A Note on Group vs. Individual Decision Making in the Analytic Hierarchy Process

Kaoru TONE*

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Abstract

When we estimate the relative distance between cities or the relative area of figures by the Analytic Hierarchy Process (AHP), it is often observed that a group decision usually outperforms an individual one. This paper addresses this phenomena and shows that the accuracy of estimates is improved in approximate proportion to the square root of the number of individuals in the group.

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1 Introduction

Saaty's AHP [2] is now being widely used for decision making purposes. One of the important factors in AHP is the pairwise comparison of alternatives in the problem. There are two kinds of pairwise comparison, i.e., by an individual and by a group. In classroom experiments for measuring the relative distance between cities on a map or the relative area of figures, the author has often experienced the result that a group decision outperforms an individual one in accuracy. This paper tries to clarify the situation and

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shows that the accuracy of estimates is improved in approximate proportion to the square root of the number of individuals in the group, if the members are unbiased and homogeneous.

2 Eigenvalue Method and Geometric Mean Method

Let the pairwise comparison matrix be

$$A = [a_{ij}], \tag{1}$$

where $a_{ii} = 1 \ \ (i = 1, \ldots, n), \ \ a_{ij} = 1/a_{ji} \ \ (\forall \ \ (i,j)),$ and $\ a_{ij} > 0 \ \ (\forall \ \ (i,j))$. There are two methods for estimating the relative weight of the alternatives.

1. Eigenvalue Method:
   This method solves the principal eigenvalue of A and its eigenvector. Let the eigenvalue and the eigenvector be λ_{max} and $\boldsymbol{v}$, respectively. We assume the eigenvector is normalized so that the sum of the elements of $\boldsymbol{v}$ is 1.

2. Geometric Mean Method:
   Geometric mean procedure works as follows: Let $\boldsymbol{g}$ be the vector composed of the geometric mean of rows of A,i.e.,

   $$g_i = \sqrt[n]{a_{i1}a_{i2}\ldots a_{in}} \quad i = 1, 2, \ldots, n. \tag{2}$$

   The vector $\boldsymbol{g}$ is normalized as

   $$g_i' \leftarrow g_i / \sum_{j=1}^{n} g_j \quad i = 1, 2, \ldots, n. \tag{3}$$

The two approaches give almost the same weights v and g', if the matrix A is nearly consistent. (See Golden and Wang [1] and Tone [5]. Also, see Takeda [4] for further extensions of the geometric mean method.) So, hereafter, we will deal with the geometric mean method (GM), since GM is more appropriate for analyzing the above mentioned subjects.

3 Perturbation of Pairwise Comparison Matrix

We assume that the true weight vector $w = (w_i)$ exists. The (i, j) element of the ideal comparison matrix is expressed as

$$\frac{w_i}{w_j}. \tag{4}$$

The estimated comparison value a_{ij} is an approximation to w_i/w_j and let relate with it by

$$a_{ij} = \frac{w_i}{w_j} e^{\varepsilon_{ij}}, \tag{5}$$

where ε_{ij} is a random variable representing the deviation from the true value. We assume that ε_{ij} has the mean zero and the variance σ_{ij}^2.

The above setting matches with the exponential scoring of pairwise comparisons. If ε_{ij} is small, then we have

$$e^{\varepsilon_{ij}} = 1 + \varepsilon_{ij} + O(\varepsilon_{ij}^2). \tag{6}$$

Therefore, (5) can be written as

$$a_{ij} = \frac{w_i}{w_j} \left(1 + \varepsilon_{ij} + O(\varepsilon_{ij}^2)\right). \tag{7}$$

Thus, ε_{ij} can be interpreted as a relative error to w_i/w_j.

