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Efficient Risk Sharing under Limited Commitment and Search Frictions*

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Abstract

This paper examines efficient risk sharing under limited commitment and search frictions. The model features a social planner and a continuum of risk-averse workers, where the planner is able to provide consumption only to workers matched with the planner and faces an aggregate resource constraint, while workers can walk away from the match in any period and search for a new match. The formation of new matches and the exogenous destruction of existing ones substantially expand the set of feasible stationary allocations, providing a role for the social welfare function. In the benchmark case of the Benthamite social welfare function, we find that the efficient stationary allocation exhibits novel consumption dynamics: Consumption begins at a relatively low level, converges toward a certain level when the participation constraint is slack, and jumps up when it binds. We then explore the role of limited commitment in generating such rich consumption dynamics.

Keywords: Dynamic contract; limited commitment; labor search and matching
JEL Classification: D86, E21, E24, J64

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1 Introduction

This paper examines efficient risk sharing under limited commitment and search frictions. The model involves a benevolent social planner who provides consumption to a continuum of risk-averse workers. To provide consumption to a worker, the planner must first employ or, equivalently, form a match with the worker through a frictional matching process. Every period, a match yields an output that is subject to idiosyncratic shocks. The planner must finance consumption to employed workers as well as the cost of creating vacancies with the output from all the matches. In any period, an employed worker has the option to leave the match with a fraction of the current output and to enter the pool of unemployment. An employed worker also becomes unemployed when the match is hit by an exogenous separation shock. In either case of match termination, the worker's past employment history is wiped out, and the worker seeks to be employed again. The main contribution of our paper is to show that the efficient stationary allocation in such an environment exhibits completely different consumption dynamics from the standard limited commitment literature.

Our paper belongs to the dynamic contracting literature that examines efficient risk sharing between a single principal and many agents. In particular, the environment of the model builds on the work of Krueger (2000), who adopts a limited commitment version of Atkeson and Lucas's (1992, 1995) model of private information and examines an efficient stationary allocation in an economy populated by a continuum of consumers who face idiosyncratic income risk. However, while, in Krueger (2000), the relationship between planner and consumers is automatically formed and never terminates in any efficient allocation, in the present paper, the relationship is formed through a frictional matching process and is subject to exogenous separation shocks, in a spirit similar to that of the Mortensen–Pissarides model.[1] This extension has two important consequences.

First, the extension enriches the contracting problem by endogenizing the agents' value of the outside option. In Krueger (2000), the outside option of an agent is autarky, over which the planner has no influence. In contrast, in our paper, a worker can seek a new match after leaving the current one and, as a result, the planner's choice of consumption affects, in any stationary allocation, not only the workers' value in the current match, but also the worker's value of the outside option through consumption in future matches.

Second, the extension substantially enlarges the set of (incentive- and resource-) feasible stationary allocations, and consequently extends the scope of welfare analysis. That is, when the principal–agent relationship is, as in Atkeson and Lucas (1995) and Krueger

[1] See, e.g., Mortensen and Pissarides (1994) and Pissarides (2000).

(2000), neither destroyed exogenously nor terminated optimally by the principal, stationarity is a rather strong requirement such that the Pareto criterion suffices to pin down the efficient allocation.[2] The situation changes dramatically once the creation and destruction of the relationship are introduced. To see the intuition, consider an allocation in which the consumption of all agents grows at a constant positive rate in the course of the principal–agent relationship. Clearly, such an allocation can never be stationary when the relationship is permanent. However, that does not apply when the relationship is continuously terminated and recreated; indeed, in our environment, any time-invariant allocation choice of the planner is consistent with stationarity, if not incentive and resource feasibility. Consequently, our model exhibits a continuum of Pareto-efficient feasible stationary allocations. This enables us to discuss how efficiency depends on the social welfare function or, equivalently, the type of social planner.

As a benchmark case, we assume a *Benthamite* social planner who maximizes the sum of individual welfare or, equivalently, the expected discounted lifetime utility of all workers in the steady state. We then examine the corresponding efficient allocation and show that the consumption of employed workers features rich dynamics, which is novel in the limited commitment literature. That is, consumption is relatively low for newly employed workers. In subsequent periods, consumption moves toward a certain level until the participation constraint binds. When the participation constraint binds, consumption jumps up, and then evolves toward this level of consumption again.

As an extension, we consider a *Rawlsian* social planner who maximizes the welfare of the least well-off workers in the steady state. We show that the consumption dynamics differ completely from the benchmark case. This time, consumption is relatively high for newly employed workers and then falls whenever the participation constraint is slack. Such a downward trend in consumption resembles the consumption dynamics in Atkeson and Lucas (1995) and Krueger (2000).

To highlight the impact of limited commitment, we then explore the consumption dynamics under full commitment. We find that, under full commitment, the Benthamite planner equalizes the consumption of all employed workers. In contrast, the Rawlsian planner allows consumption to fall throughout the employment relationship. These results indicate that limited commitment is the source of inequality among employed workers under the Benthamite planner, while consumption inequality persists regardless of the workers' commitment ability under the Rawlsian planner.

