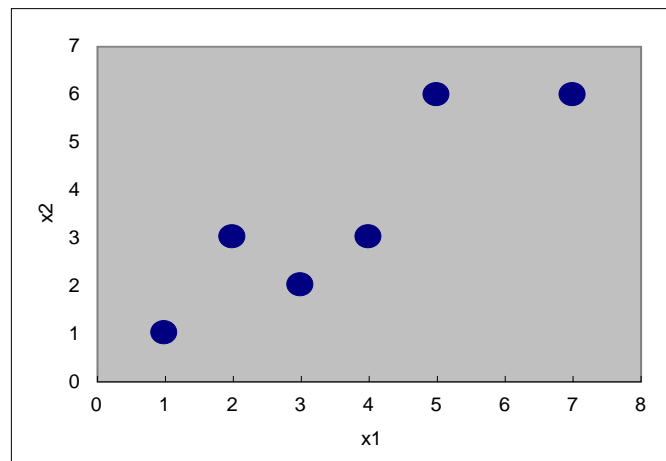


Figure 11: Example 1

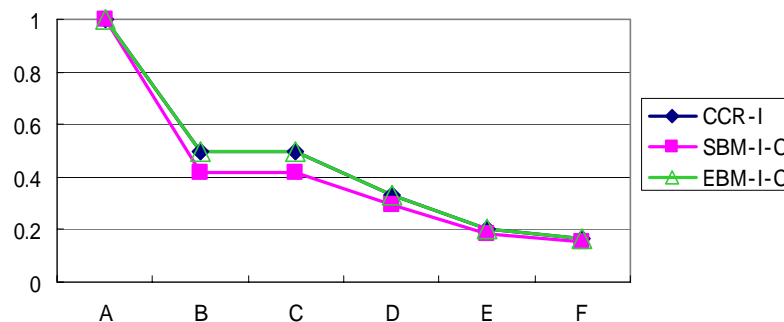


Figure 12: Comparison of scores

As can be seen, EBM scores are the same with the CCR scores. We illustrate the EBM scheme in order.

Step 1: We used [ADD] for finding slacks and projected DMUs to efficient frontiers, as shown in

Table 2. They are all projected to the only one efficient DMU A.

Table 2: Projected DMUs

	\bar{x}_1	\bar{x}_2	\bar{y}
A	1	1	1
B	1	1	1
C	1	1	1
D	1	1	1
E	1	1	1
F	1	1	1

Step 2: We calculated the diversity matrix by the formula (16). See Table 3. Since the set of efficient DMUs consists of only one DMU A, no diversity exists.

Table 3: Diversity matrix for Example 1

	x_1	x_2
x_1	0	0
x_2	0	0

Step 3: The affinity matrix is calculated by the formula (23) and displayed in Table 4.

Table 4: Affinity matrix for Example 1

	x_1	x_2
x_1	1	1
x_2	1	1

Step 4: The largest eigenvalue and eigenvector of the affinity matrix are $\rho_x = 2$ and $\mathbf{w}_x = (0.5, 0.5)$. Hence we have

$$\varepsilon_x = (m - \rho_x) / (m - 1) = 0, \quad w_1^- = 0.5, \quad w_2^- = 0.5.$$

In this case, the ellipsoid is perfectly flat.

Step 5: Using these parameter values we applied EBM-I-C to the six DMUs. Since we have $\varepsilon_x = 0$, the scores are identical with the CCR scores.

7.2 Example 2

This example has diversified DMUs as exhibited in Table 5 and Figure 13.

