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Some Computational Issues in Data Envelopment Analysis

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Abstract

This paper reflects the author's experiences in developing various DEA software, including models such as CCR, BCC, general returns to scale, categorical inputs and outputs and different systems. All the software deals with the dual side of the original CCR model and no non-Archimedian small number is used.

Introduction

This paper discusses some computational issues related to DEA, which was firstly introduced by Charnes, Cooper and Rhodes (1978). The CCR model estimates the relative efficiency of a decision making unit (DMU_o) among the whole n DMUs by solving the following linear programming problem in the weights $v \in R^m$ and $u \in R^s$.

(1) (CCR) $\max \quad u^T y_o$

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$$
\begin{array}{lrcl}
(2) & \text{subject to} \quad v^T x_o & = & 1 \\
(3) & v^T X & \geq & u^T Y \\
(4) & v & \geq & 0, \quad u \geq 0,
\end{array}
$$

where $X \in R^{m \times n}$ is the input matrix composed of x_{ij}, the ith input value of DMU_j, $Y \in R^{s \times n}$ the output matrix composed of y_{rj}, the rth output value of DMU_j and vectors x_o and y_o denote those of DMU_o concerned. We assume X and Y are positive.

Usually, n (the number of DMUs) exceeds $m + s$ (the sum of numbers of inputs and outputs) and hence it is beneficial to solve the dual side of (CCR) which is expressed as:

$$
\begin{array}{lrcl}
(5) & \text{(LP)} \quad \min \ \theta & & \\
(6) & \text{subject to} \quad \theta x_o & = & X\lambda + s_x \\
(7) & y_o & = & Y\lambda - s_y \\
(8) & \lambda \geq 0, \quad s_x & \geq & 0, \quad s_y \geq 0.
\end{array}
$$

The dual formulation has other merits which will be discussed in sequence. The expanded CCR models, e.g. Banker Charnes and Cooper (1984) and the assurance region method (Thompson *et al.* (1986)) among others, can also be solved in the dual form.

As is pointed out in Ali (1989), (1993) and Ali and Seiford (1989), it is not sufficient to use the general purpose LP programs for solving DEA problems. This paper is written in line with their viewpoint and reflects the author's experiences in developing various DEA software (Tone (1993a)).

This paper is divided into five sections. In Section 1, we describe general computational issues of DEA and define the *max-slack solution* and the *efficient DMUs*. A practical procedure for finding a max-slack solution is demonstrated. By the strong theorem of complementary slackness, there exist positive weights v and u for the efficient DMUs in the CCR model. In Section 2, we address this subject and show a method for finding positive v and u. Banker, Charnes and Cooper (1984) developed the BCC model and Banker (1984) discussed the returns to scale of efficient DMUs. In Section 3, a method for deciding the returns to scale is presented. In Section 4, we show a new method for solving the DEA model with categorical inputs and outputs introduced by Banker and Morey (1986), while in Section 5, we deal with a practical method for solving the model with different systems.

1 General Framework of DEA Computation

1.1 Remarks on the DEA Computation

Since the DEA computation is basically achieved by means of solving linear programming, we can make use of several useful techniques developed there (see Murtagh (1981)). However, the coefficient matrix of DEA problems is dense and so we need no sparse matrix considerations. The followings are points to be noted:

1. Double precision arithmetics should be applied,
2. Scaling of matrix is needed, and
3. Typical tolerances are 10^{-5} for reduced cost, 10^{-8} for pivot selection and 10^{-10} for zero criterion.

1.2 Max-Slack Solution

In DEA, the values of slacks s_x and s_y play an essential role in interpreting the solution, in addition to θ. Hence, we rewrite (LP) in a slightly different way as:

$$
\begin{array}{llrcl}
(9) & \text{(LP}') \text{ First objective} & \min\ z & = & \theta \\
(10) & \text{Second objective} & \max\ w & = & e^T s_x + e^T s_y \\
(11) & \text{subject to} & \theta x_o & = & X\lambda + s_x \\
(12) & & y_o & = & Y\lambda - s_y \\
(13) & & L & \le & e^T\lambda \ \le\ U \\
(14) & & \lambda \ge 0,\ s_x & \ge & 0,\ s_y \ge 0,
\end{array}
$$