In addition to the studies mentioned above, the present paper is particularly related to

[2] An efficient allocation is not necessarily unique in Atkeson and Lucas (1995) and Krueger (2000), but all efficient allocations share similar qualitative features.

two studies that endogenize outside option values in a limited commitment environment. Krueger and Uhlig (2006) examine risk sharing when agents are able to enter a new contract with competing principals after reneging on the original contract. Rudanko (2009) explores labor market dynamics using a model that incorporates the limited commitment environment into a competitive search model à la Moen (1997). However, these studies are concerned with market equilibrium and thus do not analyze efficient risk sharing under different social welfare functions. Moreover, these studies do not impose an aggregate resource constraint as does our paper.[3]

In terms of methodology, we differ substantially from the work of Krueger (2000) and most of the dynamic contracting literature in that we do not use the recursive formulation of the problem. This is because, in our environment, the endogenous outside option values do not allow us to readily formulate the planner's problem in a recursive fashion. We resort instead to a variational argument using the sequential formulation of the problem and obtain the conditions that characterize the efficient allocation.

2 Model

2.1 Environment

Time is discrete, and there is a single perishable consumption good. The economy is populated by a continuum of infinitely-lived workers (or agents) of mass one. Workers have a period utility function $U(c)$, where $c \geq 0$ denotes consumption, and discount the future with $\beta \in (0,1)$. The utility function U is twice continuously differentiable with $U' > 0$ and $U'' < 0$.

Throughout, the analysis focuses on the stationary (or, equivalently, steady-state) allocation. There is a benevolent social planner who aims to maximize social welfare, to be defined in Section 2.5. The planner employs workers and provides consumption during employment. Due to search frictions, the planner must post vacancies to employ, or to form matches with, workers. The flow cost of posting a vacancy is $k > 0$. The number of matches formed each period is $M(u,v)$, where u is the number of unemployed workers, and v is the number of vacancies posted by the planner. The matching function $M(u,v)$ is such that $M(0,v) = M(u,0) = 0$, increasing[4] in both arguments, concave, twice continuously differentiable, and exhibits constant returns to scale. The probability that

[3]Since Krueger and Uhlig (2006) allow the principal to consume a negative amount, their feasibility constraint has a completely distinct role from that in the present paper (or that in Atkeson and Lucas (1995) and Krueger (2000)).

[4]Throughout, *increasing* implies strictly increasing, and *decreasing* implies strictly decreasing.

each vacancy is matched with an unemployed worker is $M(u,v)/v = M(\theta^{-1},1) \equiv q(\theta)$, where $\theta \equiv v/u$ is market tightness. Similarly, the probability that an unemployed worker is matched with a vacancy is $M(u,v)/u = M(1,\theta) \equiv p(\theta) = \theta q(\theta)$.[5]

The flow output of a match is $y \in Y = \{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_N\}$, $\bar{y}_1 < \bar{y}_2 < \ldots < \bar{y}_N$, where y is idiosyncratic to each match. Let y_t denote output in the t-th period of a match, and $y^t = (y_1, y_2, \ldots, y_t)$ be the history of output up to the t-th period. Output y_t follows a first-order Markov process, and the initial output y_1 is drawn from the stationary distribution for y_t. Below, let $\pi(y^t)$ denote the unconditional probability of history y^t occurring.

Each period, the planner provides consumption to the employed worker, which may depend on the history of output in the current match but not on past employment status.[6] Thus, consumption in a match following history y^t is $c_t(y^t) \geq 0$. While unemployed, workers consume home production $b \in (0, \bar{y}_1)$. To focus on the risk-sharing implication of introducing search frictions, we preclude the provision of unemployment benefits.

Workers lack the ability to commit to stay in a match, such that, in any period, an employed worker can take away a fraction of the current output, become unemployed, and search for a new match. That is, if, in period t, a worker exits a match whose current output is y_t, the worker's consumption in period t is

$$\hat{c}(y_t) \equiv \rho y_t + (1-\rho) b \tag{1}$$

for some constant $\rho \in [0,1]$.[7] Since $y_t > b$, $\hat{c}(y_t) \geq b$ with equality if and only if $\rho = 0$.

Each period, with probability $s \in (0,1)$, a match is hit by a separation shock; the match is then terminated exogenously and the worker becomes unemployed. In an efficient stationary allocation described below, separation occurs only exogenously because, given $\bar{y}_1 > b$, it is never optimal for the planner to terminate a match.

[5] In discrete time models, it sometimes becomes necessary to truncate the matching function to avoid the probabilities $p(\theta)$ and $q(\theta)$ from exceeding one. We abstract from this issue by assuming that the model parameters are such that the efficient allocation is the interior optimum of the planner's problem.

[6] This formulation rules out the possibility of penalizing a worker who voluntarily left a previous match. We impose this restriction because, in reality, workers are generally not penalized for quitting firms.

[7] In models of limited commitment, y is typically an agent's endowment, so the natural assumption is that the agent consumes y in the period of walking away from the principal ($\rho = 1$). In labor search and matching models, it is more natural for a worker to exit the match before producing output y and consume b ($\rho = 0$). The formulation here nests both these assumptions as special cases.

