GRIPS Discussion Paper 15-05

SBM variations revisited

Kaoru Tone

May 2015

National Graduate Institute for Policy Studies
7-22-1 Roppongi, Minato-ku,
Tokyo, Japan 106-8677

SBM variations revisited

Kaoru Tone

National Graduate Institute for Policy Studies

7-22-1 Roppongi, Minato-ku, Tokyo 106-8677, Japan

tone@grips.ac.jp

5/8/2015

Abstract

Slacks-based measure (SBM) (Tone (2001), Pastor et al. (1999)) has been widely utilized as a representative non-radial DEA model. In Tone (2010), I developed four variants of the SBM model where main concerns are to search the nearest point on the efficient frontiers of the production possibility set. However, in the worst case, a massive enumeration of facets of polyhedron associated with the production possibility set is required. In this paper, I will present a new scheme for this purpose which requires a limited number of additional linear program solutions for each inefficient DMU. Although the point thus obtained is not always the nearest point, it is acceptable for practical purposes and from the point of computational loads.

Keywords: DEA, SBM, reference set, nearest point

1. Introduction

There are two types of models in DEA; radial and non-radial. Radial models are represented by the CCR (Charnes-Cooper-Rhodes) model. Basically they deal with proportional changes of inputs or outputs. As such, the CCR score reflects the proportional maximum input (output) reduction (expansion) rate which is common to all inputs (outputs). However, in real world businesses, not all inputs (outputs) behave in the proportional way. For example, if we employ labor, materials and capital as inputs, some of them are substitutional and do not change proportionally. Another shortcoming of the radial models is the neglect of slacks in reporting the efficiency score. In many cases, we find a lot of remaining non-radial slacks. So, if these slacks have an important role in evaluating managerial efficiency, the radial approaches may mislead the decision when we utilize the efficiency score as the only index for evaluating performance of DMUs.

In contrast, the non-radial SBM models put aside the assumption of proportionate changes in inputs and outputs, and deal with slacks directly. This may discard varying proportions of original inputs and outputs. The SBM models are designed to meet the following two conditions.

(1) Units invariant: The measure should be invariant with respect to the units of data

(2) Monotone: The measure should be monotone decreasing in each slack in input and output.

The original SBM model evaluates efficiency of DMUs referring to the furthest frontier point within a range. This results in the hardest score for the objective DMU and the projection may go to a remote point on the efficient frontier which may be inappropriate as the reference. In Tone (2010), I developed four variants of the SBM model where main concerns are to search the nearest point on the efficient frontiers of the production possibility set. Referring these variations, several authors published new models. Among them, I introduce two important papers.

Fukuyama et al. (2014) developed a least distance efficiency measure with the strong/weak monotonicity of the ratio form measure under several norms including 1-norm, 2-norm and ∞-norm. This model utilizes mixed-integer linear programming (MILP) to identify efficiency frontiers and hence a computational difficulty arises for large-scale problems.

Hadi-Vencheh et al. (2015) developed a new SBM model to find the nearest point on the efficient frontiers. They utilize the multiplier form model to find all supporting hyperplanes. It also utilizes software which uses fractional coefficients (high precision arithmetic) to avoid loss data. Hence, computational time increases for large-scale problems.

In order to apply DEA models to actual real world problems, we need to try many instances including selection of DMUs and input/output factors before attaining the final scheme of evaluation. For this purpose, allowable computation time and easy accessible software are desirable.

The motivation and purpose of this paper is to obtain nearly closest points on the efficient frontiers within foreseeable computation loads using only popular linear programming codes.

As a proposer of the SBM, I think that it is my duty to make this method more practical and applicable.

The rest of this paper is organized as follows. Section 2 introduces the ordinary SBM-Min model briefly. Section 3 presents the new SBM-Max model. Observations on this new model are described in Section 4. Two numerical examples are exhibited in Section 5. Section 6 concludes this paper. Although we present the model in non-oriented mode, we can treat input- and output-oriented model as well. As to returns-to-scale characteristics, we present the constant returns-to-scale (CRS) case. However we can deal with the variable returns-to-scale (VRS) model as well.

2. The SBM Min model

The SBM model was introduced by Tone (2001) (see also Pastor et al. (1999)). It has three variations, i.e. input-, output- and non-oriented. The non-oriented model indicates both input- and output-oriented.

Let the set of DMUs be $J = \{1, 2, \ldots, n\}$, each DMU having m inputs and s outputs. We denote the vectors of inputs and outputs for DMU$_j$ by $\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^T$ and $\mathbf{y}_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^T$, respectively. We define input and output matrices X and Y by

$$\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \in R^{m \times n} \text{ and } \mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n) \in R^{s \times n}. \quad (1)$$

We assume that all data are positive i.e. $\mathbf{X} > \mathbf{0}$ and $\mathbf{Y} > \mathbf{0}$.

2.1 Production Possibility Set

The production possibility set is defined using the non-negative combination of the DMUs in the set J as:

$$P=\left\{(\mathbf{x},\mathbf{y})\middle|\mathbf{x}\geq\sum_{j=1}^{n}\lambda_j\mathbf{x}_j,\ \mathbf{0}\leq\mathbf{y}\leq\sum_{j=1}^{n}\lambda_j\mathbf{y}_j,\ \boldsymbol{\lambda}\geq\mathbf{0}\right\}. \tag{2}$$

$\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_n)^T$ is called the intensity vector.

