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Several Algorithms to Determine Multipliers for Use in Cone-Ratio Envelopment Approaches to Efficiency Evaluations in DEA

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Abstract

In this paper, we will discuss subjects related to virtual multipliers in the cone-ratio model in DEA. Usually, there exists ambiguity in the virtual multipliers in the polyhedral cone-ratio method when some exemplary efficient DMUs' multipliers are employed as the admissible directions of the cone. Firstly, we will show a cell subdivision of the multiplier simplex. Then, three practical methods for resolving this ambiguity will be presented with an example. Finally, we will discuss possible applications of vertex enumeration software, based on the Double Description Method.

1 Introduction

Data Envelopment Analysis (DEA) introduced by Charnes, Cooper and Rhodes (1978) has been widely applied in evaluating the relative efficiency of

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decision making units (DMUs) with multiple inputs and outputs. Relative efficiency is measured by a ratio scale of the virtual input *vs.* the virtual output, which are the weighted sums of inputs and outputs, respectively. The weights to inputs and outputs are usually nonnegative and are decided according to which are most preferable to the DMU concerned, via a linear programming solution. Since the original Charnes, Cooper and Rhodes (CCR) model, many studies have been developed to cope with the real life problems. One such study researched the feasible region of the weights and has actually imposed some additional constraints to the weights. Such studies resulted in the Assurance Region (AR) model and the Cone-Ratio (CR) model. The assurance region method developed by Thompson, Singleton, Thrall and Smith (1986) confines the feasible region of the weights by imposing lower and upper bounds to the ratio of some selected pairs of weights. (See also Thompson, Langemeir, Lee, Lee and Thrall (1990), Roll and Golany (1993) and Thompson, Dharmapala, Rothenberg and Thrall (1994)). On the other hand, the cone-ratio model by Sun (1987), Charnes, Cooper, Wei and Huang (1989) and Charnes, Cooper, Huang and Sun (1990) solves the CCR model first and chooses a few exemplary efficient DMUs from among all the efficient ones by consulting with experts on the problem. Then, the corresponding optimal weights to the selected efficient DMUs are used to construct a convex cone as the feasible region of the weights. However, usually the optimal weights are not uniquely determined and hence there is ambiguity in selecting the weights to form the cone.

In an effort to overcome this problem, this paper will propose three practical methods for deciding the convex cone in accordance with three principles which will be explained later. In Section 2, we will discuss the subdivision of the multiplier (weight) simplex. Based on the subdivision, the AR and CR models will be conceptually compared in Section 3, with emphasis on the ambiguity of weights in the CR model. Then, in Section 4, we will propose three practical methods to resolve the ambiguity based on three principles, i.e. the most restricted, the most relaxed and the central. We will give an example of the proposed method in Section 5. Finally, in Section 6, the possible applications of Fukuda's vertex enumeration software (1993), which is based on the Double Description Method of Motzkin, Raiffa, Thompson and Thrall (1958), will be discussed. It will be seen that this software will open a rich field of research and applications in DEA.

2 Cell Subdivision of the Multiplier Simplex

Suppose there are n DMUs with m inputs and s outputs. The i-th input and the r-th output of the j-th DMU are denoted by x_{ij} and y_{rj}, respectively. Let the input and output matrices X and Y be

(1) $$X = (x_{ij}) \in R^{m \times n} \text{ and } Y = (y_{rj}) \in R^{s \times n}.$$

We assume $X > O$ and $Y > O$. The virtual input and output for DMU_j are defined by

(2) $$V_j = \sum_{i=1}^{m} v_i x_{ij} \quad (j = 1, \ldots, n)$$

and

(3) $$U_j = \sum_{r=1}^{s} u_r y_{rj}, \quad (j = 1, \ldots, n)$$

where (v_i) $((u_r))$ is the the *weight* (or *multiplier*) to the input (output) i (r). Again, we assume $v_i > 0$ $(\forall i)$ and $u_r > 0$ $(\forall r)$[1].

[1]For computational purposes, we can relax these positive constraints to nonnegativity. See Charnes, Cooper and Thrall (1991) and Tone (1993)

Now, we observe the ratio of the virtual input *vs.* output:

$$R_j = \frac{U_j}{V_j} = \frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}}. \quad (j = 1, \dots, n) \tag{4}$$

Since the ratio R_j is invariant under any multiplication by a positive scalar t to (v, u), we impose hereafter the simplex constraint to (v, u) as follows:

$$\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1. \tag{5}$$

By this constraint, together with the positiveness of multipliers, the feasible (v, u) forms the interior of the $(m + s - 1)$ dimensional simplex denoted by S. Under the above assumptions, for each $(\bar{v}, \bar{u}) \in S$, there exists at least one DMU_{j_0} that maximizes the ratio R_j $(j = 1, \dots, n)$ defined by (4). We call DMU_{j_0} *dominates* $(\bar{v}, \bar{u})$. It can be demonstrated that the $(m+s-1)$ dimensional simplex S is divided into a finite number of $(m+s-1)$ dimensional cells dominated by some DMUs. The cells are not necessarily convex. There may exist $(m+s-2)$ or less dimensional dominant DMUs, with the extremal case 0 dimensional (point) dominant DMUs. Figure 1 shows an example of the cell subdivision of the multiplier simplex S, where, for example, DMU_1, DMU_2 and DMU_3 are $(m+s-1)$ dimensional dominants, while DMU_{10} is a 0 dimensional (point) dominant. We emphasize that this figure is for explanatory purposes and not rigorous.

Figure 1

3 Assurance Region and Cone-Ratio Models

In applying DEA to actual problems, we should be conscious of the economic and socioeconomic aspects of the problems, which are closely related with the virtual multiplier (weight) v (u) to the input (output) items. Although the original DEA models impose no restriction on v and u except positivity (or nonnegativity), we can introduce the relative importance of weights by restricting their feasible regions. In this regard, two notable models have been proposed, i.e. the assurance region (AR) and the cone-ratio (CR) models.