2.2 Worker's Value Functions

Let V^u be the worker's value of being unemployed, and $V_t^e(y^t)$ be the worker's value of being employed in a match with history y^t. Note that V^u is common to all unemployed workers, because past employment status does not affect consumption in future matches. Then, V^u is expressed as

$$V^u = U(b) + \beta p(\theta)(1-s)\sum_{y^1} \pi\left(y^1\right) V_1^e\left(y^1\right) + \beta\left[1 - p(\theta)(1-s)\right] V^u. \tag{2}$$

In (2), the first term is the current utility from home production. The next period, an unemployed worker finds a match with probability $p(\theta)$ and does not face exogenous separation with probability $1-s$. The worker then receives the value of employment V_1^e, which depends on $y^1 = y_1$. With probability $1 - p(\theta)(1-s)$, the worker remains unemployed and receives V^u. We can also express V^u using the sequence of consumption $\{c_t(y^t)\}_{t=1}^{\infty}$ as

$$V^u = U(b) + \beta p(\theta)\sum_{t=1}^{\infty} \beta^{t-1}(1-s)^{t-1}\left[(1-s)\sum_{y^t} \pi\left(y^t\right) U\left(c_t\left(y^t\right)\right) + sV^u\right] \tag{3}$$
$$+ \beta(1 - p(\theta)) V^u.$$

Using V^u, we can express $V_t^e(y^t)$ as[8]

$$V_t^e\left(y^t\right) = U\left(c_t\left(y^t\right)\right) + \beta\left[(1-s)\sum_{y^{t+1}|y^t} \frac{\pi\left(y^{t+1}\right)}{\pi\left(y^t\right)} V_{t+1}^e\left(y^{t+1}\right) + sV^u\right], \tag{4}$$

which can be iterated forward to yield

$$V_t^e\left(y^t\right) = U\left(c_t\left(y^t\right)\right) + \sum_{\tau=1}^{\infty} \beta^{\tau}(1-s)^{\tau} \sum_{y^{t+\tau}|y^t} \frac{\pi\left(y^{t+\tau}\right)}{\pi\left(y^t\right)} U\left(c_{t+\tau}\left(y^{t+\tau}\right)\right) + \frac{\beta s}{1-\beta(1-s)} V^u. \tag{5}$$

Let us now consider the worker's value of the outside option, V^o. Upon exiting the match, the worker consumes $\hat{c}(y_t) = \rho y_t + (1-\rho) b$ in the current period and becomes unemployed. Thus, V^o depends on the current output y_t but not on the entire history y^t, and it can be expressed as

$$V^o(y_t) = U(\hat{c}(y_t)) - U(b) + V^u. \tag{6}$$

[8] Throughout, $y^{t+\tau}|y^t$, $\tau \geq 0$, denotes any history $\tilde{y}^{t+\tau}$ such that $\tilde{y}^t = y^t$.

Note that $V^o(y_t) \geq V^u$ since $\hat{c}(y_t) \geq b$, and that V^o rises one to one with V^u.

2.3 Excess Demand Function

For any allocation $x = (\theta, \{c_t(y^t)\}_{t=1}^{\infty}) \in \mathcal{D} \equiv \mathcal{R}_+^{\infty}$,[9] the excess demand for resources is given by

$$ED = k\theta u - \sum_{t=1}^{\infty} \sum_{y^t} e_t\left(y^t\right)\left(y_t - c_t\left(y^t\right)\right). \tag{7}$$

The first term on the right-hand side (RHS), $k\theta u = kv$, is the aggregate vacancy cost. The second term is the sum of the match output net of consumption of workers. Here,

$$e_t\left(y^t\right) = up(\theta)(1-s)^t \pi\left(y^t\right) \tag{8}$$

denotes the measure of workers employed in a match with history y^t.

Stationarity requires the number of workers exiting the pool of unemployment, $up(\theta)(1-s)$, to be equal to the number of those entering it, $(1-u)s$; hence

$$u = \frac{s}{s + p(\theta)(1-s)}. \tag{9}$$

2.4 Incentive Feasibility and Resource Feasibility

Let us now turn to the two constraints faced by the planner. The first is the participation constraint,

$$V_t^e\left(y^t\right) \geq V^o(y_t), \ \forall y^t, \tag{10}$$

which implies that at any history, the worker's value of staying in a match must weakly exceed the value of the outside option. The second is the resource constraint, $ED \leq 0$, or

$$k\theta u - \sum_{t=1}^{\infty} \sum_{y^t} e_t\left(y^t\right)\left(y_t - c_t\left(y^t\right)\right) \leq 0. \tag{11}$$

An allocation is *incentive feasible* if it satisfies the participation constraint, *resource feasible* if it satisfies the resource constraint, and *feasible* if it is both incentive and resource feasible. An *efficient allocation* is a feasible allocation that maximizes social welfare.

