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A Logarithmic Least Squares Method for Incomplete Pairwise Comparisons in the Analytic Hierarchy Process

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Abstract

This paper describes a logarithmic least squares method or a geometric mean method for estimating the relative weight of alternatives when some entries of the pairwise comparisons matrix are missing.

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1 Introduction

Saaty's Analytic Hierarchy Process (AHP) [3] is now being widely used for decision making purposes. At each level in the hierarchy, the AHP uses a pairwise comparisons matrix $A = (a_{ij}) \in R^{n \times n}$ representing the relative importance of alternatives i against j, with the properties:

(1) $$a_{ii} = 1 \ (i = 1, \ldots, n), \ a_{ij} = 1/a_{ji} \ (\forall(i,j)), \ a_{ij} > 0 \ (\forall(i,j)).$$

There are two frequently used methods for estimating the relative weight of n alternatives from the matrix A. One is the eigenvalue method (EM) which

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solves the principal eigenvector $v = (v_i) \in R^n$ in $Av = \lambda_{max} v$ with λ_{max} the principal eigenvalue of A. The other is the geometric mean method (GM) or the logarithmic least squares method (LLSM) which minimizes, with respect to $g = (g_i) \in R^n$,

$$\sum_{i,j=1}^{n} \left(\log a_{ij} - \log g_i/g_j\right)^2 . \tag{2}$$

It turns out that the GM (LLSM) solution g is given by the normalized geometric mean of elements in each row:

$$g_i = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj}}}. \quad (i = 1, \ldots, n) \tag{3}$$

The two methods are not the same and the solutions v and g are different generally. However, it was showed that the two approaches give almost the same weight v and g, if the matrix A is nearly consistent. (See Golden and Wang [1] and Tone [5]. See Saaty [4], too, for further discussions.)

One major drawback of the AHP is that at each level in the hierarchy, $n(n-1)/2$ questions must be answered. The number of questions grows very large with n. In addition, for certain pairs (i, j), it is very difficult to answer the question "compare i against j". This results in some entries of A being missing. Therefore, methods for estimating the weight of alternatives from the incomplete matrix are requested. Harker [2] solved this problem effectively in the framework of the eigenvalue method. The main purpose of this paper addresses the solution to the incomplete matrix problem by the logarithmic least squares principle.

2 Logarithmic Least Squares for Incomplete Pairwise Comparisons Matrix

We can define an undirected graph corresponding to the paired comparisons with the vertices 1, 2, ..., n and with arcs $(i,\ j)$ if i and j are compared directly.

Definition 1 *We call a pairwise comparisons matrix* incomplete, *if*

1. *the corresponding graph is connected and*

2. *is not a perfect graph.*

Let an incomplete matrix be $A = (a_{ij})$. For each vertex i, we define P_i as the set of vertices adjacent to i and N_i as the degree of i , i.e., the number of arcs connected to i. Since the graph is connected, for each i, P_i is nonempty and $N_i \geq 1$. For the missing matrix entries a_{ij}, let us approximate their value by the ratio of the (yet unknown) weights g_i/g_j. For the purpose of obtaining the weight g, we solve the following logarithmic least squares problem:

$$\text{(4)} \qquad \textit{minimize} \qquad \sum_{i,j} (\log a_{ij} - \log g_i + \log g_j)^2$$

$$\text{(5)} \qquad = \sum_{i=1}^{n} \left[\sum_{j \in P_i} (\log a_{ij} - \log g_i + \log g_j)^2 \right]$$

The problem results in a set of linear equations in $(\log g_j)$ as

$$\text{(6)} \qquad N_i \log g_i - \sum_{j \in P_i} \log g_j = \sum_{j \in P_i} \log a_{ij}. \quad (i = 1, 2, \ldots, n)$$

Example 1

The matrix below has entries $(1,3)$, $(2,4)$ and $(3,4)$ missing.

$$A = \begin{pmatrix} 1 & a_{12} & g_1/g_3 & a_{14} \\ a_{21} & 1 & a_{23} & g_2/g_4 \\ g_3/g_1 & a_{32} & 1 & g_3/g_4 \\ a_{41} & g_4/g_2 & g_4/g_3 & 1 \end{pmatrix}.$$

The corresponding linear equations are

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \log g_1 \\ \log g_2 \\ \log g_3 \\ \log g_4 \end{pmatrix} = \begin{pmatrix} \log a_{12}a_{14} \\ \log a_{21}a_{23} \\ \log a_{32} \\ \log a_{41} \end{pmatrix}.$$

The rule for constructing the coefficient matrix of the linear equations is:
(1) Put -1 on the compared entries and 0 on the missing ones, and
(2) on the diagonal entries, put the number of -1s on the row.