3.1 Assurance Region Model

The AR model imposes lower and upper bounds to the ratio of some selected pairs of weights. For example, we may add a constraint on the ratio of weights to Input 1 and Input 2 as follows:

$$l_{12} \leq \frac{v_2}{v_1} \leq u_{12}, \tag{6}$$

where l_{12} and u_{12} are the lower and upper bounds to the ratio, respectively. Likewise, similar constraints may be added to pairs of some output multipliers and even to multipliers between some selected input and output multipliers (Thompson et al. (1990) named this type 'linkage constraints'). See Thompson et al. (1986), (1990), (1994) and Roll and Golany (1993) for more details and further extensions. The constraint such as (6) restricts the feasible region of multipliers to the polyhedral cone originated from a vertex of the simplex S. Thus, the assurance region, which satisfies all the ratio constraints will come to form a polyhedron as in $ABCD$ in Figure 2, for example.

Figure 2

3.2 The Intersection Form Cone-Ratio Model

On the other hand, in the cone-ratio model, especially in the polyhedral cone-ratio model, some exemplary DMUs will be chosen from among the CCR efficient DMUs as a result of expert knowledge. Then, the optimal weights corresponding to the selected DMUs will be used to form a polyhedral cone for an admissible region of multipliers. We will now briefly describe the polyhedral cone-ratio model. (See Charnes et al. (1990) for details). Let us assume that the feasible input weight v is in the polyhedral convex cone spanned by the k admissible nonnegative direction (row) vectors $a_j \in R^m \ (j = 1, \ldots, k)$. Thus, a feasible v can be expressed as

$$\begin{aligned} v &= \sum_{j=1}^{k} \alpha_j a_j \quad \text{with} \quad \alpha_j \geq 0 \ (\forall j) \\ &= \alpha A, \end{aligned}$$

where $A^T = (a_1^T, \ldots, a_k^T) \in R^{m \times k}$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$.

Let the convex cone thus defined be

$$V = \{v | v = \alpha A, \ \alpha \geq 0\}. \tag{7}$$

Likewise, based on the l admissible nonnegative direction (row) vector $b_j \in R^s \ (j = 1, \ldots, l)$, we assume that the feasible output weight u is in the polyhedral convex cone defined by

$$U = \{u | u = \beta B, \ \beta \geq 0\}, \tag{8}$$

where $B^T = (b_1^T, \ldots, b_l^T) \in R^{s \times l}$ and $\beta = (\beta_1, \ldots, \beta_l)$.

Then, the cone-ratio model for evaluating the relative efficiency of the n DMUs coincides with the CCR model that evaluates the same DMUs with the following transformed data set $(\bar{X}, \bar{Y})$.

$$\bar{X} = AX \in R^{k \times n} \quad \text{and} \quad \bar{Y} = BY \in R^{l \times n}. \tag{9}$$

This corresponds to the CR model in 'intersection form' as the direct product of V and U, i.e. the components v and u range independently over the input and output cones. Thus, the intersection form CR model evaluates the relative efficiency of DMU_o by the following primal and dual linear programs:

$$
\begin{array}{llrcl}
(10) & (CRP_o) & \max\ u\bar{y}_o & & \\
(11) & & \text{subject to}\quad v\bar{x}_o & = & 1 \\
(12) & & -v\bar{X} + u\bar{Y} & \leq & 0 \\
(13) & & v \geq 0,\quad u & \geq & 0.
\end{array}
$$

$$
\begin{array}{llrcl}
(14) & (CRD_o) & \min\ \theta & & \\
(15) & \text{subject to} & \theta\bar{x}_o - \bar{X}\lambda - s_x & = & 0 \\
(16) & & \bar{y}_o - \bar{Y}\lambda + s_y & = & 0 \\
(17) & & \lambda & \geq & 0,\quad s_x \geq 0,\quad s_y \geq 0.
\end{array}
$$

3.3 The Linked-Cone Model

As with the linkage constraints in the AR model, we can define a linked CR model as follows. Let the linked multipliers (v, u) be in a cone described by

$$(v, u) = \gamma C, \tag{18}$$

where $\gamma \in R^q$, $C \in R^{q \times (m+s)}$ and the row vector $c_j \in R^{m+s}$ $(j = 1, \ldots, q)$ corresponds to a linked admissible direction. From (18), the multipliers can be expressed as

$$v = \gamma C_1 \ \text{ and } \ u = \gamma C_2 \ \text{ with } \ C = (C_1 C_2). \tag{19}$$

Let us define

$$\bar{X} = C_1 X \in R^{q \times n} \ \text{ and } \ \bar{Y} = C_2 Y \in R^{q \times n}. \tag{20}$$

Based on the data set $(\bar{X}, \bar{Y})$, we have a linked CR model as follows:

$$
\begin{array}{llrcl}
(21) & (LCRP_o) & \max \gamma \bar{y}_o & & \\
(22) & & \text{subject to} \quad \gamma \bar{x}_o & = & 1 \\
(23) & & \gamma(-\bar{X} + \bar{Y}) & \leq & 0 \\
(24) & & \gamma & \geq & 0.
\end{array}
$$

$$
\begin{array}{llrcl}
(25) & & (LCRD_o) \quad \min \ \theta & & \\
(26) & \text{subject to} & \theta \bar{x}_o + (-\bar{X} + \bar{Y})\lambda - s & = & \bar{y}_o \\
(27) & & \lambda & \geq & 0, \quad s \geq 0.
\end{array}
$$

The programs have a finite optimum if the cone generated by C includes at least one feasible weight (v, u) for the original CCR model. Otherwise, $(LCRP_o)$ has no feasible solution and hence we would be obliged to change the problem to a 'linked-cone profit model' by neglecting the constraint $\gamma(-\bar{X} + \bar{Y}) \leq 0$ in the primal and the λ- term in the dual, in a similar way to Thompson et al. (1994) in the extended AR model. However, the latter extension is applicable to the CR model, regardless of the feasibility of $(LCRP_o)$.