In the problems we consider, it can be shown that the planner uses up all available

[9] Throughout, x denotes an infinite dimensional allocation vector $\left(\theta, \{c_t(y^t)\}_{t=1}^{\infty}\right)$. Similarly, x' denotes an allocation vector $\left(\theta', \{c_t'(y^t)\}_{t=1}^{\infty}\right)$.

resources and thus (11) holds with equality.[10] Dividing the expression by θu, we obtain

$$k = q(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi(y^t) (y_t - c_t(y^t)). \tag{12}$$

For later use, note that (12), $p(\theta) = \theta q(\theta)$, and $\theta p'(\theta)/p(\theta) = 1 + \theta q'(\theta)/q(\theta)$ yield

$$k - p'(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi(y^t) (y_t - c_t(y^t)) = -k \frac{\theta q'(\theta)}{q(\theta)}. \tag{13}$$

2.5 Social Welfare Function

As a benchmark case, we consider a Benthamite planner who maximizes the sum of the welfare of all workers in the steady state. More precisely, the planner maximizes the social welfare function

$$V^B \equiv uV^u + \sum_{t=1}^{\infty} \sum_{y^t} e_t(y^t) V_t^e(y^t) \tag{14}$$

by choosing an allocation $x = (\theta, \{c_t(y^t)\}_{t=1}^{\infty})$ subject to the participation constraint, (10), and the resource constraint, (11). In other words, the planner maximizes V^B by choosing $x \in \mathcal{D}'$, where $\mathcal{D}' \subset \mathcal{D}$ is the set of feasible allocations.[11] In much of the analysis, we resort to the following sequential form of V^B, whose derivation is shown in Appendix A:

$$V^B = \frac{1}{1-\beta} u \left[U(b) + p(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi(y^t) U(c_t(y^t)) \right]. \tag{15}$$

In the next section, we discuss the properties of the *Benthamite efficient allocation*, which is the constrained efficient allocation chosen by the Benthamite planner.

3 Benthamite Efficient Allocation

3.1 Efficiency Conditions

A popular approach for analyzing dynamic contracting problems is to formulate recursive problems by using, for example, promised utilities or Pareto weights as state

[10] Lemma A2 in Appendix B, which collects lemmas that hold for general social welfare functions, proves this result.

[11] Throughout, we let $\subset$ denote a proper subset.

variables.[12] However, in our environment, complications arise from the fact that the value of the worker's outside option is endogenous, making this approach not readily applicable.[13] Instead, our approach is to resort to a variational argument using the sequential formulation of the problem; we take a candidate optimal allocation and derive conditions that must hold to rule out welfare-improving perturbations.

The discussions below require the introduction of some notations and definitions. Let (y^t, y_{t+1}) denote the continuation history of y^t in which output in the $t+1$-th period is y_{t+1}. Let $c^b(y_t)$ be the value of optimal consumption at y^t when $V_t^e(y^t) = V^o(y_t)$, and let $\lambda_t(y^t)$ be defined by

$$\lambda_t\left(y^t\right) \equiv 1/U'\left(c_t\left(y^t\right)\right). \tag{16}$$

Unless otherwise noted, *cost* implies the expected resource cost, evaluated using the planner's subjective prices. The *direct cost of* $V_t^e(y^t)$ implies the cost of providing $V_t^e(y^t)$ to a single worker with history y^t, taking V^u as given. Similarly, the *direct cost of* V^u refers to the cost of providing V^u to a single unemployed worker, taking V^u in future periods as given. The *direct marginal cost of* $V_t^e(y^t)$ is the increase in the direct cost of $V_t^e(y^t)$ as $V_t^e(y^t)$ is raised by one infinitesimal unit, which ignores the indirect costs and benefits arising from the effect of the change in $V_t^e(y^t)$ on V^u and $V_\tau^e(y^\tau)$, $y^\tau \neq y^t$. The *direct marginal cost of* V^u is defined similarly. The *shadow cost of the participation constraint at* (y^t, y_{t+1}) is the additional cost required to satisfy the participation constraint at (y^t, y_{t+1}) as this constraint is tightened by one infinitesimal unit, when there is a single worker with history (y^t, y_{t+1}). These three types of costs are proportional to the relevant population of workers and, therefore, for example, the descriptions above hold when "a single worker" is replaced by "measure one of workers". Finally, the *marginal cost of* V^B is the additional cost required to increase social welfare V^B by one infinitesimal unit.

We are now ready to present Proposition 1, which summarizes the conditions that characterize the Benthamite efficient allocation.[14]

[12]See, e.g., Abreu, Pearce, and Stacchetti (1986, 1990) and Marcet and Marimon (1994).

[13]Another potential approach, which is to set up the Lagrangian from the planner's sequential problem and to take the first-order conditions, faces two challenges. First, the sequential problem is an infinite-dimensional problem, in which case the Lagrangian may not be expressed as a sum of an infinite series (see Dechert (1982) and Rustichini (1998)). Second, first-order conditions are not sufficient for an optimum, because of the non-convexity of the problem due to endogenously determined outside option values; moreover, the infinite-dimensionality of the problem makes it difficult to show that a constraint qualification is satisfied or, equivalently, that first-order conditions are necessary for an optimum. The conditions obtained from this approach, however, do coincide with those in Propositions 1 and 3 below.