The inequalities in (2) can be transformed into equalities by introducing slacks as follows:

$$\begin{aligned}&\mathbf{x}=\sum_{j=1}^{n}\lambda_j\mathbf{x}_j+\mathbf{s}^-\\&\mathbf{y}=\sum_{j=1}^{n}\lambda_j\mathbf{y}_j-\mathbf{s}^+\\&\mathbf{s}^-\geq\mathbf{0},\ \mathbf{s}^+\geq\mathbf{0},\end{aligned} \tag{3}$$

where $\mathbf{s}^-=(s_1^-,s_2^-,\ldots,s_m^-)^T\in R^m$ and $\mathbf{s}^+=(s_1^+,s_2^+,\ldots,s_s^+)^T\in R^s$ are respectively called input and output slacks.

2,2 Non-oriented SBM

Non-oriented or both-oriented SBM efficiency $\rho_o^{\min}$ is defined by

$$\text{[SBM-Min]}\quad \rho_o^{\min}=\min_{\boldsymbol{\lambda},\mathbf{s}^-,\mathbf{s}^+}\frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}}$$

subject to

$$\begin{aligned}&x_{io}=\sum_{j=1}^{n}x_{ij}\lambda_j+s_i^-\ (i=1,\ldots,m)\\&y_{ro}=\sum_{j=1}^{n}y_{rj}\lambda_j-s_r^+\ (r=1,\ldots,s)\\&\lambda_j\geq 0\ (\forall j),\ s_i^-\geq 0(\forall i),\ s_r^+\geq 0(\forall r).\end{aligned} \tag{4}$$

[Definition 1] (SBM-efficient) A DMU $_o=(\mathbf{x}_o,\mathbf{y}_o)$ is called SBM-efficient if $\rho_o^{\min}=1$ holds.

This means $\mathbf{s}^-=\mathbf{0}$ and $\mathbf{s}^{+*}=\mathbf{0}$, i.e. all input and output slacks are zero.

[SBM-Min] can be transformed into a linear program using the Charnes-Cooper transformation as follows:

$$\text{[SBM-Min-LP]} \quad \tau^* = \min_{t,\Lambda,S^-,S^+} t - \frac{1}{m}\sum_{i=1}^{m} \frac{S_i^-}{x_{io}}$$

subject to

$$1 = t + \frac{1}{s}\sum_{r=1}^{s} \frac{S_r^+}{y_{ro}}$$
$$tx_{io} = \sum_{j=1}^{n} x_{ij}\Lambda_j + S_i^- \ (i = 1,\ldots,m)$$
$$ty_{ro} = \sum_{j=1}^{n} y_{rj}\Lambda_j - S_r^+ \ (r = 1,\ldots,s)$$
$$\Lambda_j \geq 0 \ (\forall j),\ S_i^- \geq 0 (\forall i),\ S_r^+ \geq 0 (\forall r),\ t > 0. \tag{5}$$

Let an optimal solution be $(\tau^*, t^*, \mathbf{\Lambda}^*, \mathbf{S}^{-*}, \mathbf{S}^{+*})$. Then, we have an optimal solution of [SBM-Min] as defined by

$$\rho_o^{\min} = \tau^*,\ \boldsymbol{\lambda}^* = \mathbf{\Lambda}^* / t^*,\ \mathbf{s}^{-*} = \mathbf{S}^{-*} / t^*,\ \mathbf{s}^{+*} = \mathbf{S}^{+*} / t^*. \tag{6}$$

3. The SBM Max Model

In this section, we introduce the new non-oriented SBM-Max model.

Step 1. Solve SBM-Min

First, we solve the ordinary SBM (SBM-Min) model as represented by the program (4) for DMU $(\mathbf{x}_o, \mathbf{y}_o)(o = 1,\ldots,n)$. Let an optimal solution be $(\boldsymbol{\lambda}^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$.

Step 2. Define efficient DMUs

We define the set R^{eff} of all efficient DMUs as

$$R^{eff} = \left\{ j \middle| \rho_j^{\min} = 1, j = 1,\ldots,n \right\}. \tag{7}$$

We denote these efficient DMUs as $(\mathbf{x}_1^{eff}, \mathbf{y}_1^{eff}), (\mathbf{x}_2^{eff}, \mathbf{y}_2^{eff}), \ldots, (\mathbf{x}_{Neff}^{eff}, \mathbf{y}_{Neff}^{eff})$, wher *Neff* is the number of efficient DMUs.

Step 3. Local reference set

For an inefficient DMU $(\mathbf{x}_o, \mathbf{y}_o)$, we define the local reference set R_o^{local}, i.e., efficient DMUs set for DMU $(\mathbf{x}_o, \mathbf{y}_o)$, by (8).

$$R_o^{local} = \left\{ j \middle| \lambda_j^* > 0, j = 1,\ldots,n \right\}. \tag{8}$$

Step 4. Pseudo-Max score

For each inefficient DMU, i.e., $\rho_o^{\min} < 1$, we solve the following program.

$$\text{[Pseudo-1]} \qquad \max \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_{io}^{-}}{x_{io}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_{r}^{+}}{y_{ro}}}$$

subject to

$$\mathbf{x}_o = \sum_{j \in R_o^{local}} \mathbf{x}_j \lambda_j + \mathbf{s}^- \tag{9}$$

$$\mathbf{y}_o = \sum_{j \in R_o^{local}} \mathbf{y}_j \lambda_j - \mathbf{s}^+$$

$$\mathbf{s}^-, \mathbf{s}^+, \boldsymbol{\lambda} \geq \mathbf{0}.$$

Let an optimal slacks be $\left(\mathbf{s}^{-*}, \mathbf{s}^{+*}\right)$. We solve the following program with variables $(\boldsymbol{\lambda}, \mathbf{s}^-, \mathbf{s}^+)$.