where e is a vector of ones and L and U are the lower and upper bounds to $e^T\lambda$, respectively. The bounds L and U are introduced with the purpose of extending the CCR model and have to satisfy the relation $0 \le L \le 1 \le U$. (See Adolphson *et al.* (1991).) Since (LP′) has an evident feasible solution, $\theta = 1,\ \lambda_o = 1,\ \lambda_j = 0\ (\forall j \neq o),\ s_x = 0,\ s_y = 0$, no phase 1 process is needed. The initial feasible basis can be easily found by selecting the columns corresponding to λ_o, θ and linearly independent slack variables. Then, first, we solve (LP′) with the first objective function. Let the optimal objective function value be $z = \theta^*$ and then we solve the problem with the second objective function under the additional constraint $\theta = \theta^*$. Let the optimal solution be $(\theta^*, \lambda^*,\ s_x^*,\ s_y^*)$.

Definition 1 (Max-Slack Solution) *The solution* $(\theta^*, \lambda^*,\ s_x^*,\ s_y^*)$ *is called the max-slack solution of* (LP′)[1] .

Definition 2 (Slackless DMU) *If the max-slack solution has* $s_x^* = 0$ *and* $s_y^* = 0$*, then we describe the DMU as slackless.*

[1]We can modify the definition by changing the second objective to $w_x^T s_x + w_y^T s_y$, where w_x and w_y are the weights to the slacks s_x and s_y, respectively. Furthermore, we can define a new measure of efficiency, which takes account of slacks in the inputs and shortage in the outputs. See Tone (1993b) for details.

Definition 3 (Efficient DMU) *We call a DMU as efficient if its max-slack solution has $\theta^* = 1$ and is slackless.*

The above procedure and the definition of efficiency are given in Charnes *et al.* (1992), as well.

1.3 Details of the Max-Slack Solution Procedure

Let us express (LP′) into a more general form as:

$$\text{First objective} \quad \min\ z = c^T x \tag{15}$$

$$\text{Second objective} \quad \min\ w = d^T x \tag{16}$$

$$\text{subject to} \quad Ax = b \tag{17}$$

$$x \geq 0. \tag{18}$$

Let the optimal basis of the first objective function be B and the shadow price vector of the nonbasic variables be q^N. At the next step, we form the *restricted problem* by deleting the nonbasic columns with a negative shadow price ($q_j^N < 0$) from the original problem and proceed to minimize the second objective of the restricted problem, starting from the basis B. This process can be achieved merely by neglecting those columns ($q_j^N < 0$) for candidates of basic variables in the succeeding steps of the simplex method.

Theorem 1 *The optimal second objective function value is equal to that of the original problem.*

(Proof.) First, we show that for any feasible solution of the restricted problem there exists a feasible solution of the original problem with the same first objective function value z^*. At the end of the optimization of the first objective, we have the objective row as:

$$z = z^* + \sum_{j \in N} q_j^N x_j^N, \tag{19}$$

where N is the set of the nonbasic variables at the end of the first stage. Since the restricted problem contains no nonbasic variables with $q_j^N < 0$, any feasible solution of the restricted problem consists of the basic variables and the nonbasic variables with $q_j^N = 0$ at the end of the first stage. Hence, for any feasible solution of the restricted problem, we have $z = z^*$. By adding the exiled variables with the value $x_j^N = 0$ to the restricted problem, we have a feasible solution of the original problem with the same first objective function value.

On the other hand, for any feasible solution x to the second stage of the original problem, the exiled variables x_j^N with $q_j^N < 0$ must be zero, since otherwise we have $z > z^*$ for the x from (19). Thus, for any feasible solution to the second stage of the original problem there corresponds a feasible solution of the restricted problem with the same first and second objective values. The same relation holds between optimal solutions. Hence, we have the theorem. □

2 Positive Weights in the CCR Model

In this Section, we restrict the subject in the CCR model, although it can be naturally extended to other models of (LP′). As is easily seen, the variables (v, u) in the CCR model and the slacks (s_x, s_y) are mutually complementary and at optimality they satisfy the complementary slackness condition:

$$v^T s_x^* = 0 \quad \text{and} \quad u^T s_y^* = 0. \tag{20}$$

By the strong theorem of complementary slackness, a slackless DMU ($s_x^* = 0$ and $s_y^* = 0$) is guaranteed to have positive weights v and u in the corresponding CCR model. However, if we solve the model in the (LP) form and use the multipliers as dual variables v and u, it is not certain that they satisfy the strong complementary relation. The reason is that the multipliers are not unique and we arrived at one of them at the end of the procedure in the previous section. A positive (v, u) for an efficient DMU can be obtained as follows.