3.4 Three Principles for Choosing Cones

It is not easy to choose the admissible directions, since usually the optimal virtual weights for an efficient DMU are not uniquely determined. For example, let us observe the case when the experts chose DMUs 1, 2 and 3 in Figure 1 as exemplary. Obviously, any point in the cell 1, 2 or 3 makes the DMU efficient. Therefore, we need some other criteria for selecting a reasonable point in the cell. There may be at least three principles for this purpose. The first one, the most restricted case, is to choose the cone as the minimum diameter convex set which makes the exemplary DMUs efficient. The next one, the most relaxed case, is to choose the cone as the convex hull of the exemplary cells. The last one chooses the cone generated by the 'central points' of each exemplary DMUs.

However, from computational points of view, the above three principles cannot be easily implemented. In fact, the first two might be NP-hard problems and the last one depends on the method of choosing the central point for each cell.

4 Practical Methods for the Three Cases

Corresponding to the above mentioned general principles, we will propose three practical methods which approximately implement them.

4.1 The Most Restricted Case

Let the chosen exemplary DMUs be $DMU_{\alpha_1}, \ldots, DMU_{\alpha_p}$. We solve the following fractional program (FP_k) for each DMU_{α_k} $(k = 1, \ldots, p)$.

$$
\begin{aligned}
&(28) \qquad (\mathrm{FP}_k) \quad \max \quad \frac{\sum_{r=1}^{s} u_r \sum_{j \neq k} y_{r\alpha_j}}{\sum_{i=1}^{m} v_i \sum_{j \neq k} x_{i\alpha_j}} \\
&(29) \qquad \text{subject to} \quad \sum_{i=1}^{m} v_i x_{i\alpha_k} = \sum_{r=1}^{s} u_r y_{r\alpha_k} = 1 \\
&(30) \qquad \sum_{i=1}^{m} v_i x_{ij} \geq \sum_{r=1}^{s} u_r y_{rj} \quad (j = 1, \ldots, n) \\
&(31) \qquad v_i \geq 0 \ (\forall i) \qquad u_r \geq 0 \ (\forall r).
\end{aligned}
$$

The objective of this fractional program is to find a point (v^*, u^*) among the optimal points for DMU_{α_k} that maximizes the ratio scale of the DMU formed by aggregating other exemplary DMUs.

The optimal objective value in $(\mathrm{FP})_k$ is not greater than 1 and attains 1 if, and only if, the intersection of the cells corresponding to $DMU_{\alpha_1}, \ldots, DMU_{\alpha_p}$ is not empty. In this case, it is sufficient to solve only (FP_1) and the vertex found will be used as the cone, (actually the ray) for the cone-ratio model. Otherwise, we will solve (FP_k) for $k = 1, \ldots, p$ and find optimal vertices $(v_1^*, u_1^*), \ldots, (v_p^*, u_p^*)$ which will be used to form the cone. The objective term in $(\mathrm{FP})_k$ corresponds to the aggregated DMU with $(\sum_{j\neq k} x_{i\alpha_j}, \sum_{j\neq k} y_{r\alpha_j})$ and hence the scaling of each component $(x_{\alpha_j}, y_{\alpha_j})$ deserves consideration. In any case, there remains ambiguity in the optimal solution if it is degenerate.

The fractional program (FP_k) can be solved as a linear programming problem via the Charnes and Cooper transformation (1962), which is briefly described in Appendix A.

4.2 The Most Relaxed Case

Instead of maximizing the objective function in (FP_k), we try to minimize it, subject to the same constraints. Thus the objective is:

$$\text{minimize} \quad \frac{\sum_{r=1}^{s} u_r \sum_{j\neq k} y_{r\alpha_j}}{\sum_{i=1}^{m} v_i \sum_{j\neq k} x_{i\alpha_j}}. \tag{32}$$

This program will find a vertex in the cell DMU_k which is, in a sense, farthest from the other exemplary DMUs. Let the optimal solution be $(\bar{v}_k^*, \bar{u}_k^*)$ $(k = 1, \ldots, p)$, which will be utilized to form the cone for the cone-ratio model.

4.3 The Central Case

This case corresponds to finding a central (relatively interior) point for each CCR-efficient DMU.

The CCR model is formulated for each DMU_o $(o = 1, \ldots, n)$ as the following LP:

$$(CCR_o) \quad \max \ uy_o \tag{33}$$
$$\text{subject to} \quad vx_o = 1 \tag{34}$$
$$-vX + uY \leq 0 \tag{35}$$
$$v \geq 0, \quad u \geq 0. \tag{36}$$

The computation is usually done on the dual side of (CCR_o) which is :

$$(LP_o) \quad \min \ \theta \tag{37}$$
$$\text{subject to} \quad \theta x_o - X\lambda - s_x = 0 \tag{38}$$
$$y_o - Y\lambda + s_y = 0 \tag{39}$$
$$\lambda \geq 0, \quad s_x \geq 0, \quad s_y \geq 0. \tag{40}$$

Let optimal solutions for (CCR_o) and (LP_o) be (v^*, u^*) and $(\theta^*, \lambda^*, s_x^*, s_y^*)$, respectively, for which we have the complementarity:

$$v^* s_x^* = 0 \ \text{ and } \ u^* s_y^* = 0. \tag{41}$$

Furthermore, for a CCR-efficient DMU, we have

$$\theta^* = 1, \quad s_x^* = 0 \ \text{ and } \ s_y^* = 0.$$

However, by the strong theorem of complementarity, it can be seen that for a CCR-efficient DMU, there exists, in addition, an optimal solution of (CCR_o) with

$$v^* > 0 \ \text{ and } \ u^* > 0. \tag{42}$$

We will present two methods for finding a strictly complementary solution for a CCR-efficient DMU.

4.3.1 The Primal-Dual Interior Point Method

The primal-dual interior point methods for linear programming (see Kojima, Mizuno and Yoshise (1989), McShane, Monma and Shanno (1989) and Choi, Monma and Shanno (1990), among others) will theoretically converge to the center of the optimal facet of the problem and the solution is strictly complementary. However, practical implementations of interior point methods usually employ a long step size to the boundary of the feasible region for attaining the next interior iterate, and hence the optimal solution is strictly complementary but not central. Therefore, in order to follow the central trajectory, we need to take special care in choosing the step size.