$$\text{[Pseudo-2]} \qquad \min \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_{io}^{-}}{x_{io}-s_{i}^{-*}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_{r}^{+}}{y_{ro}+s_{r}^{+*}}}$$

subject to

$$\mathbf{x}_o - \mathbf{s}^{-*} = \sum_{j \in R^{eff}} \mathbf{x}_j^{eff} \lambda_j + \mathbf{s}^- \tag{10}$$

$$\mathbf{y}_o + \mathbf{s}^{+*} = \sum_{j \in R^{eff}} \mathbf{y}_j^{eff} \lambda_j - \mathbf{s}^+$$

$$\mathbf{s}^-, \mathbf{s}^+, \boldsymbol{\lambda} \geq \mathbf{0}.$$

Let the optimal slacks be $\left(\mathbf{s}^{-**}, \mathbf{s}^{+**}\right)$. We define the Pseudo-Max score $\rho_o^{pseudo\max}$ by

$$\text{[Pseudo-Max]} \qquad \rho_o^{pseudo\max} = \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_{io}^{-*}+s_{io}^{-**}}{x_{io}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_{r}^{+*}+s_{r}^{+**}}{y_{ro}}}. \tag{11}$$

Step 5. Distance and SBM-Max score

For each inefficient DMU $\left(\mathbf{x}_o, \mathbf{y}_o\right)$, i.e., $\rho_o^{\min} < 1$, we calculate the distance between $\left(\mathbf{x}_o, \mathbf{y}_o\right)$ and $\left(\mathbf{x}_h^{eff}, \mathbf{y}_h^{eff}\right)(h = 1, \ldots, Neff)$ by

$$\text{[Distance]} \quad d_h = \sum_{i=1}^{m} \frac{\left| x_{ih}^{eff} - x_{io} \right|}{x_{io}} + \sum_{i=1}^{s} \frac{\left| y_{ih}^{eff} - y_{io} \right|}{y_{io}}. \tag{12}$$

This distance is units-invariant.

Step 5.1. Reorder the distance

We renumber the efficient DMUs in the ascending order of d_h, so that

$$d_1 \leq d_2 \leq \ldots \leq d_{Neff}. \tag{13}$$

We define the set R_h by

$$R_h = \{1, \ldots, h\} \; (h = 1, \ldots, Neff). \tag{14}$$

Step 5.2. Find slacks and max-score for the set R_h

We evaluate the efficiency score of the inefficient DMU $(\mathbf{x}_o, \mathbf{y}_o)$ referring to the set R_h by solving the following program.

$$\begin{aligned} \text{[Max-1]} \quad & \max \frac{1 - \frac{1}{m} \sum_{i=1}^{m} \frac{s_{io}^{-}}{x_{io}}}{1 + \frac{1}{s} \sum_{r=1}^{s} \frac{s_r^{+}}{y_{ro}}} \\ & \text{subject to} \\ & \mathbf{x}_o = \sum_{j \in R_h} \mathbf{x}_j^{eff} \lambda_j + \mathbf{s}^{-} \\ & \mathbf{y}_o = \sum_{j \in R_h} \mathbf{y}_j^{eff} \lambda_j - \mathbf{s}^{+} \\ & \mathbf{s}^{-}, \mathbf{s}^{+}, \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned} \tag{15}$$

(a) If this program is infeasible, we define $\rho_{oh}^{*} = 0$. Otherwise, let an optimal slacks be $(\mathbf{s}^{-*}, \mathbf{s}^{+*})$.

(b) If the optimal objective value is 1, i.e., $\mathbf{s}^{-*} = \mathbf{0}$ and $\mathbf{s}^{+*} = \mathbf{0}$, we define $\rho_{oh}^{*} = 0$. This indicates that DMU $(\mathbf{x}_o, \mathbf{y}_o)$ can be expressed as a non-negative combination of DMUs in R_h and hence, in view of $\rho_o^{\min} < 1$, it is inside the production possibility set.

(c) If the optimal objective value is less than 1, we again solve the following program with the variables $(\boldsymbol{\lambda}, \mathbf{s}^{-}, \mathbf{s}^{+})$.

$$[\text{Max-2}] \quad \min \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_{io}^{-}}{x_{io}-s_i^{-*}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^{+}}{y_{ro}+s_r^{+*}}}$$

subject to

$$\mathbf{x}_o - \mathbf{s}^{-*} = \sum_{j\in R^{eff}} \mathbf{x}_j^{eff}\lambda_j + \mathbf{s}^{-} \tag{16}$$

$$\mathbf{y}_o + \mathbf{s}^{+*} = \sum_{j\in R^{eff}} \mathbf{y}_j^{eff}\lambda_j - \mathbf{s}^{+}$$

$$\mathbf{s}^{-}, \mathbf{s}^{+}, \boldsymbol{\lambda} \geq \mathbf{0}.$$

Let the optimal slacks be $\left(\mathbf{s}^{-**}, \mathbf{s}^{+**}\right)$. We define ρ_{oh}^{*} by

$$[\rho_{oh}^{*}] \qquad \rho_{oh}^{*} = \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_{io}^{-*}+s_{io}^{-**}}{x_{io}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^{+*}+s_r^{+**}}{y_{ro}}}. \tag{17}$$

We assign ρ_{oh}^{*} as the max-score referring to the set R_h.