4 Effect of Perturbation on Weight

If we calculate the weight by GM, using the perturbed matrix (5), we have

$$g_i = \frac{w_i}{\sqrt[n]{\prod_{j=1}^n w_j}} e^{\left(\sum_{j=1}^n \varepsilon_{ij}\right)/n}, \quad (i = 1, \ldots, n) \tag{8}$$

where $\varepsilon_{ii} = 0 \ (\forall i)$. By normalizing g, we have the estimated weight

$$g'_i = \frac{w_i e^{\left(\sum_{j=1}^n \varepsilon_{ij}\right)/n}}{\sum_{j=1}^n w_j e^{\left(\sum_{k=1}^n \varepsilon_{jk}\right)/n}}. \quad (i = 1, \ldots, n) \tag{9}$$

Under the small ε_{ij} hypothesis, g'_i can be approximated by

$$\begin{aligned} g'_i &= w_i \Bigg[1 - \frac{1}{n} \Big\{ \sum_{k=1}^{i-1} (w_k - w_i + 1)\, \varepsilon_{ki} + \sum_{k=i+1}^{n} (w_i - w_k - 1)\, \varepsilon_{ik} \\ &\quad + \sum_{j<k,(j,k\neq i)} (w_j - w_k)\, \varepsilon_{jk} \Big\} + \frac{1}{n^2} O\left(\varepsilon_{jk}^2\right) \Bigg]. \end{aligned} \tag{10}$$

Let us observe the first order term in ε in (10), which can be regarded as the relative error of the estimated g'_i from w_i under the small ε hypothesis:

$$\begin{aligned} \delta_i &= -\frac{1}{n} \Bigg[\sum_{k=1}^{i-1} (w_k - w_i + 1)\, \varepsilon_{ki} + \sum_{k=i+1}^{n} (w_i - w_k - 1)\, \varepsilon_{ik} \\ &\quad + \sum_{j<k,(j,k\neq i)} (w_j - w_k)\, \varepsilon_{jk} \Bigg]. \quad (i = 1, \ldots, n) \end{aligned} \tag{11}$$

If we assume that ε_{jk} s distribute independently with the mean 0 and the variance σ^2, then δ_i is a random variable with the mean 0 and the variance V_i as

$$V_i = \frac{\sigma^2}{n} \left(\sum_{j=1}^n w_j^2 - 2w_i + 1 \right). \tag{12}$$

(See Appendix for derivation). Since,

$$\sum_{j=1}^n w_j^2 - 2w_i + 1 = \sum_{j=1, j\neq i}^n w_j^2 + (1 - w_i)^2 \leq 2, \tag{13}$$

we have:

Proposition 1 *The estimated g_i' has a relative error approximately proportional to $\sigma/\sqrt{n}$.*

5 Effect of Group Decision on Weight

We observe the case where m individuals do the pairwise comparisons independently and make the matrix A by their geometric mean. Thus, we have

$$(14) \qquad a_{ij} = \frac{w_i}{w_j} e^{\left(\sum_{k=1}^{m} \varepsilon_{ijk}\right)/m},$$

where ε_{ijk} is a random variable corresponding to the error term of the k-th individual. The group decision weight can be determined by the row-wise geometric mean of A:

$$(15) \qquad \bar{g}_i = \frac{w_i}{\sqrt[n]{\prod_{j=1}^{n} w_j}} e^{\left(\sum_{j=1}^{n} \sum_{k=1}^{m} \varepsilon_{ijk}\right)/nm}. \quad (i = 1, \ldots, n : \varepsilon_{iik} = 0)$$

In the same way as (10), we can approximate $\bar{g}_i$ by

$$(16) \qquad \begin{aligned} \bar{g}_i &= w_i \Bigg[1 - \frac{1}{nm} \sum_{k=1}^{m} \Big\{ \sum_{h=1}^{i-1} (w_h - w_i + 1)\, \varepsilon_{hik} \\ &\quad + \sum_{h=i+1}^{n} (w_i - w_h - 1)\, \varepsilon_{ihk} + \sum_{j<h,(j,h \neq i)} (w_j - w_h)\, \varepsilon_{jhk} \Big\} \\ &\quad + \frac{1}{(nm)^2} O\left(\varepsilon_{jhk}^2\right) \Bigg]. \end{aligned}$$