4.3.2 The Parametric Linear Programming Approach

If a strictly complementary solution, i.e. $v^* > 0$ and $u^* > 0$, is required instead of the central one, we can obtain one by a simplex based method, as follows:

If a CCR-efficient solution is not strictly complementary, we will solve the following parametric linear program in a scalar t:

$$\begin{aligned}
&(43) & \max w &= t(es_x + es_y) \\
&(44) & \text{subject to } x_o &= X\lambda + s_x \\
&(45) & y_o &= Y\lambda - s_y \\
&(46) & \lambda &\geq 0, \quad s_x \geq 0, \quad s_y \geq 0.
\end{aligned}$$

Since we have the optimal solution $w^* = 0$ for $t = 0$ at the end of the (CCR_o) solution, we try to make t positive, while keeping $s_x = 0$ and $s_y = 0$. A positive t^* is guaranteed to exist by the strong theorem of complementarity, which turns out, in the (CCR_o) side, positive weights such that

$$v^* \geq t^*e > 0 \quad \text{and} \quad u^* \geq t^*e > 0. \tag{47}$$

From the (v^*, u^*), we can get a strictly complementary solution for (CCR_o).

5 An Example

Table 1 exhibits data for 14 general hospitals operated under similar environments. As input items, we employ the working hours per month of doctors and nurses, while as outputs the amounts of medical expense insurance for outpatients and inpatients are used.

Table 1: Data of General Hospital and Results

	Input		Output	
Hospital	Doctor	Nurse	Outpatient	Inpatient
H1	3008	20980	97775	101225
H2	3985	25643	135871	130580
H3	4324	26978	133655	168473
H4	3534	25361	46243	100407
H5	8836	40796	176661	215616
H6	5376	37562	182576	217615
H7	4982	33088	98880	167278
H8	4775	39122	136701	193393
H9	8046	42958	225138	256575
H10	8554	48955	257370	312877
H11	6147	45514	165274	227099
H12	8366	55140	203989	321623
H13	13479	68037	174270	341743
H14	21808	78302	322990	487539

The CCR-efficiency, along with weights to inputs and outputs, is shown in Table 2. Now, suppose that we chose H6 and H10 as exemplary among the

Table 2: Efficiency and Weight by CCR Model

Hospi-tal	CCR Eff.	Doctor v_1	Nurse v_2	Outpatient u_1	Inpatient u_2	Reference Set	
H1	0.955	.332E-03	0	.959E-05	.167E-06	H2	H6
H2	1	.242E-03	.140E-05	.714E-05	.225E-06	H2	
H3	1	.104E-03	.204E-04	.339E-05	.325E-05	H3	
H4	0.702	.282E-03	.128E-06	0	.699E-05	H6	H8
H5	0.827	0	.245E-04	0	.384E-05	H10	
H6	1	.784E-04	.154E-04	.256E-05	.245E-05	H6	
H7	0.844	.133E-03	.102E-04	0	.505E-05	H3	H6
H8	1	.209E-03	.458E-07	.111E-07	.516E-05	H8	
H9	0.995	0	.233E-04	.426E-05	.136E-06	H2	H10
H10	1	.319E-04	.149E-04	.215E-05	.143E-05	H10	
H11	0.913	.162E-03	.739E-07	0	.402E-05	H6	H8
H12	0.969	.793E-04	.611E-05	0	.301E-05	H3	H6
H13	0.786	.0	.147E-04	0	.230E-05	H10	
H14	0.974	0	.128E-04	0	.200E-05	H10	

five CCR-efficient DMUs. The three approaches mentioned in the preceding section would give the following results.

5.1 The Most Restricted Case

We solved the fractional program (FP) in Subsection 4.1 for H6 and obtained the optimal solution:

$$\begin{array}{ll} v_1^* = 7.8408 \times 10^{-5} & v_2^* = 1.5401 \times 10^{-5} \\ u_1^* = 2.5567 \times 10^{-6} & u_2^* = 2.4503 \times 10^{-6}, \end{array}$$

with the optimal objective value = 1.

Thus, the cells of H6 and H10 have the above common vertex, which was used as the admissible direction for solving the intersection form cone-ratio model. The results are exhibited in the 'Restricted' column of Table 3.

5.2 The Most Relaxed Case

We solved the fractional program (FP) with the minimizing objective function (32) for H6 and H10 and obtained the following results:

For H6

$$\begin{array}{ll} v_1^* = 1.8601 \times 10^{-4} & v_2^* = 0 \\ u_1^* = 5.3658 \times 10^{-6} & u_2^* = 9.3452 \times 10^{-8}, \end{array}$$

with the optimal objective value = 0.8863.

For H10

$$\begin{array}{ll} v_1^* = 0 & v_2^* = 2.0427 \times 10^{-5} \\ u_1^* = 0 & u_2^* = 3.1961 \times 10^{-6}, \end{array}$$

with the optimal objective value = 0.9065.

The two directions thus obtained were used to form the intersection cone and the results are exhibited in the 'Relaxed' column of Table 3.

5.3 The Central Case

For the central case, we used the solutions obtained by an interior point code 'NUOPT'. This was developed by Mathematical Systems Institute Inc., (Yamashita(1992)). The solutions are as follows:

For H6

$$\begin{array}{llll} v_1^* = 1.6534 \times 10^{-4} & (5.5703 \times 10^{-13}) & v_2^* = 2.9591 \times 10^{-6} & (3.1123 \times 10^{-11}) \\ u_1^* = 3.5461 \times 10^{-6} & (2.5971 \times 10^{-11} & u_2^* = 1.6201 \times 10^{-6} & (5.6845 \times 10^{-11}) \end{array}$$

For H10

$v_1^* = 1.5924 \times 10^{-5}$ (7.9234×10^{-11}) $v_2^* = 1.7645 \times 10^{-5}$ (7.1506×10^{-11})
$u_1^* = 2.1308 \times 10^{-6}$ $(5.9213 \times 10^{-11}$ $u_2^* = 1.4434 \times 10^{-6}$ (8.7413×10^{-10})

The numbers in the parentheses designate the dual slacks. Both solutions are sufficiently central in the sense that the complementary slackness conditions between each variable and slack turned out to give almost equal values. Using these two directions, we have the 'Central' column of Table 3.