Table 5: Example 2

x_1	x_2	y	CCR-I	SBM-I-C	EBM-I-C
-------	-------	-----	-------	---------	---------

A	2	6	1	1	1	1
B	6	3	1	1	1	1
C	10	3	1	1	0.8	0.8
D	2	10	1	1	0.8	0.8

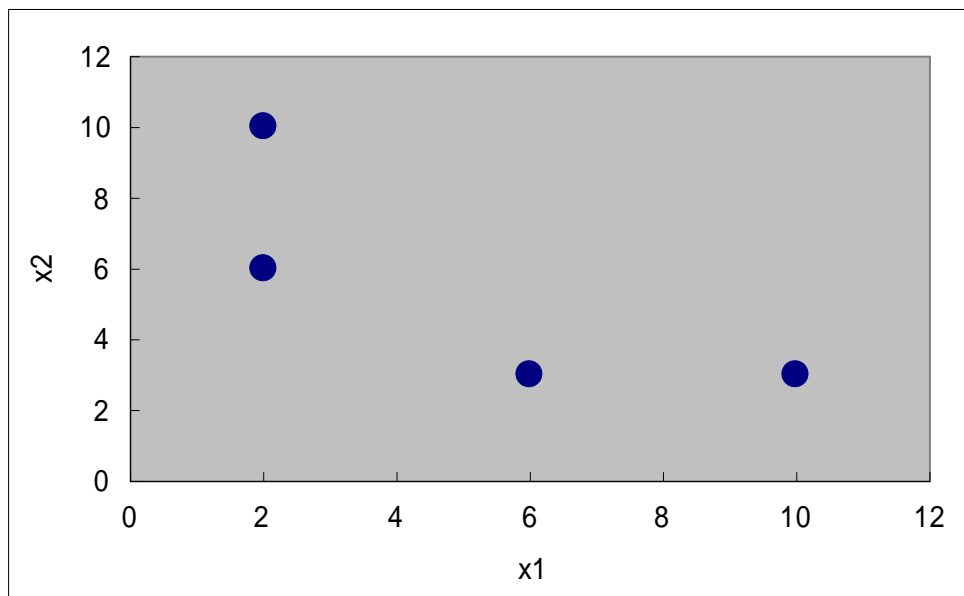


Figure 13: Example 2

The projected data are exhibited in Table 6.

Table 6: Projected data for Example 2

	\bar{x}_1	\bar{x}_2	\bar{y}
A	2	6	1
B	6	3	1
C	6	3	1
D	2	6	1

The diversity matrix for the EBM-I-C model is displayed in Table 7 along with the affinity matrix in Table 8.

Table 7: Diversity matrix for Example 2

	x_1	x_2
x_1	0	0.5
x_2	0.5	0

Table 8: Affinity matrix for Example 2

	x_1	x_2
x_1	1	0
x_2	0	1

The largest eigenvalue and eigenvector of the affinity matrix are $\rho_x = 1$ and $\mathbf{w}_x = (0.5, 0.5)$.

Hence we have

$$\varepsilon_x = (m - \rho_x) / (m - 1) = 1$$

$$w_1^- = 0.5, \quad w_2^- = 0.5.$$

The value $\varepsilon_x = 1$ is the largest one showing the diversity of the data set and EBM-I-C results are identical with the SBM results.

7.3 Example 3

Table 9 reports efficiency scores of 12 hospitals. We utilized numbers of doctors and nurses as inputs, and numbers of outpatients and inpatients per month as outputs. Figure 14 displays comparisons of three scores: CCR-I, SBM-I-C and EBM-I-C.

Table 9: Hospital data and efficiency scores

	(I)Doctor	(I)Nurse	(O)Outpatient	(O)Inpatient	CCR-I	SBM-I-C	EBM-I-C
A	20	151	100	90	1	1	1
B	19	131	150	50	1	1	1
C	25	160	160	55	0.883	0.852	0.868
D	27	168	180	72	1	1	0.986
E	22	158	94	66	0.763	0.756	0.761
F	55	255	230	90	0.835	0.704	0.771
G	33	235	220	88	0.902	0.895	0.898
H	31	206	152	80	0.796	0.774	0.788
I	30	244	190	100	0.960	0.905	0.931
J	50	268	250	100	0.871	0.781	0.829
K	53	306	260	147	0.955	0.866	0.912
L	38	284	250	120	0.958	0.936	0.946