Step 5.3. SBM-Max and projection

Finally, we define the max-score $\rho_o^{\max}$ of inefficient DMU $(\mathbf{x}_o, \mathbf{y}_o)$ by

$$[\text{SBM-Max}] \qquad \rho_o^{\max} = \max\left\{\rho_o^{pseudo\max}, \rho_{o1}^{*}, \cdots, \rho_{oNeff}^{*}\right\}. \tag{18}$$

We also hold the slacks $\left(\mathbf{s}^{-**}, \mathbf{s}^{+**}\right)$ corresponding to the maximum $\rho_o^{\max}$. The projection of DMU $(\mathbf{x}_o, \mathbf{y}_o)$ onto efficient frontiers is given by

$$[\text{Projection}] \qquad \mathbf{x}_o^{*} = \mathbf{x}_o - \mathbf{s}^{-*} - \mathbf{s}^{-**}, \mathbf{y}_o^{*} = \mathbf{y}_o + \mathbf{s}^{+*} + \mathbf{s}^{+**}. \tag{19}$$

4. Observations

In this section, we discuss several characteristics of the algorithm in Section 3.

4.1. Distance and choice of the set R_h

The set R_h plays a central role in choosing referent DMUs for inefficient DMUs. Because our main concern is the projection to the nearest point on the efficient frontiers, we evaluate the distance between the DMU $(\mathbf{x}_o, \mathbf{y}_o)$ and efficient DMUs by (12), and choose the shortest distance DMU as the first candidate DMU. Then, we expand the referent set in the ascending order of distances. Thus, we can expect a close efficient point on the frontiers with high probability. If tie occurs in distances, we can choose any one at random.

4.2. The role of Programs (10) and (16)

For example, Program (16) is necessary to project the point $\left(\mathbf{x}_o-\mathbf{s}^{-*},\mathbf{y}_o+\mathbf{s}^{+*}\right)$ on the efficient frontiers. Thus, $\left(\mathbf{x}_o^*=\mathbf{x}_o-\mathbf{s}^{-*}-\mathbf{s}^{-**},\mathbf{y}_o^*=\mathbf{y}_o+\mathbf{s}^{+*}+\mathbf{s}^{+**}\right)$ is the projected point on the efficient frontiers and it is expected to be close to the DMU $\left(\mathbf{x}_o,\mathbf{y}_o\right)$ by the selection rule of R_h.

4.3. Computational amount

Computations needed for this algorithm for an inefficient DMU are as follows.

Let t_1 and t_2 be the CPU time for solving a LP problem, respectively, with the (m + s) rows and n columns, and (m + s) rows and *Neff* columns. Since LP solution time is proportional to the number of columns. We can estimate roughly $t_1 = (n / Neff)\, t_2$.

(1) Program (4) or (5) needs $n * t_1$ CPU time.

(2) Programs (9) and (10) need at most 2 * (n –*Neff*) * t_2 CPU time.

(3) Programs (15) and (16) need at most 1.5 * (n – *Neff*) * *Neff* * t_2 CPU time, because the member of R_h in (15) varies from 1 to *Neff*.

However, if Step 5.2 (c) occurs at some set R_h, we can skip the computations for the succeeding Programs (15) and (16) for h +1,…, *Neff*.

Overall, the total time for LP computation is at most

$$T= n * t_1 + 2 * (n - Neff) * t_2 + 1.5 * (n - Neff) * Neff * t_2$$
$$= [n + (2 + 1.5 * Neff) * (n - Neff) * (Neff / n)] * t_1. \qquad (20)$$

Thus, the computational amount is polynomial order and we do not need other software, e.g., MILP and fractional arithmetic.

4.4. Consistency with the super-efficiency SBM measure

The SBM-Max model aims at getting to the nearest point on the efficient frontiers. This concept is in line with the super-efficiency SBM model (Tone (2002)) which solves the following program for an efficient DMU $\left(\mathbf{x}_o,\mathbf{y}_o\right)$ to mesure the minimum ratio-scale distance from the efficient frontier excluding the DMU $\left(\mathbf{x}_o,\mathbf{y}_o\right)$.

$$\text{[Super-SBM]} \qquad \delta^* = \min \frac{1+\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1-\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}}$$

subject to

$$\mathbf{x}_o + \mathbf{s}^- = \sum_{j=1, j\neq o}^{n} \mathbf{x}_j \lambda_j \tag{21}$$

$$\mathbf{y}_o - \mathbf{s}^+ = \sum_{j=1, j\neq o}^{n} \mathbf{y}_j \lambda_j$$

$$\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{s}^- \geq \mathbf{0}, \mathbf{s}^+ \geq \mathbf{0}.$$

We can solve the super-efficiency SBM model by applying LP code just once, because this problem belongs to a convex programming, i.e., minimization of a convex function over a convex region. However, SBM-Max problem cannot be solved in this manner, because it is a maximization of a convex function over a convex region.

5. Numerical examples

In this section, we show two numerical examples, the first one is illustrative and the other deals with real data. All computations are executed using a PC with Intel Core i7-3770 CPU at 3.40 GHz 16 GB (RAM) and Microsoft Excel VBA (Visual Basic for Applications). A LP soft (revised simplex method) is coded by the author. We checked the results of the first example using LINGO (LINDO Systems Inc.) and had the same figures.