On an average, the restricted case is the most severe in efficiency and the central case is between the restricted and the relaxed cases. Numbers in the last column 'Fukuda' will be explained in the next section.

Table 3: Efficiency measured by the Intersection Form Cone-Ratio Models:

Hospital	Restricted	Relaxed	Central	Fukuda
H1	0.8910	0.9546	0.9137	0.8870
H2	0.9434	1	0.9816	0.9359
H3	1	1	0.9961	1
H4	0.5456	0.7018	0.5251	0.5826
H5	0.7418	0.8270	0.7991	0.7812
H6	1	1	1	1
H7	0.7361	0.8441	0.7146	0.7601
H8	0.8428	1	0.8962	0.9033
H9	0.9318	0.9946	0.9652	0.9452
H10	1	1	1	1
H11	0.8276	0.9125	0.8402	0.8556
H12	0.8701	0.9690	0.8498	0.8921
H13	0.6096	0.7859	0.6110	0.6576
H14	0.6929	0.9742	0.8051	0.7937
Average	0.8309	0.9260	0.8498	0.8567

5.4 The Linked-Cone Case

The results of the linked-cone case, using the same directions, are exhibited in Table 4. It is observed that the efficiency is less than that in the intersection case, since the feasible region of (v, u) is more restricted in this case.

Table 4: Efficiency by the Linked Cones Models:

Hospital	Restricted	Relaxed	Central	Fukuda
H1	0.8910	0.9545	0.9130	0.8776
H2	0.9434	1	0.9437	0.9118
H3	1	0.9771	0.9690	0.9815
H4	0.5456	0.6195	0.4954	0.5824
H5	0.7418	0.8269	0.7991	0.7810
H6	1	1	1	1
H7	0.7361	0.7910	0.6818	0.7498
H8	0.8428	0.8462	0.8816	0.9028
H9	0.9318	0.9345	0.9593	0.9452
H10	1	1	1	1
H11	0.8276	0.7942	0.8288	0.8554
H12	0.8701	0.9126	0.8126	0.8782
H13	0.6096	0.7859	0.6110	0.6369
H14	0.6929	0.9742	0.8051	0.7819
Average	0.8309	0.8869	0.8357	0.8489

6 Enumeration of Optimal Vertices

It is interesting to identify all the vertices (v, u) of the convex polyhedron, which makes a DMU efficient, not only for the purposes mentioned in the preceding sections but also in order to arrive at an understanding of the overall positioning of the efficient DMU in the (v, u) space.

Recently, Fukuda (1993) has developed an algorithm and software[2] for enumerating all vertices of a convex polyhedron defined by a system of linear inequalities, base on the Double Description Method (1958). This software is useful for the above mentioned purposes in several ways.

6.1 Enumeration of Optimal Vertices for Selected DMUs

Let the convex polyhedron corresponding to a CCR-efficient DMU_o be

$$C_o = \{(v, u) \mid vx_o = uy_o = 1,\ vx_j \geq uy_j\ (\forall j),\ v \geq 0,\ u \geq 0\}.$$

We can apply this software to enumerate all the vertices of the convex polyhedron C_o. It works efficiently for medium size problems. In fact, for an actual problem with 42 DMUs, 4 inputs and 2 outputs, all vertices of an efficient DMU could be obtained instantly on the SUN SparkServer 1000 and in 15 seconds on the Mac Powerbook Duo 210 (68030, 25mhz). The problem has 6 efficient DMUs among the total of 42 DMUs and the polyhedron corresponding to the said DMU has 22 vertices. It has been observed that the number of vertices increases exponentially with the number of tight (active) constraints in the optimal solution of (CCR_o). It increases also as the number of inputs and outputs does.

We applied this software to the sample problem above, especially for DMUs H6 and H10 and obtained 6 and 5 vertices for H6 and H10, respectively, as exhibited in Table 5 (CPU-time was negligible).

For the purpose of comparing vertices of the polyhedra corresponding to different DMUs, it is convenient to normalize $(v,\ u)$ so that

$$\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_i = 1.$$

[2]The free software "cdd.c" is available via anonymous ftp from ftp.epfl.ch (directory incoming/dma).

Table 5: All Vertices of H6 and H10

All Vertices of H6				
No.	v_1	v_2	u_1	u_2
1	1.8542E-4	8.4472E-8	0	4.5953E-6
2	1.8601E-4	0	1.9038E-8	4.5793E-6
3	1.8601E-4	0	5.3658E-6	9.3452E-8
4	1.2090E-4	9.3195E-6	0	4.5953E-6
5	7.3557E-5	1.6095E-5	4.0202E-6	1.2224E-6
6	7.8408E-5	1.5401E-5	2.5567E-6	2.4503E-6
All Vertices of H10				
No.	v_1	v_2	u_1	u_2
1	5.5037E-5	1.0810E-5	1.7946E-6	1.7199E-6
2	5.1905E-5	1.1357E-5	2.8369E-6	8.6257E-7
3	0	2.0427E-5	3.7409E-6	1.1890E-7
4	0	2.0427E-5	0	3.1961E-6
5	3.2349E-5	1.4774E-5	0	3.1961E-6

Table 6 shows the normalized vertices for H6 and H10.

It is observed that H6 and H10 have two vertices in common ((No.5 in H6 and No.2 in H10) and (No.6 in H6 and No.1 in H10)) and 'the most restricted case' in Subsection 5.1 identified the second of these. The last two columns of Table 6 show the CCR-ratio scale of H6 and H10 for each vertex, respectively. As expected, 'the most relaxed case' in Subsection 5.2 found the vertex that gives the lowest score to the counterpart DMU, in this example.

From Table 5, we can obtain the 'center', as the average of each coordinate value, for each polyhedron as shown in Table 7. By using these two directions as admissible for the cone, we obtained the CCR efficiency as denoted in the columns 'Fukuda' in Tables 3 and 4.