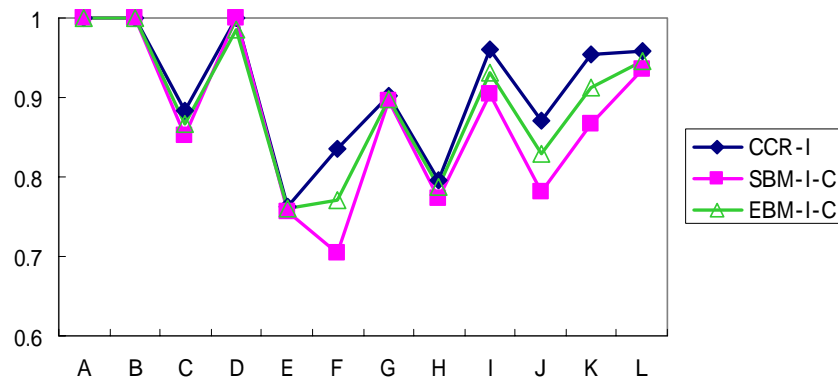


Figure 14: Comparisons of scores (hospital)

We utilized [ADD] for projecting the dataset to the VRS-efficient frontiers and obtained the new dataset exhibited in Table 10.

Table 10: Projected DMUs (hospital)

	Doctor	Nurse	Outpatient	Inpatient
A	20.00	151.00	100.00	90.00
B	19.00	131.00	150.00	50.00
C	24.59	160.00	160.00	72.98
D	27.00	168.00	180.00	72.00
E	22.00	156.79	158.26	66.00
F	35.06	255.00	230.00	108.90
G	33.00	235.00	220.00	88.04
H	27.44	206.00	162.03	102.41
I	30.00	223.45	190.00	102.29
J	50.00	268.00	250.00	100.00
K	53.00	306.00	260.00	147.00
L	38.00	284.00	250.00	120.00

The diversity matrix is displayed in Table 11 along with the affinity matrix in Table 12.

Table 11: Diversity matrix for Example 3

	Doctor	Nurse
--	--------	-------

Doctor	0	0.265
Nurse	0.265	0

Table 12: Affinity matrix for Example 3

	Doctor	Nurse
Doctor	1	0.471
Nurse	0.471	1

This affinity matrix has the largest eigenvalue and eigenvector:

$$\rho_x = 1.471, \mathbf{w}_x = (0.5, 0.5)$$

Hence we have:

$$\varepsilon_x = (m - \rho_x) / (m - 1) = 0.529$$

$$w_1^- = 0.5, \quad w_2^- = 0.5.$$

The EBM scores were obtained using these ε_x and \mathbf{w}^- values.

Table 13 exhibits θ and slacks s_1^-, s_2^-, s_1^+ and s_2^+ in the solution of the EBM-I-C model. It is interesting to notice that hospital D is inefficient with the score 0.986, in contrast to the CCR and SBM score 1 (efficient). The EBM model imposes no restriction on θ , and D has an optimal $\theta = 1.016 (> 1)$. Thus the optimal solution insists that all inputs are multiplied by 1.016 and further no. of doctor is decreased by the slacks $s_1^- = 3.078$. The projected inputs for D are $27 \times 1.016 - 3.078 = 24.35$ for Doctor and $168 \times 1.016 = 170.68$ for Nurse. D's references are A ($\lambda_A = 0.2118$) and B ($\lambda_B = 1.0588$). D is recommended to reduce doctors from 27 to 24 and increase nurses from 168 to 171 in order to improve efficiency. This is one of characteristics of the composite model EBM, whereas such substitution of inputs cannot occur in the CCR or the SBM models.

Table 13: θ and slacks

DMU	Score	Rank	θ	s_1^-	s_2^-	s_1^+	s_2^+
A	1	1	1	0	0	0	0
B	1	1	1	0	0	0	0
C	0.868	8	0.885	1.644	0	0	0
D	0.986	3	1.016	3.078	0	0	0
E	0.761	12	0.766	0.461	0	0	0
F	0.771	11	0.846	15.696	0	0	0
G	0.898	7	0.902	0	3.349	0	0

H	0.788	10	0.804	1.887	0	0	0
I	0.931	5	0.960	0	27.206	0	0
J	0.829	9	0.885	10.404	0	0	0
K	0.912	6	0.964	10.328	0	0	0
L	0.946	4	0.958	0	12.6	0	0

8. Extensions to other orientations and variable returns-to-scale models

So far, we have developed the EBM in the input-orientation under the constant returns-to-scale environment. However, we can extend it to other orientations and returns-to-scale environment as follows. In every variation we follow Step 1 in Section 5 and obtain the set of projected VRS-efficient DMUs (22).