5.1. An illustrative example

We deal with the same data as one in Tone (2010). Table 1 displays the data with two inputs (Doctor and Nurse) and two outputs (Outpatient and Inpatient).

Table 1: Illustrative example

DMU	(I)Doctor	(I)Nurse	(O)Outpatient	(O)Inpatient
A	20	151	100	90
B	19	131	150	50
C	25	160	160	55
D	27	168	180	72
E	22	158	94	66
F	55	255	230	90
G	33	235	220	88
H	31	206	152	80
I	30	244	190	100

J	50	268	250	100
K	53	306	260	147
L	38	273	250	133

5.1.1. Solution of SBM-Min model

First, we solved SBM-Min model and obtained the results exhibited in Table 2.

Table 2: Results of SBM-Min model

DMU	Score	Rank	Reference(Lambda)			
A	1	1	*A*	1		
B	1	1	*B*	1		
C	0.8265	8	*B*	0.449	*L*	0.371
D	1	1	*D*	1		
E	0.7277	11	*B*	0.667	*L*	0.246
F	0.6857	12	*A*	0.092	*L*	0.883
G	0.8765	6	*B*	0.16	*L*	0.784
H	0.7713	9	*L*	0.755		
I	0.9016	5	*A*	0.233	*L*	0.667
J	0.7653	10	*B*	0.152	*L*	0.909
K	0.8619	7	*B*	0.15	*L*	1.049
L	1	1	*L*	1		

We find four efficient DMUs, i.e. $R^{eff} = \{A, B, D, L\}$.

5.1.2. The case of inefficient DMU *I*

We explain the case of inefficient DMU *I*, step by step.

(1) Steps 1, 2 and 3: $\rho_I^{\min} = 0.9016, R_I^{local} = \{A, L\}, R^{eff} = \{A, B, D, L\}$.

(2) Step 4: $\rho_I^{pseudo\max} = 0.9016$.

(3) Step 5.1: Distances from efficient DMUs are d_A=1.28816, d_B=1.54030, d_D=0.74411, and d_L=1.03131. Thus, we have $R_1 = \{D\}, R_2 = \{D, L\}, R_3 = \{D, L, A\}, \text{and}\, R_4 = \{D, L, A, B\}$.

(4) Step 5.2: We solve (16) and (17), and find: $\rho_{I1}^* = 0.859885, \rho_{I2}^* = 0.910900, \rho_{I3}^* = 0.921168, \text{and}\, \rho_{I4}^* = 0.920198$.

(5) Step 5.3: From (18), we find $\rho_I^{\max} = \max\{\rho_I^{pseudo\max}, \rho_{I1}^*, \rho_{I2}^*, \rho_{I3}^*, \rho_{I4}^*\} = 0.921168 = \rho_{I3}^*$ with the reference set $R_3 = \{A, D, L\}$. Its projection is $(x_{1I}^* = 30, x_{2I}^* = 205.53, y_{1I}^* = 190, y_{2I}^* = 100)$ with the slacks $(s_1^- = 0, s_2^- = 38.47, s_1^+ = 0, s_2^+ = 0)$. The SBM-Min model has $\rho_I^{\min} = 0.9016$ with slacks $(s_1^- = 0, s_2^- = 26.767, s_1^+ = 0, s_2^+ = 9.667)$. This indicates that SBM-Min model requires

a reduction of Nurse by 26.767 and an increase of Inpatient by 9.667 to attain the efficiency status, whereas SBM-Max model requires a reduction of Nurse by 38.47 to attain the efficiency status.

5.1.3. Comparisons among SBM-Max, Pseudo-Max and SBM-Min scores

Table 3 compares results with SBM-Max, SBM-Pseudo and SBM-Min scores. Inefficient DMUs increased their efficiency from SBM-Min to SBM-Max.

Table 3: Comparisons

DMU	SBM-Max	Rank	Pseudo	Rank	SBM-Min	Rank
A	1	1	1	1	1	1
B	1	1	1	1	1	1
C	0.87507	8	0.855	8	0.8265	8
D	1	1	1	1	1	1
E	0.7682	11	0.7391	11	0.7277	11
F	0.72648	12	0.6868	12	0.6857	12
G	0.93688	5	0.9052	5	0.8765	6
H	0.80918	10	0.7714	10	0.7714	9
I	0.92117	6	0.9016	6	0.9016	5
J	0.81032	9	0.7898	9	0.7653	10
K	0.88894	7	0.8622	7	0.8619	7
L	1	1	1	1	1	1
Average	0.8947		0.8759		0.8681	
Max	1		1		1	
Min	0.7265		0.6868		0.6857	
St Dev	0.0982		0.1114		0.115	

Table 4 exhibits $\rho_o^{pseudo\max}, \rho_{o1}^*, \cdots, \rho_{o4}^*$ for inefficient DMUs. Shaded portions indicate the max. The SBM-Max scores are found at several stage of R_h.

Table 4

DMU	ρ0	ρ1	ρ2	ρ3	ρ4
C	0.85495	0.87507	0.87507	0.87507	0.87507
E	0.73911	0.7682	0.7682	0.7682	0.7682
F	0.68681	0.68681	0.72648	0.72648	0.72648
G	0.90516	0.93688	0.93688	0.93688	0.93688
H	0.77135	0.80918	0.80918	0.80918	0.80918
I	0.90163	0.85989	0.9109	0.92117	0.92019
J	0.78982	0.75731	0.81032	0.81032	0.81032
K	0.86221	0.86221	0.86615	0.88894	0.88894

5.1.4. Comparisons of average differences between SBM-Max and SBM-Min

Figure 1 shows the average of percentage deviations, │Data - Projection│ * 100 / Data. It is observed large differences exist in SBM-Min, while small differences in SBM-Max.