Table 6: Normalized Vertices of H6 and H10

Normalized Vertices of H6					Efficiency	
No.	v_1	v_2	u_1	u_2	H6	H10
1	0.9754	0.0004	0.0000	0.0242	1.0000	0.9041
2	0.9759	0.0000	0.0001	0.0240	1.0000	0.9035
3	0.9715	0.0000	0.0280	0.0005	1.0000	0.8863
4	0.8968	0.0691	0.0000	0.0341	1.0000	0.9647
5	0.7751	0.1696	0.0424	0.0129	1.0000	1.0000
6	0.7935	0.1559	0.0259	0.0248	1.0000	1.0000
Normalized Vertices of H10					**Efficiency**	
No.	v_1	v_2	u_1	u_2	H6	H10
1	0.7935	0.1559	0.0259	0.0248	1.0000	1.0000
2	0.7751	0.1696	0.0424	0.0129	1.0000	1.0000
3	0.0000	0.8411	0.1540	0.0049	0.9239	1.0000
4	0.0000	0.8647	0.0000	0.1353	0.9065	1.0000
5	0.6429	0.2936	0.0000	0.0635	0.9543	1.0000

Table 7: 'Center' of Polyhedron:

No.	v_1	v_2	u_1	u_2
H6	1.3838E-4	6.8166E-6	1.9936E-6	2.9227E-6
H10	2.7858E-5	1.5559E-5	1.6745E-6	1.8187E-6

6.2 Enumeration of All Optimal Vertices

Instead of enumerating all vertices corresponding to selected DMUs, we can enumerate optimal vertices (v, u) of all efficient DMUs as follows.

Let us define the convex polyhedron C by

$$C = \{(v, u) \mid vx_j \geq uy_j \ \ (\forall j), v \geq 0, \ u \geq 0\}. \tag{48}$$

C is a convex cone pointed to $(0, \ 0)$ and represented by a set of extreme rays $\{(v_{r_i}, \ u_{r_i})\}$. If an extreme ray $(v_{r_i}, \ u_{r_i})$ satisfies the relation $vx_k = uy_k$

for some k, then the DMU $(x_k,\ y_k)$ is CCR-efficient and the $(v_{r_i},\ u_{r_i})$ corresponds to an optimal vertex for the DMU_k. Table 8 shows the list of all twelve optimal extreme rays that make at least one DMU efficient. Notice that the rays are normalized so that the sum of elements is equal to one. Fukuda's software *cdd* provides several useful options, of which we will mention two functions (i) to identify the incidence relations between the vertices/rays and the inequalities that are satisfied by equality, and (ii) to find the adjacent vertices/rays for each vertex/ray. The column 'DMUs' in Table 8 exhibits DMUs that are efficient for the corresponding vertex and the column 'Adjacency' shows the list of adjacent vertices. The graph of the optimal vertices are displayed in Figure 3, where the dominating DMUs correspond to the facets.

Table 8: Enumeration of All Optimal Vertices

No.	v_1	v_2	u_1	u_2	DMUs	Adjacency
V1	0.0000	0.8411	0.1540	0.0049	H2 H10	V3 V5 V12
V2	0.7935	0.1559	0.0259	0.0248	H3 H6 H10	V3 V4 V8
V3	0.7751	0.1696	0.0424	0.0129	H2 H6 H10	V1 V2 V9
V4	0.6429	0.2936	0.0000	0.0635	H3 H10	V2 V5 V8
V5	0.0000	0.8647	0.0000	0.1353	H10	V1 V4
V6	0.9754	0.0004	0.0000	0.0242	H6 H8	V7 V8 V10
V7	0.9759	0.0000	0.0001	0.0240	H6 H8	V6 V9 V10
V8	0.8968	0.0691	0.0000	0.0341	H3 H6	V2 V4 V6
V9	0.9715	0.0000	0.0280	0.0005	H2 H6	V3 V7 V11
V10	0.9759	0.0000	0.0000	0.0241	H8	V6 V7
V11	0.9715	0.0000	0.0285	0.0000	H2	V9 V12
V12	0.0000	0.8412	0.1588	0.0000	H2	V1 V11

Figure 3

6.3 From Cone-Ratio to Hull-Constraints

The 'hull' option of Fukuda's *cdd* does the reverse operation. That is, given a set of rays, this option finds a minimal system of linear inequalities to represent the convex cone spanned by the rays. As a matter of course, the given set of rays must be full dimensional. We tried two applications of this option for our research in the cone-ratio model. One is an equivalent formulation of the model with the separate (intersection) cones V and U and the other is the linked cone case in Section 3.

6.3.1 The Separate Cone Case

Let the admissible input and output directions of v and u be $\{a_j\}$ $(j = 1, \ldots, k)$ and $\{b_j\}$ $(j = 1, \ldots, l)$, respectively. (The corresponding cones V and U are defined by (7) and (8).) We assume that $\{a_j\}$ $(\{b_j\})$ spans the $m(s)$ dimensional space. By applying the 'hull' option, we obtain the equivalent linear inequalities representation of cones as

$$vH \leq 0 \quad \text{and} \quad uG \leq 0. \tag{49}$$

Thus, the cone-ratio model for evaluating DMU_o can be expressed by the following primal and dual programs

$$\text{(50)} \qquad \begin{array}{rrcl} \text{(Primal)} & \max\ uy_o & & \\ & \text{subject to} \quad vx_o & = & 1 \\ & -vX + uY & \leq & 0 \\ & vH & \leq & 0 \\ & uG & \leq & 0 \\ & v \geq 0, \quad u & \geq & 0. \end{array}$$

$$(51) \qquad \text{(Dual)} \quad \min \ \theta$$

$$\begin{aligned} \text{subject to} \quad \theta x_o \ - \ X\lambda + H\mu - s_x &= 0 \\ y_o \ - \ Y\lambda - G\nu + s_y &= 0 \\ \lambda \geq 0, \ \ \mu \geq 0, \ \ \nu &\geq 0, \ s_x \geq 0, \ \ s_y \geq 0. \end{aligned}$$