If an efficient DMU_o has some zero multipliers, then we formulate the following parametric linear programming problem in a scaler t.

$$\begin{aligned} &(21) & \max \ w &= t\ (e^T s_x + e^T s_y) \\ &(22) & \text{subject to} \ \ x_o &= X\lambda + s_x \\ &(23) & y_o &= Y\lambda - s_y \\ &(24) & \lambda \geq 0, \ s_x &\geq 0, \ s_y \geq 0. \end{aligned}$$

Since we have the optimal solution $w^* = 0$ for $t = 0$ at the end of the second stage, we try to tend t positive. A positive t^* is guaranteed to exist by the previous theorem, which turns out, in the (CCR) side, positive weights such that

$$(25) \qquad v \geq t^* e > 0 \ \text{ and } \ u \geq t^* e > 0.$$

Notice that the positive weights are generally not unique, reflecting the fact that the supporting hyperplane to a polyhedral convex set at a vertex is not unique. We should be careful in discussions using the optimal weights v^* and u^*. The so-called *cross efficiency matrix* (see, for example, Sextone *et al.* (1986), Boussofiane *et al.* (1991)) is misleading as the index for evaluating overall efficiency performance of DMUs.

The discussions in this Section are closely related with the controversial *non-Archimedian infinitesimal*, which should not be used for any computational purposes. (See Ali and Seiford (1989), Färe and Hunsaker (1986), Boyd and Färe (1984) and Charnes and Cooper (1984), among others.)

3 BCC Model and Returns to Scale

In (LP′), the case $L = U = 1$ corresponds to the BCC model (Banker, Charnes and Cooper (1984)), whose dual is:

$$
\begin{aligned}
&(26) \qquad \text{(BCC)} \quad \max\ z &=&\ u^T y_o - u_0 \\
&(27) \qquad \text{subject to} \quad v^T x_o &=&\ 1 \\
&(28) \qquad -v^T X + u^T Y - u_0 e &\leq&\ 0 \\
&(29) \qquad v &\geq&\ 0, \quad u \ \geq\ 0,
\end{aligned}
$$

where $u_0 \in R$ is unconstrained in sign. As is pointed out by Banker (1984) and by Banker and Thrall (1992), the value of u_0 for an efficient DMU relates to its returns to scale. Specifically, they showed that the lower (upper) bound $\underline{u}_0$ ($\bar{u}_0$) can be found by solving:

$$
\begin{aligned}
&(30) \qquad \text{(RS)} \quad \min\ (\max) &=&\ u_0 \\
&(31) \qquad \text{subject to} \quad v^T x_o &=&\ 1 \\
&(32) \qquad -v^T X + u^T Y - u_0 e &\leq&\ 0 \\
&(33) \qquad u^T y_o - u_0 &=&\ 1 \\
&(34) \qquad v &\geq&\ 0, \quad u \ \geq\ 0.
\end{aligned}
$$

If $\bar{u}_0 < 0$ ($\underline{u}_0 > 0$), then DMU_o is under *increasing (decreasing) returns to scale* and if $\bar{u}_0 \geq 0 \geq \underline{u}_0$, then DMU_o is under *constant returns to scale.*

This process can be dualized as follows and be more effectively solved:

$$(35) \qquad (\text{RS1}) \qquad \bar{u_0} = \min\ (\theta - \lambda_0)$$
$$(36) \qquad \text{subject to} \quad \theta x_o \geq X\lambda$$
$$(37) \qquad \lambda_0 y_o \leq Y\lambda$$
$$(38) \qquad -e^T\lambda + \lambda_0 = 1$$
$$(39) \qquad \lambda \geq 0,$$

and

$$(40) \qquad (\text{RS2}) \qquad \underline{u}_0 = \max\ (\lambda_0 - \theta)$$
$$(41) \qquad \text{subject to} \quad \theta x_o \geq X\lambda$$
$$(42) \qquad \lambda_0 y_o \leq Y\lambda$$
$$(43) \qquad e^T\lambda - \lambda_0 = 1$$
$$(44) \qquad \lambda \geq 0,$$

where θ and λ_0 are unconstrained in sign.