[14]The proofs of all the propositions are given in Appendix C.

Proposition 1 *The Benthamite efficient allocation exists and satisfies*

$$\lambda_{t+1}\left(y^t, y_{t+1}\right) = \alpha + \beta\lambda_t\left(y^t\right) + \psi_{t+1}\left(y^t, y_{t+1}\right), \tag{17}$$

$$\lambda_1\left(y^1\right) = \alpha + \beta\gamma + \psi_1\left(y^1\right), \tag{18}$$

$$\begin{aligned} -ku\frac{\theta q'(\theta)}{q(\theta)} =& \gamma u\beta p'(\theta)(1-s)\left(\sum_{y^1}\pi\left(y^1\right)V_1^e\left(y^1\right) - V^u\right) \\ &+ \alpha u p'(\theta)\sum_{t=1}^{\infty}(1-s)^t\sum_{y^t}\pi\left(y^t\right)V_t^e\left(y^t\right) \\ &- \alpha u(1-u)\frac{p'(\theta)}{p(\theta)}\left[V^u + p(\theta)\sum_{t=1}^{\infty}(1-s)^t\sum_{y^t}\pi\left(y^t\right)V_t^e\left(y^t\right)\right], \end{aligned} \tag{19}$$

$$\gamma u\left[(1-\beta) + \beta(1-s)p(\theta)\right] = \alpha u + \sum_{t=1}^{\infty}\sum_{y^t}e_t\left(y^t\right)\left(\beta s\lambda_t\left(y^t\right) - \psi_t\left(y^t\right)\right). \tag{20}$$

Here, $\alpha > 0$ is the marginal cost of V^B, while $\lambda_t(y^t) > 0$ and $\gamma > 0$ are the direct marginal costs of $V_t^e(y^t)$ and V^u, respectively, and $\psi_t(y^t) \geq 0$, defined by

$$\psi_{t+1}\left(y^t, y_{t+1}\right) \equiv \max\left\{0, \frac{1}{U'\left(c^b\left(y_{t+1}\right)\right)} - \alpha - \beta\lambda_t\left(y^t\right)\right\}, \tag{21}$$

$$\psi_1\left(y^1\right) \equiv \max\left\{0, \frac{1}{U'\left(c^b\left(y_1\right)\right)} - \alpha - \beta\gamma\right\}, \tag{22}$$

is the shadow cost of the participation constraint at y^t, for the Benthamite planner. Further, $\gamma \leq \alpha/(1-\beta)$, with strict inequality if there is any binding participation constraint.

Let us now explain these conditions. First, (17) can be seen as the optimality condition for $V_{t+1}^e(y^t, y_{t+1})$, given $V_t^e(y^t)$. To see this, suppose the planner raises $V_{t+1}^e(y^t, y_{t+1})$ by one infinitesimal unit for a measure $e_{t+1}(y^t, y_{t+1})$ of workers with history (y^t, y_{t+1}), keeping $V_t^e(y^t)$ unchanged. Then, social welfare V^B rises by $e_{t+1}(y^t, y_{t+1})$ units, whose value in resource units is $\alpha e_{t+1}(y^t, y_{t+1})$. Further, while the planner incurs cost $e_{t+1}(y^t, y_{t+1})\lambda_{t+1}(y^t, y_{t+1})$ from raising $V_{t+1}^e(y^t, y_{t+1})$, the rise in $V_{t+1}^e(y^t, y_{t+1})$ reduces the cost of providing the same $V_t^e(y^t)$ as before by $e_t(y^t)\beta(1-s)(\pi_{t+1}(y^t, y_{t+1})/\pi_t(y^t))\lambda_t(y^t)$, as can be seen from (4). When the participation constraint is slack at (y^t, y_{t+1}), such a perturbation as well as the reverse perturbation are both incentive feasible. Thus, the net gain from such a perturbation must be zero, or

$$e_{t+1}\left(y^t, y_{t+1}\right) \lambda_{t+1}\left(y^t, y_{t+1}\right) = \alpha e_{t+1}\left(y^t, y_{t+1}\right) + e_t\left(y^t\right) \beta (1-s) \frac{\pi_{t+1}\left(y^t, y_{t+1}\right)}{\pi_t\left(y^t\right)} \lambda_t\left(y^t\right), \tag{23}$$

which agrees with the condition obtained by setting $\psi_{t+1}(y^t, y_{t+1}) = 0$ in (17). When the participation constraint binds at (y^t, y_{t+1}), (17) and (21) yield $\lambda_{t+1}(y^t, y_{t+1}) = 1/U'\left(c^b(y_{t+1})\right)$, consistently with $V^e_{t+1}(y^t, y_{t+1}) = V^o(y_{t+1})$. The argument for (18) is similar.