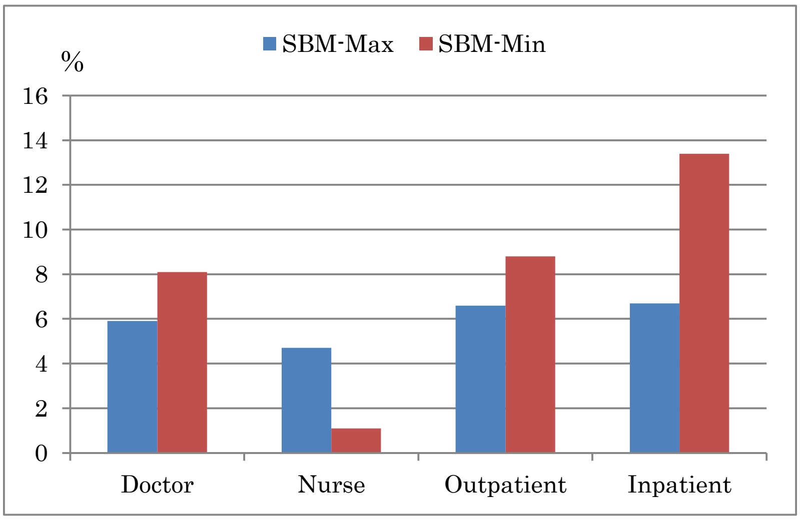

Figure 1: Average deviations (%)

Table 5 reports data and projections along with deviations (%) in the case of SBM-Max.

Table 5: Data and projection by SBM-Max

DM	Score	Rank	Doc.			Nur.			Outpatient			Inpatient		
			Data	Proj.	Diff.(%)	Data	Proj.	Diff.(%)	Data	Proj.	Diff.(%)	Data	Proj.	Diff.(%)
A	1	1	20	20	0	151	151	0	100	100	0	90	90	0
B	1	1	19	19	0	131	131	0	150	150	0	50	50	0
C	0.8751	8	25	25	0	160	155.6	-2.78	160	166.7	4.17	55	66.7	21.21
D	1	1	27	27	0	168	168	0	180	180	0	72	72	0
E	0.7682	11	22	20.9	-4.88	158	158	0	94	104.6	11.32	66	94.2	42.69
F	0.7265	12	55	34.5	-37.27	255	214.7	-15.82	230	230	0	90	92	2.22
G	0.9369	5	33	33	0	235	205.3	-12.62	220	220	0	88	88	0
H	0.8092	10	31	30	-3.23	206	186.7	-9.39	152	200	31.58	80	80	0
I	0.9212	6	30	30	0	244	205.5	-15.77	190	190	0	100	100	0
J	0.8103	9	50	43.1	-13.86	268	268	0	250	287.1	14.86	100	115	14.86
K	0.888	7	53	46.5	-12.26	306	306	0	260	289.1	11.2	147	147	0
L	1	1	38	38	0	273	273	0	250	250	0	133	133	0

5.2. Japanese municipal hospitals

The data were collected from the Annual Databook of Local Public Enterprise published by the Ministry of Internal Affairs and Communications Japanese Government, 2005.

5.2.1. Data

Number of DMUs: 707 hospitals (n = 707).

Number of inputs: 5. (1) No. of beds (Bed), (2) Expenses for outsourcing (Outsource), (3) No. of doctors (Doctor), (4) No. of nurses (Nurse) and (5) Expenses for other medical materials (Material). (m = 5)

Number of outputs: 4. (1) Revenue from operation per day (Operation), (2) Revenue from first consultation per day (1st time), (3) Revenue from return to clinic per day (Follow-up) and (4) Revenue from hospitalization per day (Hotel). (s = 4)

Table 6 exhibits statistics of the dataset.

Table 6: Statistics of dataset (n = 707)

	Bed	Outsource	Doctor	Nurse	Material	Operation	1st time	Follow-up	Hotel
Max	1063	2231247	215.562	955.464	3395791	1.7E+07	1432079	3359160	1.8E+07
Min	25	7767	0.98	11	9197	8979	2706	13636	109650
Average	255.924	312686	33.2783	175.709	491909	2128506	211538	415872	3263845
SD	191.764	334184	34.1647	149.678	598474	2533050	212439	327749	3020809

5.2.2. SBM scores

The SBM-Min model found that 66 hospitals among 707 are efficient ($Neff$ = 66). Table 7 compares three scores. We found large differences between Max and Min models.

Table 7: Comparisons of three scores

	SBM-Max	Pseudo	SBM-Min
Average	0.7835	0.6997	0.4515
Max	1	1	1
Min	0.1889	0.0394	0.0118
St Dev	0.1339	0.211	0.229

Figures 2 (SBM-Max) and 3 (SBM-Min) exhibit respectively scores of 707 hospitals in ascending order where we can observe big differences.