4 DEA Model with Categorical Inputs and Outputs

Banker and Morey (1986) developed a model corresponding to the case where at least one input or output variable is *categorical.* For example, all the DMUs are categorized into three classes, 'class I', 'class II' and 'class III', where the 'class I' group is operating under the most difficult environment, the 'class II' under the moderate one, and the 'class III' under the most favorable one. Banker and Morey proposed to evaluate the efficiency of a DMU in 'class I' only within the group and that of 'class II' with reference to DMUs in 'class I' plus 'class II', while a DMU in 'class III' should be evaluated by using the whole DMUs in the model. In order to formulate this situation, Banker and Morey introduced sophisticated binary variables to represent the hierarchy structure.

However, if we design DEA software for this problem, it is easily able to cope with these categorical variables without introducing any binary variables. Here, we consider the case (LP) (the case (LP′) can be dealt with in the same way).

1. Start the simplex method from the initial feasible basis composed of the columns corresponding to the variables λ_o, θ and linearly independent slack variables.

2. Never choose the DMUs in the higher classes than DMU_o as the candidates for basic variables in the succeeding procedures of the simplex method.

The above computational process can be implemented by introducing an index showing the class number for each DMU.

5 DEA with Different Systems

There are situations where the DMUs in the problem are divided into several different systems. For example, a set of DMUs belong to the system A, while the remaining DMUs to the system B. In the production feasibility correspondence, the convex combination of a DMU in the system A with a DMU in the system B has no meaning at all, since there is no system between A and B.

Let the input and output data for DMUs in the systems A and B be $(X_A,\ Y_A)$ and $(X_B,\ Y_B)$, respectively. Then, the production possibility set

$(x,\ y)$ satisfies:

$$x \geq X_A\lambda_A + X_B\lambda_B \tag{45}$$

$$y \leq Y_A\lambda_A + Y_B\lambda_B \tag{46}$$

$$Lz_A \leq e^T\lambda_A \leq Uz_A \tag{47}$$

$$Lz_B \leq e^T\lambda_B \leq Uz_B \tag{48}$$

$$z_A + z_B = 1 \tag{49}$$

$$\lambda_A \geq 0,\ \lambda_B \geq 0 \tag{50}$$

$$z_A,\ z_B = 0 \text{ or } 1. \tag{51}$$

The set can easily expanded to the case where more than two systems exist.

The relative efficiency of DMU_o is represented by the following mixed binary programming:

$$\text{(SP)} \quad \min\theta \tag{52}$$

$$\text{subject to} \quad \theta x_o \geq X_A\lambda_A + X_B\lambda_B \tag{53}$$

$$y_o \leq Y_A\lambda_A + Y_B\lambda_B \tag{54}$$

$$Lz_A \leq e^T\lambda_A \leq Uz_A \tag{55}$$

$$Lz_B \leq e^T\lambda_B \leq Uz_B \tag{56}$$

$$z_A + z_B = 1 \tag{57}$$

$$\lambda_A \geq 0,\ \lambda_B \geq 0 \tag{58}$$

$$z_A,\ z_B = 0 \text{ or } 1. \tag{59}$$

We can solve this problem by enumeration rather than by mixed integer 0-1 programming. This is shown below.

1. Set $z_A = 1$, $z_B = 0$ and solve (SP) above. Let the optimal objective be θ_A which is infinity if (SP) is infeasible.

2. Set $z_A = 0$, $z_B = 1$ and solve (SP) above. Let the optimal objective be θ_B which is infinity if (SP) is infeasible.

3. We have the optimal solution of the mixed integer problem (SP) by

$$\theta^* = \min\{\theta_A,\ \theta_B\}. \tag{60}$$

The results can be used to compare overall performance of the systems A and B or to estimate the territory of preference one system over the other in the production feasibility set. Banker and Morey (1986) (see also Kamakura (1988)) discussed the treatment of controllable categorical variables which is similar to ours but quite different from ours in that they allow an undefined system between two or more systems.

6 Concluding Remarks

In this paper, we have discussed the computational issues related to the general CCR models, the returns to scale, categorical variables and different systems, based on the author's experiences in developing DEA software. In addition, we warned against erroneous usage of the optimal weight $(v,\ u)$ obtained.

Although we have concentrated on the input-oriented DEA, the output-oriented DEA can be dealt with similarly.

There are other directions of research on this subject, e.g. reduction of computation time. It is to be noted that Ali (1993) has succeeded in realizing this objective significantly for CCR, BCC and Additive models and Sueyoshi (1992) for the assurance region model.

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