8.1 Output-oriented EBM

Step 2. Formation of affinity matrix

In the output-oriented case, we calculate the affinity matrix $S = [s_{ij}] \in R^{s \times s}$ with the elements

$$s_{ij} = S(\bar{y}_i, \bar{y}_j) \quad (i, j = 1, \dots, s) \quad (27)$$

Step 3. Solving the largest eigenvalue and eigenvector of the affinity matrix

We solve the largest eigenvalue ρ_y and eigenvector w_y of the affinity matrix S in (27).

Step 4. Calculation of ε_y and w^+ .

We define

$$\varepsilon_y = \frac{s - \rho_y}{s - 1} \quad (\text{if } s > 1), \quad = 0 \quad (\text{if } s = 1) \quad (28)$$

$$w^+ = \frac{w_y}{\sum_{i=1}^s w_{yi}}$$

Using ε_y and w^+ we solve the following linear program:

[EBM-O-C]

$$1/\tau^* = \max_{\eta, \lambda, s^+} \eta + \varepsilon_y \sum_{i=1}^s \frac{w_i^+ s_i^+}{y_{io}} \quad (29)$$

$$\text{subject to } \mathbf{X}\lambda \leq \mathbf{x}_o \quad (30)$$

$$\begin{aligned} \eta \mathbf{y}_o - \mathbf{Y}\lambda + \mathbf{s}^+ &= \mathbf{0} \\ \lambda &\geq \mathbf{0} \\ \mathbf{s}^+ &\geq \mathbf{0}. \end{aligned} \quad (31)$$

8.2 Non-oriented (both-oriented) EBM

We apply Steps 2 and 3 for the input-oriented and the output-oriented affinity matrix separately, and

obtain ε_x , ε_y , w^- , and w^+ . The non-oriented EBM can be formulated in the following fractional program which can be solved as a linear program using the Charnes-Cooper transformation. (See Cooper et al. [6].)

$$\kappa^* = \min_{\theta, \eta, \lambda, s^-, s^+} \frac{\theta - \varepsilon_x \sum_{i=1}^m \frac{w_i^- s_i^-}{x_{io}}}{\eta + \varepsilon_y \sum_{i=1}^s \frac{w_i^+ s_i^+}{y_{io}}} \quad (32)$$

subject to $\theta \mathbf{x}_o - \mathbf{X}\lambda - \mathbf{s}^- = \mathbf{0}$
 $\eta \mathbf{y}_o - \mathbf{Y}\lambda + \mathbf{s}^+ = \mathbf{0}$
 $\lambda \geq \mathbf{0}, \mathbf{s}^- \geq \mathbf{0}, \mathbf{s}^+ \geq \mathbf{0}.$

8.3 Variable returns-to-scale EBM

All models can be modified to variable returns-to-scale (VRS) ones by adding the condition:

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = 1. \quad (33)$$

9. Concluding remarks

In this paper, we have proposed EBM as a third pole of technical efficiency in DEA by combining radial and non-radial models in a unified framework. Since DEA is a data driven method, we need to measure technical efficiency from the observed data under less assumptions on its distribution. For this purpose we introduced a new index called “affinity index” for measuring similarity between two vectors for use in DEA. Using this index, we defined a scalar measure epsilon (ε) that represents the diversity or the scattering of the observed dataset. We proposed a scheme for setting weights to slacks based on the principal component analysis. We also extended it to other orientations and returns-to-scale assumptions.

Future research subjects include search for other measure of affinity index that satisfies the properties (P1) to (P4), extensions to Super-EBM and identifications of returns-to-scale and scale efficiency under this model.