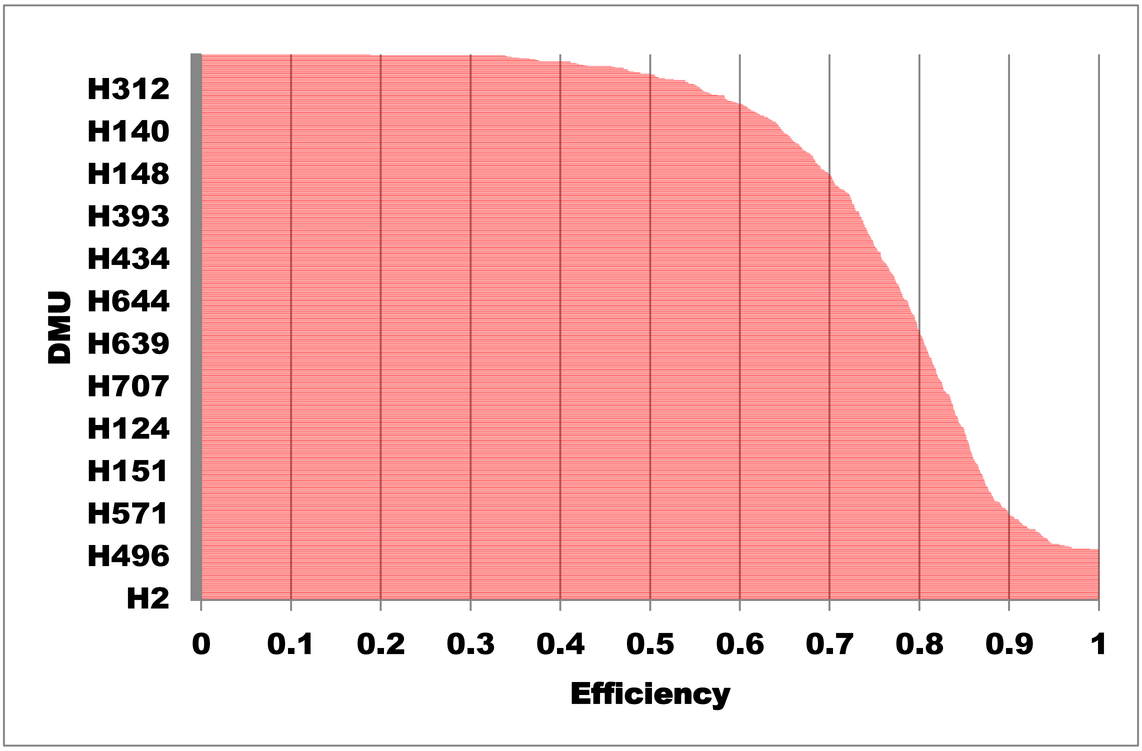

Figure 2: Distribution of SBM-Max scores

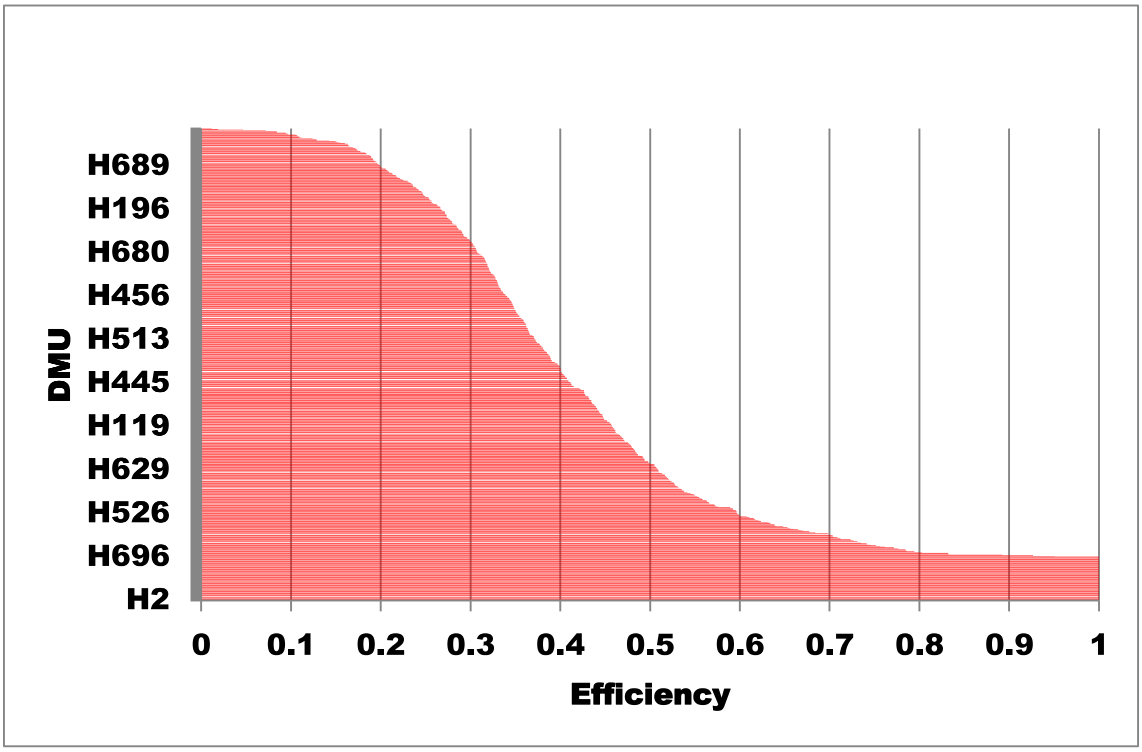

Figure 3: Distribution of SBM-Min scores

Figure 8 shows the average of percentage deviations, │Data - Projection│* 100 / Data. It is observed large differences exist in SBM-Min, while small differences in SBM-Max.