Let the coefficient matrix be D. Then, we have

Theorem 1 *The rank of the matrix D is $n-1$, if and only if the graph of the pairwise comparisons is connected.*

Proof. First, we show the 'if-part'. Since the sum of n row vectors of D is zero, the rank of D is less than $n-1$. Let D_{n-1} be the left upper $(n-1)\times(n-1)$ matrix of D. For a vector $x = (x_j) \in R^{n-1}$, the quadratic form associated with D_{n-1} is:

$$(7) \qquad Q = x^T D_{n-1} x = \sum_{i=1}^{n-1} N_i x_i^2 + 2 \sum_{1\le i<j\le n-1} d_{ij}x_i x_j$$

$$(8) \qquad = \sum_{1\le i<j\le n-1,\ d_{ij}=-1} (x_i - x_j)^2 + \sum_{i=1}^{n-1}\left(N_i + \sum_{j=1,\ne i}^{n-1} d_{ij}\right) x_i^2$$

We observe the case $Q=0$.

(i) If the first term $(x_i - x_j)^2$ on the right-hand side of (8) is not vacant, then, for each i, we have, under the condition $Q = 0$,

$$x_i = x_j. \quad (\forall j \in P_i) \tag{9}$$

Furthermore, at least one of x_i and (x_j) $(j \in P_i)$ has the term x_i^2 or x_j^2 in the second term on the right-hand side of (8), since otherwise the vertices x_i and (x_j) $(j \in P_i)$ are disconnected to the remaining ones and this contradicts the connected graph hypothesis. Thus, we have, for each i in the first term,

$$x_i = x_j \quad (\forall j \in P_i) = 0. \tag{10}$$

(*ii*) For x_k not included in the first term, we have x_k^2 in the second term. Hence $x_k = 0$.

Thus, if $Q = 0$, then $\boldsymbol{x} = \boldsymbol{0}$. Therefore, all the eigenvalue of D_{n-1} is positive and the rank of D_{n-1} is $n - 1$.

The 'only-if' part can be demonstrated as follows. Suppose the graph is disconnected. Then, the matrix D can be decomposed, after rearrangement, into

$$D = \begin{pmatrix} D_1 & O \\ O & D_2 \end{pmatrix}, \tag{11}$$

where $D_1 \in R^{n_1 \times n_1}$, $D_2^{(n-n_1)\times(n-n_1)}$ and $n_1 > 0$. The ranks of D_1 and D_2 are less than or equal to $n_1 - 1$ and $n - n_1 - 1$, respectively. Hence, the rank of D must be less than or equal to $n - 2$, since the rank is the maximum number of linearly independent columns (or rows) of the matrix. □

3 A Geometric Mean Method for Incomplete Pairwise Comparisons

Based on the preceding theorem, a geometric mean method for incomplete pairwise comparisons goes as follows:

1. Let any one of $(\log g_j)$ $(j = 1, \ldots, n)$ be zero and solve the equations (6) in remaining $(n-1)$ unknowns.

2. Obtain the weight g_j from $\log g_j$ for $(j = 1, \ldots, n)$.

3. Normalize (g_j) so that

$$g'_j = \frac{g_j}{\sum_{k=1}^{n} g_k}. \quad (j = 1, \ldots, n) \tag{12}$$

Example 2

Let an incomplete pairwise comparisons matrix A be as below, where the symbol $-$ stands for uncompared entries:

$$A = \begin{pmatrix} 1 & - & 3 & 2 \\ - & 1 & 9 & 6 \\ 1/3 & 1/9 & 1 & - \\ 1/2 & 1/6 & - & 1 \end{pmatrix}.$$

The corresponding linear equations are

$$\begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} \log g_1 \\ \log g_2 \\ \log g_3 \\ \log g_4 \end{pmatrix} = \begin{pmatrix} \log 3 \times 2 \\ \log 9 \times 6 \\ \log(1/3) \times (1/9) \\ \log(1/2) \times (1/6) \end{pmatrix}.$$

We assume $\log g_4 = 0$ and solve the system for $\log g_j$ $(j = 1, 2, 3)$ which gives

$$\log g_1 = \log 2, \; \log g_2 = \log 6, \; \log g_3 = \log(2/3), \; \log g_4 = 0.$$

Thus, we obtain the normalized weight

$$g' = (0.207, \; 0.621, \; 0.069, \; 0.103). \tag{13}$$

4 Concluding Remarks

This paper dealt with the incomplete pairwise comparisons in the AHP within the framework of the logarithmic least squares method. It is easy to see that the weight thus obtained has perfect consistency, if the estimates in the compared entries are consistent. A measure of consistency can be defined by

$$G = \frac{\sum_{i=1}^{n}\left(\sum_{j\in P_i} a_{ij}g_j/g_i - N_i\right)}{\sum_{i=1}^{n} N_i}, \tag{14}$$

which is an average deviation of the compared estimate a_{ij} from g_i/g_j. Obviously, G is nonnegative and equal to zero if and only if the estimate a_{ij} satisfies

$$a_{ij} = \frac{g_i}{g_j} \quad (\forall(i,j)). \tag{15}$$

However, it should be noted that, if, in the most incomplete case, the graph is a spanning tree, the calculated weight (g_i) is always consistent and hence $G = 0$. This observation suggests the need for other indices of accuracy of measurement for incomplete comparisons. This is a future research subject.

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