Next, (19) is the optimality condition for θ. The left-hand side (LHS) shows the effect of a marginal increase in θ on the excess demand.[15] On the RHS, the first line, as can be observed from (2), shows the benefit of an increase in V^u from the increase in θ, taking future V^u and all $V^e_1(y^1)$ as given, converted into resource units by multiplying by γ. The second and third lines represent, as seen from (8), (9), and (14), the impact of an increase in θ on V^B, taking V^u and all $V^e_t(y^t)$ as given, measured in resource units. At the optimum, these effects must be equalized as in (19).

Finally, (20) is the optimality condition for V^u, which ensures that, in the efficient allocation, there is no net gain from perturbing V^u. Here, the LHS shows the effect of a marginal increase in V^u on the cost of providing V^u to a measure u of unemployed workers, taking into account the effect of the change in V^u in future periods on the current V^u. On the RHS, the first term is the direct impact of an increase in V^u on V^B, measured in resource units. The second term shows the effects on the cost of providing $V^e_t(y^t)$ and on the shadow cost of the participation constraint at y^t, summed over the relevant population of workers.

3.2 Consumption Dynamics

We now use the efficiency conditions to characterize the consumption dynamics.

Proposition 2 *Let c_∞ be defined by $U'(c_\infty) = (1-\beta)/\alpha$, and assume that the participation constraint binds at some y^t. In the Benthamite efficient allocation, (i) for some $\bar{c}_1 > 0$, $c_1(y^1) = \bar{c}_1 < c_\infty$ for all y^1 at which the participation constraint is slack, (ii) if the participation constraint is slack at (y^t, y_{t+1}), then $1/U'(c_{t+1}(y^t, y_{t+1})) = \alpha + \beta/U'(c_t(y^t))$ such that $c_{t+1}(y^t, y_{t+1}) \in (c_t(y^t), c_\infty)$ for $c_t(y^t) < c_\infty$ and $c_{t+1}(y^t, y_{t+1}) \in (c_\infty, c_t(y^t))$ for $c_t(y^t) > c_\infty$, and (iii) there is at least one $\bar{y}_n \in Y$ such that $c^b(\bar{y}_n) > c_\infty$.*

[15] To see this, differentiate (7) with respect to θ, taking $\{c_t(y^t)\}_{t=1}^{\infty}$ as given. The terms capturing the effects of the change in θ on the excess demand, which work through the change in u, cancel out given (12). Using (13), we obtain the LHS of (19).

Let us explain Proposition 2. First, for all y^1 at which the participation constraint is slack, initial consumption equals some $\bar{c}_1$, which is a relatively low value. Second, whenever the participation constraint is slack, consumption converges toward some c_∞; thus, consumption rises if it was previously below c_∞, and falls otherwise. Third, when the participation constraint binds, consumption jumps up; moreover, there exists at least one output realization such that the resulting consumption exceeds c_∞. Following such output realization, consumption remains above c_∞ throughout the employment spell.

As shown above, the Benthamite efficient allocation exhibits rich consumption dynamics, which, to the best of our knowledge, is a novel result in the limited commitment literature. In particular, the increasing consumption profile at the early stages of a match contrasts with the downward drift in consumption observed in Krueger (2000). This difference arises because, in our environment, the workers' outside option values are endogenous, which generates benefits from relaxing the participation constraint by lowering $\bar{c}_1$ and thus V^u. We discuss in more detail the impact of limited commitment on consumption dynamics in Section 4.3.

4 Discussion

4.1 Rawlsian Efficient Allocation

As an extension, we now consider the *Rawlsian efficient allocation* chosen by the Rawlsian planner, who maximizes the welfare of the least well-off agents based on the *difference principle* (Rawls, 1971). Then, the social welfare function is

$$V^R \equiv V^u, \tag{24}$$

since the participation constraint (10) and $V^o(y_t) \geq V^u$ imply that unemployed workers are the least well off.[16] As in the baseline case, we often use the sequential form of $V^R = V^u$, given by (3) or the expression below, whose derivation is given in Appendix A:

$$V^R = V^u = \frac{1}{(1-\beta)\left[1+\frac{\beta p(\theta)(1-s)}{1-\beta+\beta s}\right]}\left[U(b) + p(\theta)\sum_{t=1}^{\infty}\beta^t(1-s)^t\sum_{y^t}\pi\left(y^t\right)U\left(c_t\left(y^t\right)\right)\right]. \tag{25}$$

[16]Our mathematical formulation of the Rawlsian efficient allocation has some similarities with that of Grout (1977), who proposes a *Rawlsian intertemporal consumption rule* in an overlapping-generations framework. Under this rule, each generation maximizes utility subject to the constraint that all subsequent generations are at least as well off as the current generation, which amounts to maximizing the minimum of the utility of current and future generations.

Besides Rawlsian ethics, this social welfare function is also motivated by the approach from the search and matching literature; in the standard Mortensen-Pissarides model, the efficient allocation is usually assumed to be the allocation that maximizes the discounted present value of the aggregate output net of vacancy costs, which turns out to also maximize the value of unemployed workers.[17]

Proposition 3 summarizes the efficiency conditions in this case.