Here again, we assume that $\varepsilon_{jhk}(\forall (jhk))$ subjects to a distribution with the mean 0 and the variance σ^2,i.e., unbiased and homogeneous. Under the small ε hypothesis, the first order terms of ε in (16) correspond to the relative error of $\bar{g}_i$ to w_i, whose mean is 0 and variance is:

$$\bar{V}_i = \frac{\sigma^2}{nm}\left(\sum_{j=1}^{n} w_j^2 - 2w_i + 1\right). \tag{17}$$

By comparing (17) with the individual case (12) discussed in the preceding section, we have:

Proposition 2 *The group decision by m individuals reduces the error of the estimated weight by the factor $1/\sqrt{m}$, if the members of the group are unbiased and homogeneous.*

6 Concluding Remarks

This paper discussed the relative error of judgements by the geometric mean method in terms of the relative error in the pairwise comparisons and evaluated those of individual and group decisions. As a consequence, we showed that the group decision improves the accuracy of estimated weight in proportion to the square root of the number of individuals in the group, if the members are 'unbiased and homogeneous'. On the 'unbiased' issue, the approximate *Consistency Index* (C.I.) below can be usefully applied.

$$\text{C.I.} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(g'_j/g'_i) - n^2}{n(n-1)}. \tag{18}$$

If a member's C.I. (or the corresponding C.R.) is greater than 0.1, his comparison matrix must be retried or deleted from the group decision. As to the 'homogeneity' issue, Saaty [3] will contribute to a better understanding of the matter.

References

[1] Golden, B.L. and Q. Wang,"An Alternate Measure of Consistency", in B.L.Golden, E.A.Wasil and P.T.Harker (Eds), *The Analytic Hierarchy Process:Application and Studies*, pp.68-81, Springer, (1989).

[2] Saaty, T.L.,*The Analytic Hierarchy Process*, McGraw-Hill, (1980).

[3] Saaty, T.L., "Group Decision Making and the AHP",in B.L.Golden, E.A.Wasil and P.T.Harker (Eds), *The Analytic Hierarchy Process:Application and Studies*, pp.59-67, Springer, (1989).

[4] Takeda, E., "A Note on Consistent Adjustments of Pairwise Comparison Judgments ", *Mathematical and Computer Modelling*, **17**, 7, 29-35 (1993).

[5] Tone,K.,"A Practical Procedure for Finding Copy-like Alternatives in AHP", (in Japanese),*Communications of the Operations Research Society of Japan*, **36**, 4, 190-194, 1991.

Appendix: Derivation of (12)

$$
\begin{aligned}
V_i &= \frac{\sigma^2}{n^2}\left[\sum_{k=1}^{i-1}(w_k - w_i + 1)^2 + \sum_{k=i+1}^{n}(w_i - w_k - 1)^2 + \sum_{j<k,(j,k\neq i)}(w_j - w_k)^2\right] \\
&= \frac{\sigma^2}{n^2}\left[\sum_{k=1(k\neq i)}^{n}(w_i - w_k - 1)^2 + \sum_{j<k,(j,k\neq i)}(w_j - w_k)^2\right] \\
&= \frac{\sigma^2}{n^2}\left[\sum_{j<k}(w_j - w_k)^2 + (n-1) - 2(n-1)w_i + 2\sum_{k=1,\neq i}^{n} w_k\right] \\
&= \frac{\sigma^2}{n^2}\left[(n-1)\sum_{j=1}^{n} w_j^2 - 2\sum_{j<k} w_j w_k - 2nw_i + n + 1\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sigma^2}{n^2}\left[n\sum_{j=1}^{n} w_j^2 - \left(\sum_{j=1}^{n} w_j\right)^2 - 2nw_i + n + 1\right] \\
&= \frac{\sigma^2}{n}\left[\sum_{j=1}^{n} w_j^2 - 2w_i + 1\right].\square
\end{aligned}
$$