Example

Let the two admissible directions for input weights be

$$a_1 = (0.7935, \ 0.1559) \ \text{ and } \ a_2 = (0.7751, \ 0.1696).$$

Then, we have the hull expression as follows:

$$v_1 - 5.090v_2 \leq 0 \ \text{ and } \ -v_1 + 4.570v_2 \leq 0.$$

Let the two admissible directions for output weights be

$$b_1 = (0.0259, \ 0.0248) \ \text{ and } \ b_2 = (0.0424, \ 0.0129).$$

Then, we have the hull expression as follows:

$$u_1 - 3.287u_2 \leq 0 \ \text{ and } \ -u_1 + 1.044u_2 \leq 0.$$

6.3.2 The Linked-Cone Case

Let the set of linked directions be $\{c_j\}$ $(j = 1, \ldots, q)$ with each $c_j \in R^{m+s}$. Again, we assume that $\{c_j\}$ spans the $(m + s)$ dimensional space. Then, via the 'hull' option, we will obtain a minimal set of linear inequalities to represent the cone spanned by $\{c_j\}$, such as

$$(52) \qquad vH + uG \leq 0.$$

This constraint will be utilized as the linked constraint in the (Primal) and consequently, in the (Dual) side, we have the constraints as below.

$$\begin{aligned} \theta x_o &- X\lambda + H\mu - s_x &= 0 \\ y_o &- Y\lambda - G\mu + s_y &= 0 \end{aligned}$$
$$\lambda \geq 0, \quad \mu \geq 0, \quad s_x \geq 0, \quad s_y \geq 0.$$

Example

Let the linked admissible directions be

$$\begin{array}{rlllll} c_1 = & (& 0.7935 & 0.1559 & 0.0259 & 0.0248 &) \\ c_2 = & (& 0.7751 & 0.1696 & 0.0424 & 0.0129 &) \\ c_3 = & (& 0.7751 & 0.1696 & 0.0350 & 0.0180 &) \\ c_4 = & (& 0.7935 & 0.1559 & 0.0405 & 0.0148 &) \\ c_5 = & (& 3.1372 & 0.6510 & 0.1438 & 0.0705 &) \end{array}$$

Then, we have the hull expression as follows:

$$\begin{array}{rrrrcl} 2.197\ v_1 & +v_2 & -30.58\ u_1 & -44.65\ u_2 & \leq & 0 \\ -2.134\ v_1 & -v_2 & +29.84\ u_1 & +43.30\ u_2 & \leq & 0 \\ -2.017\ v_1 & -v_2 & +28.28\ u_1 & +41.29\ u_2 & \leq & 0 \\ 1.824\ v_1 & +v_2 & -25.91\ u_1 & -37.59\ u_2 & \leq & 0 \end{array}$$

6.3.3 Notes on Hull Representation

Theoretically, the cone form DEA is equivalent to the hull form one, if the matrix C of the admissible directions is full dimensional. However, the hull form is more directly connected to the original DEA problem, in that the optimal solution can be interpreted within the framework of the original input/output data, similarly to the assurance region model. Furthermore, we can extend the model to the 'linked-cone profit model', which is a future research subject.

7 Concluding Remarks

In the first half of this paper, we proposed three approaches to the cone-ratio model with respect to the virtual input/output multipliers. The choice of the admissible directions for the cone is crucial to the successful application of the cone-ratio model. If expert knowledge on the virtual multipliers (weights) could be combined with the three cases proposed, progress could be made in the evaluation of the relative efficiency of DMUs. In the latter half of the paper, we described the possible applications of Fukuda's vertex enumeration algorithm for efficient DMUs, which would be an interesting future research subject.

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I would like to thank Komei Fukuda and Hiroshi Yamashita for applying their software 'cdd.c' and 'NUOPT' to my DEA problems and would encourage others to use these efficient programs. Also, I am grateful to Professors W.W. Cooper and R.M. Thrall for their helpful comments and corrections.

Appendix A. Solution of Fractional Programming

The fractional program (FP_k) can be transformed into equivalent linear program as follows:

$$(53) \qquad (\mathrm{LP}_k) \quad \max \quad \sum_{r=1}^{s} U_r \sum_{j \neq k} y_{r\alpha_j}$$

$$(54) \qquad \text{subject to} \quad \sum_{i=1}^{m} V_i \sum_{j \neq k} x_{i\alpha_j} = 1$$

$$(55) \qquad \sum_{i=1}^{m} V_i x_{i\alpha_k} = \sum_{r=1}^{s} U_r y_{r\alpha_k} = t$$

$$\sum_{i=1}^{m} V_i x_{ij} \geq \sum_{r=1}^{s} U_r y_{rj} \quad (j = 1, \ldots, n) \tag{56}$$

$$V_i \geq 0 \ \ (\forall i), \qquad U_r \geq 0 \ \ (\forall r), \qquad t \geq 0. \tag{57}$$

Let an optimal solution of (LP_k) be ($\boldsymbol{V}^*$, $\boldsymbol{U}^*$, t^*). Then we have the optimal solution of (FP_k) by

$$v^* = \boldsymbol{V}^*/t^* \ \text{ and } \ u^* = \boldsymbol{U}^*/t^*. \tag{58}$$

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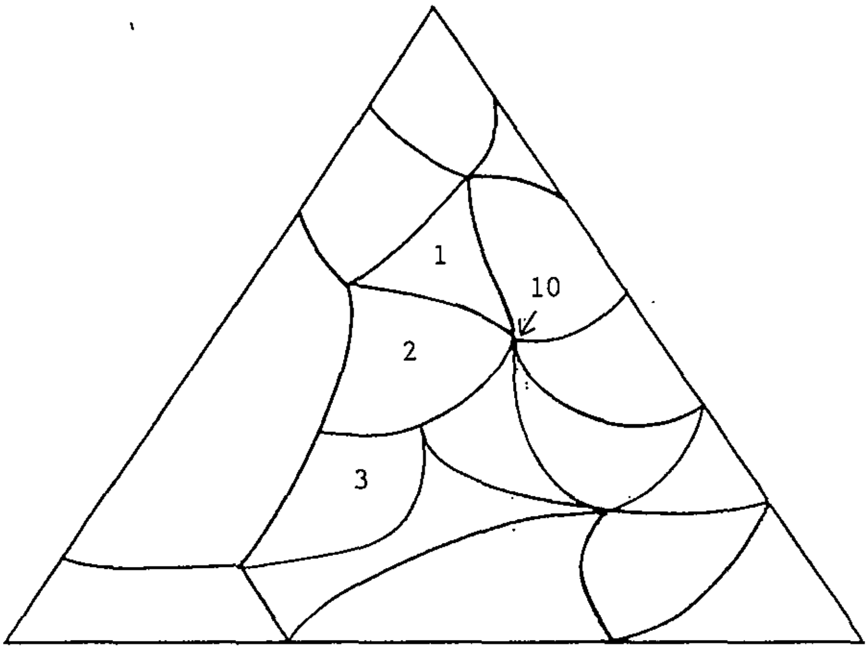