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Appendix A. Proof of Proposition 7

Since $(\mathbf{x}_o, \mathbf{y}_o)$ is EBM input-inefficient, it holds that

$$\gamma^* = \theta^* - \varepsilon_x \sum_{i=1}^m \frac{w_i^- s_i^-}{x_{io}} < 1. \quad (\text{A1})$$

Let an optimal solution for $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ be $(\gamma^{**}, \theta^{**}, \boldsymbol{\lambda}^{**}, \mathbf{s}^{-**})$. The EBM objective function value is:

$$\gamma^{**} = \theta^{**} - \varepsilon_x \sum_{i=1}^m \frac{w_i^- s_i^-}{x_{io}^*}. \quad (\text{A2})$$

The corresponding constraints for $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ are:

$$\theta^{**} \mathbf{x}_o^* = \mathbf{X} \boldsymbol{\lambda}^{**} + \mathbf{s}^{-**}, \quad \mathbf{y}_o^* \leq \mathbf{Y} \boldsymbol{\lambda}^{**}. \quad (\text{A3})$$

This reduces to:

$$\theta^{**} \theta^* \mathbf{x}^* = \mathbf{X} \boldsymbol{\lambda}^{**} + \mathbf{s}^{-**} + \theta^{**} \mathbf{s}^{-*}, \quad \mathbf{y}_o \leq \mathbf{y}_o^* \leq \mathbf{Y} \boldsymbol{\lambda}^{**}. \quad (\text{A4})$$

This is another expression for $(\mathbf{x}_o, \mathbf{y}_o)$ and its objective function value is:

$$f = \theta^{**} \theta^* - \varepsilon_x \sum_{i=1}^m \frac{w_i^- (s_i^- + \theta^{**} s_i^-)}{x_{io}} = \theta^{**} \gamma^* - \varepsilon_x \sum_{i=1}^m \frac{w_i^- s_i^-}{x_{io}}. \quad (\text{A5})$$

We have three possibilities as follows:

- i) The case $\theta^{**} < 1$. In this case, it holds that $f < \gamma^*$. This contradicts the optimality of γ^* for $(\mathbf{x}_o, \mathbf{y}_o)$. Thus, this case never occurs.
- ii) The case $\theta^{**} = 1$. In this case, by the optimality of γ^* for $(\mathbf{x}_o, \mathbf{y}_o)$, we have $s_i^{-**} = 0 (\forall i)$. Thus, $\gamma^{**} = 1$ and $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ is EBM input-efficient.
- iii) The case $\theta^{**} > 1$. From the optimality of γ^* for $(\mathbf{x}_o, \mathbf{y}_o)$, it holds that

$$\theta^{**} \gamma^* - \varepsilon_x \sum_{i=1}^m \frac{w_i^- s_i^-}{x_{io}} \geq \gamma^*$$

Hence we have

$$\theta^{**} \geq 1 + \frac{\varepsilon_x}{\gamma^*} \sum_{i=1}^m \frac{w_i^- s_i^-}{x_{io}}. \quad (\text{A6})$$

Suppose that $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ is EBM-inefficient, i.e. $\gamma^{**} = \theta^{**} - \varepsilon_x \sum_{i=1}^m \frac{w_i^- s_i^-}{x_{io}^*} < 1$. Then we have:

$$\theta^{**} < 1 + \varepsilon_x \sum_{i=1}^m \frac{w_i^- s_i^-}{x_{io}^*}. \quad (\text{A7})$$

We compare the terms $\gamma^* x_{io}^*$ in (A6) and x_{io}^* in (A7). Since $x_{io}^* = \theta^* x_{io} - s_i^-$, we have

$$\gamma^* x_{io}^* - x_{io}^* = (\gamma^* - \theta^*) x_{io} - s_i^- = -\varepsilon_x x_{io} \left(\sum_{k=1}^m \frac{w_k^- s_k^-}{x_{ko}^*} \right) - s_i^- \leq 0.$$

Thus, it holds that

$$\gamma^* x_{io}^* \leq x_{io}^*. \quad (\text{A8})$$

Comparing (A6) and (A7), we have

$$\theta^{**} \geq 1 + \varepsilon \sum_{i=1}^m \frac{w_i^- s_i^-}{\gamma^* x_{io}^*} \geq 1 + \varepsilon \sum_{i=1}^m \frac{w_i^- s_i^-}{x_{io}^*} > \theta^{**}. \quad (\text{A9})$$

This cannot occur. Thus, in this case, $(\mathbf{x}_o^*, \mathbf{y}_o^*)$ is EBM input-efficient. Q.E.D.