Proposition 3 *The Rawlsian efficient allocation exists and satisfies*

$$\lambda_{t+1}\left(y^t, y_{t+1}\right) = \beta\lambda_t\left(y^t\right) + \psi_{t+1}\left(y^t, y_{t+1}\right), \tag{26}$$

$$\lambda_1\left(y^1\right) = \beta\gamma + \psi_1\left(y^1\right), \tag{27}$$

$$-ku\frac{\theta q'(\theta)}{q(\theta)} = \gamma u\beta p'(\theta)(1-s)\left(\sum_{y^1}\pi\left(y^1\right)V_1^e\left(y^1\right) - V^u\right). \tag{28}$$

Here, $\lambda_t\left(y^t\right) > 0$ and $\gamma > 0$ are the direct marginal costs of $V_t^e\left(y^t\right)$ and V^u, respectively, and $\psi_t\left(y^t\right) \geq 0$, defined by

$$\psi_{t+1}\left(y^t, y_{t+1}\right) \equiv \max\left\{0, \frac{1}{U'\left(c^b\left(y_{t+1}\right)\right)} - \beta\lambda_t\left(y^t\right)\right\}, \tag{29}$$

$$\psi_1\left(y^1\right) \equiv \max\left\{0, \frac{1}{U'\left(c^b\left(y_1\right)\right)} - \beta\gamma\right\}, \tag{30}$$

is the shadow cost of the participation constraint at y^t, for the Rawlsian planner.

Let us again explain these conditions. As for (17), (26) can be understood as the optimality condition for $V_{t+1}^e\left(y^t, y_{t+1}\right)$, given $V_t^e\left(y^t\right)$. Unlike in the benchmark case, however, if $V_{t+1}^e\left(y^t, y_{t+1}\right)$ is increased by one infinitesimal unit for all workers with history $\left(y^t, y_{t+1}\right)$ while $V_t^e\left(y^t\right)$ is kept unchanged, there is no direct impact on social welfare V^R. Thus, the condition corresponding to (23) is

$$e_{t+1}\left(y^t, y_{t+1}\right)\lambda_{t+1}\left(y^t, y_{t+1}\right) = e_t\left(y^t\right)\beta(1-s)\frac{\pi_{t+1}\left(y^t, y_{t+1}\right)}{\pi_t\left(y^t\right)}\lambda_t\left(y^t\right), \tag{31}$$

which agrees with the condition obtained by setting $\psi_{t+1}\left(y^t, y_{t+1}\right) = 0$ in (26). The argument for (27) is similar.

[17]See Pissarides (2000), Chapter 4. There is yet another interpretation for this social welfare function. It is straightforward to introduce entry and exit by assuming that workers die with a constant probability and are replaced by new-born workers who enter the labor market as unemployed workers. In such an extended model, maximizing V^u is equivalent to maximizing the ex ante welfare of new-born workers.

Finally, (28) is the optimality condition for θ. Note that in (28), the terms that appear on the second and third lines on the RHS of (19) are absent because, this time, an increase in θ has no direct impact on V^R.[18]

Proposition 4 describes the consumption dynamics in the Rawlsian efficient allocation.

Proposition 4 *In the Rawlsian efficient allocation, (i) for some $\tilde{c}_1 > 0$, $c_1(y^1) = \tilde{c}_1$ for all y^1 at which the participation constraint is slack, (ii) if the participation constraint is slack at (y^t, y_{t+1}), then $U'(c_t(y^t)) = \beta U'(c_{t+1}(y^t, y_{t+1}))$ such that $c_t(y^t) > c_{t+1}(y^t, y_{t+1})$, and (iii) if the participation constraint binds at (y^t, y_{t+1}), then $U'(c_t(y^t)) > \beta U'(c_{t+1}(y^t, y_{t+1}))$.*

In the Rawlsian efficient allocation, $c_1(y^1)$ is equalized across all y^1 at which the participation constraint is slack, just as for the benchmark Benthamite case. Unlike the benchmark case, however, consumption is relatively high at the beginning of the match and falls in subsequent periods whenever the participation constraint is slack. Such consumption dynamics are similar to those in Krueger (2000), which implies that in the Rawlsian case, the fact that the workers' outside option values are endogenous does not qualitatively affect the consumption profile. We explain the reason in Section 4.3.

4.2 Examples: Constant Relative Risk Aversion Utility and Two Values of Output

To illustrate the consumption dynamics in the Benthamite and Rawlsian efficient allocations, Figures 1 and 2 plot the typical paths of (logged) consumption in a match, assuming $U(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma > 0$, and two values for match output, y_1 and y_2.[19]

In the Benthamite efficient allocation, $1/U'(c_{t+1}(y^t, y_{t+1})) = \alpha + \beta/U'(c_t(y^t))$ in Proposition 2(ii) yields

$$\frac{c_{t+1}(y^t, y_{t+1})}{c_t(y^t)} = \left[\alpha \frac{1}{(c_t(y^t))^{\frac{1}{\sigma}}} + \beta\right]^{\frac{1}{\sigma}}. \qquad (32)$$

Thus, as shown in Figure 1, the consumption growth rate when the participation constraint is slack is decreasing in the previous consumption and is positive (negative) when consumption is previously below (above) $c_\infty = [\alpha/(1-\beta)]^{\frac{1}{\sigma}}$.