Figure 1. Cell Subdivision

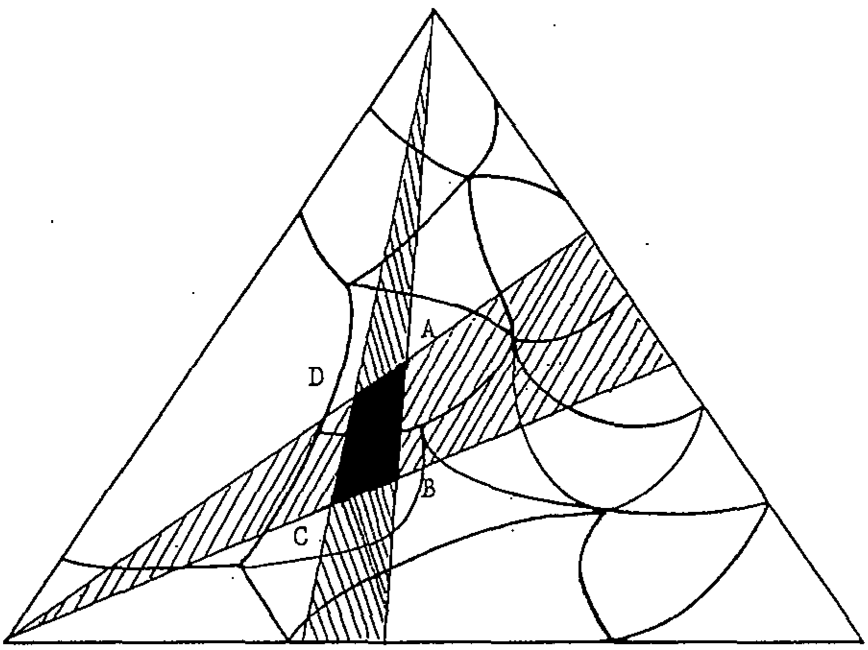

Figure 2. Assurance Region

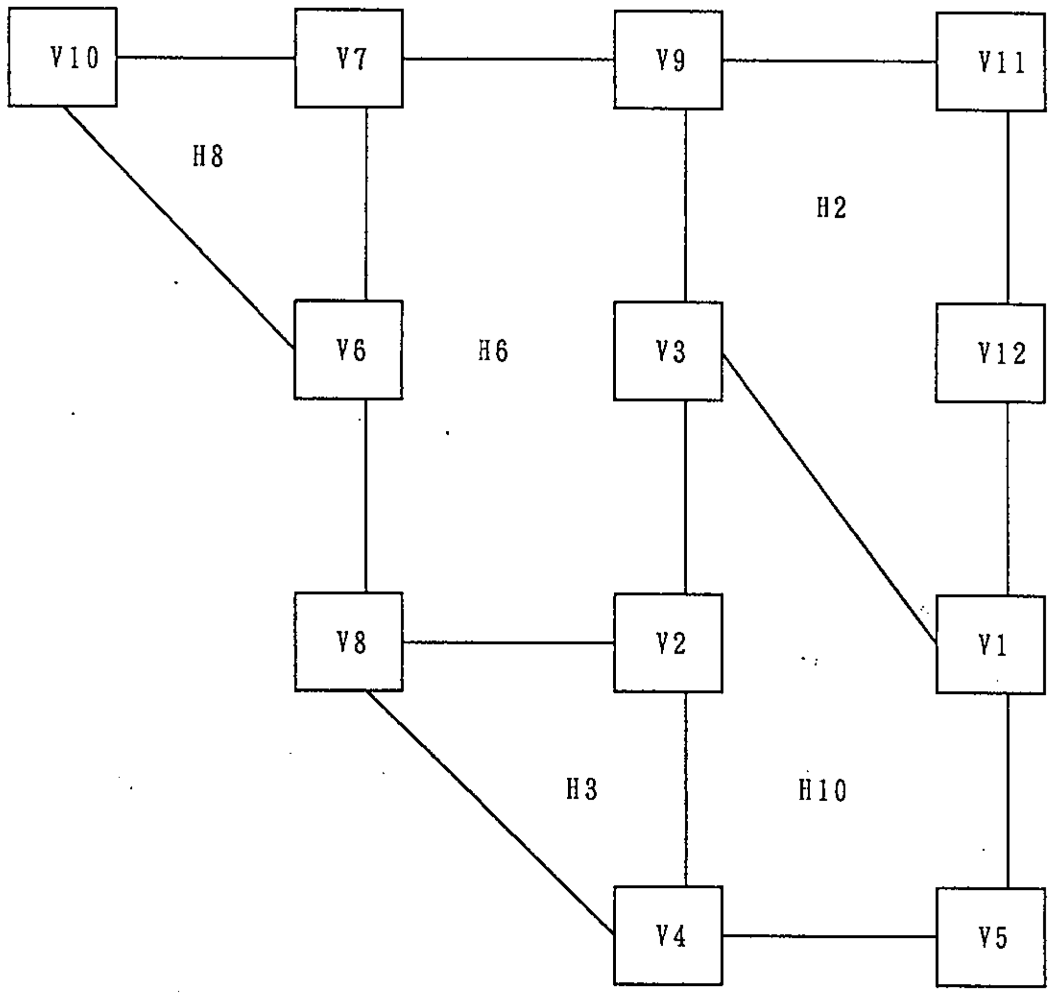

Figure 3. Graph of All Optimal Vertices