Appendix B. Proof of Proposition 8

(a) Proof of $D(\mathbf{a}, \mathbf{b}) = D(\mathbf{b}, \mathbf{a})$

Let $d_j = \ln(a_j / b_j)$ ($j=1, \dots, n$), $d_{\max} = \max\{d_j\}$, $d_{\min} = \min\{d_j\}$, and $\bar{d} = \frac{1}{n} \sum d_j$. Then it holds that

$d_j = -c_j$ ($j=1, \dots, n$), $d_{\max} = -c_{\min}$, $d_{\min} = -c_{\max}$, and $\bar{d} = -\bar{c}$. Hence, we have

$$D(\mathbf{b}, \mathbf{a}) = \frac{\sum |d_j - \bar{d}|}{n(d_{\max} - d_{\min})} = \frac{\sum |-c_j + \bar{c}|}{n(-c_{\min} + c_{\max})} = D(\mathbf{a}, \mathbf{b}). \quad (\text{B1})$$

(b) Proof of $D(\mathbf{a}, \mathbf{b}) \leq 1/2$

If \mathbf{a} and \mathbf{b} are proportional, then it holds that $c_{\max} = c_{\min}$ and $D(\mathbf{a}, \mathbf{b}) = 0$. Otherwise if \mathbf{a} and \mathbf{b} are not proportional, then $c_{\max} > c_{\min}$ and $D(\mathbf{a}, \mathbf{b}) > 0$. Let N_1 and N_2 be respectively the set of j such that $c_j \leq \bar{c}$ and $c_j > \bar{c}$, and $n_1 = |N_1|$ and $n_2 = |N_2|$. We have $n = n_1 + n_2$. The numerator of $D(\mathbf{a}, \mathbf{b})$ can be transformed into the following:

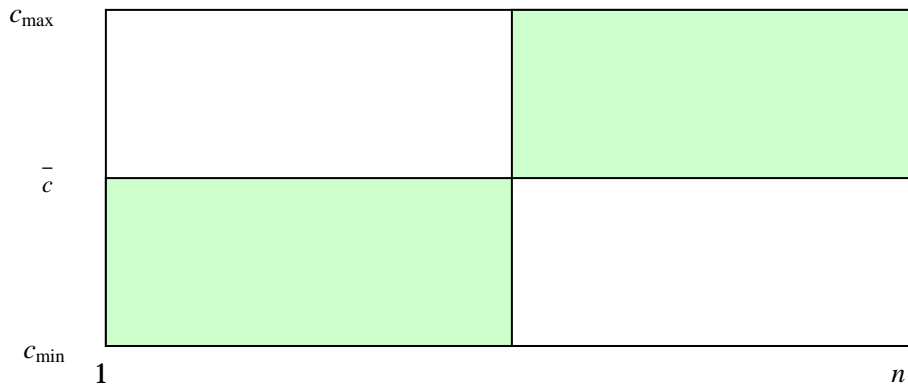
$$\begin{aligned} \sum_{j=1}^n |c_j - \bar{c}| &= \sum_{j \in N_1} (-c_j + \bar{c}) + \sum_{j \in N_2} (c_j - \bar{c}) \\ &\leq n_1(-c_{\min} + \bar{c}) + n_2(c_{\max} - \bar{c}) \\ &= \frac{2}{n}(c_{\max} - c_{\min})(n n_1 - n_1^2). \end{aligned} \quad (\text{B2})$$

The last term in the last expression attains the maximum $n^2/4$ at $n_1 = n/2$

Hence, we have

$$D(\mathbf{a}, \mathbf{b}) = \frac{\sum_{j=1}^n |c_j - \bar{c}|}{n(c_{\max} - c_{\min})} \leq \frac{1}{2}. \quad (\text{B3})$$

$D(\mathbf{a}, \mathbf{b}) \doteq 1/2$ holds when $\{c_j\}$ distributes as exemplified in Figure B1.

Figure B1: The case $D(\mathbf{a}, \mathbf{b}) \doteq 1/2$