Table 8: Average deviation (%)

	SBM-Max	SBM-Min
Bed	13.1644	3.4014
Outsource	20.0291	24.8771
Doctor	12.0707	9.9367
Nurse	9.6586	5.7022
Material	10.7101	8.144
Operation	12.7155	48.2925
1st time	9.4884	407.243
Follow-up	13.3537	192.406
Hotel	17.4181	1.1889

Figure 4 illustrates deviations graphically. Large differences found in SBM-Min, while balanced deviations in SBM-Max.

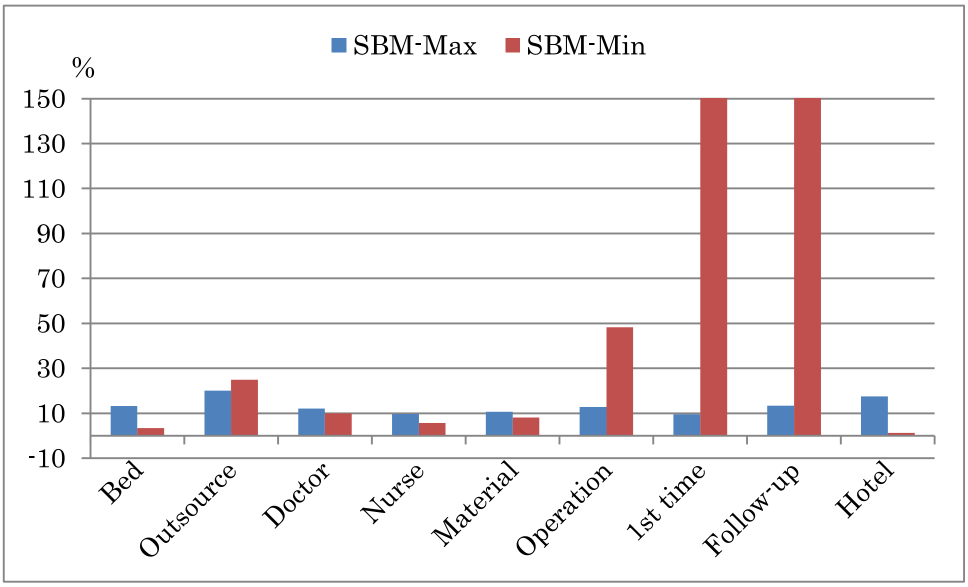

Figure 4: Average deviations (%) of inputs and outputs (cut at 150%)

5.2.3. Computational time

The computational time increases as the number of efficient DMUs (*Neff*) increases, because number of facets increases accordingly and we need to solve additional *Neff* LPs. In this example we have:

(1) CPU time for SBM-Min and SBM-Pseudo = 12 seconds.

(2) CPU time for SBM-Max = 179 seconds.

SBM-Max needs about 15 times of SBM-Min and SBM-Pseudo. This number is reasonable and consistent with the formula (20).

6. Conclusions

In this paper, we have developed the SBM-Max model which attempts to find nearly closest reference point on the efficient frontiers so that slacks are minimized, while the scores are maximized. Sacrificing the rigorous solutions, the proposed model utilizes a standard LP code and finds approximate solutions in allowable (polynomial) times.

Many applications of the SBM-Min models have been developed over the world. According to Google Citation Index, 1348 articles cited Tone (2001) at now (2015/5/4). Also many DEA models are developed based on this model. Above all, Network SBM (NSBM) (Tone and Tsutsui (2009)), Dynamic SBM (DSBM) (Tone and Tsutsui (2010)), Dynamic and Network SBM (DNSBM) (Tone and Tsutsui (2014)), and Malmquist SBM (Cooper at al. (2007)) are representative. Revisions of these models based on the SBM-Max model are imperative future research subjects.

References

[1] Cooper WW, Seiford LM, Tone K. (2007) Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software, Second Edition, Springer.

[2] Fukuyama H, Masaki H, Sekitani K, Shi J. (2014) Distance optimization approach to ratio-form efficiency measures in data envelopment analysis, Journal of Productivity Analysis 42, 175-186.

[3] Hadi-Vencheh A, Jablonsky J, Esmaeilzadeh A. (2015) The slack-based measure model based on supporting hyperplanes of production possibility set, Expert Systems with Applications, doi:http://dx.doi.org/10.1016/j.eswa.2015.03.032.

[4] Pastor JT, Ruiz JL, Sirvent I. (1999) An enhanced DEA Russell graph efficiency measure, European Journal of Operational Research 115, 596-607.

[5] Tone K. (2001) A slacks-based measure of efficiency in data envelopment analysis, European Journal of Operational Research 130, 498-509.

[6] Tone K. (2002) A slacks-based measure of super-efficiency in data envelopment analysis, European Journal of Operational Research 143, 32-41.

[7] Tone K. (2010) Variations on the theme of slacks-based measure of efficiency in DEA, European Journal of Operational Research 200, 901-907.

[8] Tone K, Tsutsui M. (2009) Network DEA: A slacks-based measure approach, European Journal of Operational Research 197, 243-252.

[9] Tone K, Tsutsui M. (2010) Dynamic DEA: A slacks-based measure approach, Omega 38, 145-156.

[10] Tone K, Tsutsui M. (2014) Dynamic DEA with network structure: A slacks-based measure approach, Omega 42, 124-131.