[18] The counterpart to (20) exists also for the Rawlsian efficient allocation. However, this time, the condition is not necessary to pin down the efficient allocation; rather, it determines the value of α, given the efficient allocation.

[19] In Figures 1 and 2, "PC" refers to the participation constraint.

In the Rawlsian efficient allocation, $U'(c_t(y^t)) = \beta U'(c_{t+1}(y^t, y_{t+1}))$ in Proposition 4(ii) yields

$$\frac{c_{t+1}(y^t, y_{t+1})}{c_t(y^t)} = \beta^{\frac{1}{\sigma}} < 1. \tag{33}$$

Thus, consumption grows at a constant negative rate $\beta^{\frac{1}{\sigma}} - 1$ when the participation constraint is slack, which corresponds to downward sloping line segments in Figure 2.

4.3 Limited Commitment and the Drift in Consumption

To highlight the impact of limited commitment on consumption dynamics, we now consider the situation in which workers are able to fully commit to stay in a match except in the case of exogenous separation.

Let us begin with the benchmark Benthamite case. Substituting (17) and (18) into (20) yields

$$\gamma + \frac{p(\theta)}{1 - \beta(1-s)(1-p(\theta))} \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) \psi_t\left(y^t\right) = \frac{\alpha}{1-\beta}. \tag{34}$$

Under full commitment, $\psi_t(y^t) = 0$ for all y^t. Thus, (34) implies $\gamma = \alpha/(1-\beta)$; but then, (17) and (18) imply $\lambda_t(y^t) = \alpha/(1-\beta)$ for all y^t, such that $c_t(y^t)$ is the same for all y^t.[20] The intuition is straightforward: if there is no commitment problem, the Benthamite planner simply equates the marginal utility of consumption of all employed workers, which implies a flat consumption path for each worker.

A typical result in the one-sided limited commitment literature is that whenever the agent's participation constraint is slack, the agent's marginal utility of consumption grows at a constant rate.[21] Since this growth rate is determined solely by the relative size of the discount factor of the principal and the agent, it applies also under full commitment. Our results above indicate that such a typical result in the literature no longer applies in our environment. That is, in the Benthamite case, limited commitment generates the drift in consumption when the participation constraint is slack, not just the jump in consumption when the participation constraint binds.

Let us now consider what happens in the Rawlsian case under full commitment.[22]

[20] While this $c_t(y^t)$ satisfies $U'(c_t(y^t)) = (1-\beta)/\alpha$, in general, it does not equal c_∞ in the limited commitment case, since α is an endogenous variable.

[21] See, e.g., Krueger and Uhlig (2006). When the utility function exhibits constant relative risk aversion in consumption, this implies a constant consumption growth rate.

[22] Under full commitment, calling a planner who maximizes $V^R = V^u$ as Rawlsian is somewhat misleading because, in the absence of the participation constraint, the unemployed workers need not be the least well-off agents. This observation, however, is inessential to our argument that limited commitment

This time, if $\psi_t(y^t) = 0$ for all y^t, then (26) and (27) imply $\lambda_t(y^t) = \beta^t\gamma$ for all y^t. Thus, consumption is common across y^1 and then falls over time, such that the marginal utility of consumption grows at rate $\beta^{-1} - 1 < 0$. Such a fall in consumption is the same as when the participation constraint is slack under limited commitment. Therefore, in the Rawlsian case, limited commitment plays no role in generating the drift in consumption; the planner chooses to front-load the consumption of workers because, given that workers discount the future, this is the least costly way to provide a given level of utility.

It is worthwhile explaining why limited commitment generates the drift in consumption in the Benthamite case but not in the Rawlsian case. The difference arises because, under limited commitment, the Benthamite planner has an incentive to lower V^u relative to the full commitment case; this is achieved by lowering $V_1^e(y^1)$ and, therefore, $c_1(y^1)$ for y^1 at which the participation constraint is slack. The resulting relaxation of the participation constraint improves efficiency and raises V^B. The Rawlsian planner, however, does not reap the benefit of lowering V^u to relax the participation constraint, since the objective function V^R is V^u itself.

We conclude this section by noting that the consumption dynamics described above also have implications on inequality among employed workers. Under the Benthamite planner, consumption inequality arises precisely because of limited commitment. It is easily observed that such inequality in consumption translates into inequality in welfare. In contrast, somewhat paradoxically, inequalities in consumption and welfare exist under the Rawlsian planner irrespective of the presence of the commitment problem.

4.4 Decentralization

Krueger (2000) discusses how, through an argument similar to that of Atkeson and Lucas (1992, 1995), the efficient allocation under limited commitment can be decentralized as an equilibrium. The idea is to consider financial intermediaries who compete in providing long-term insurance contracts to clients. These financial intermediaries freely borrow or lend the consumption good with other intermediaries at a gross interest rate R, and R adjusts to clear this consumption loan market.

We now explore the possibility of extending the argument above to decentralize the efficient allocation in our environment. Consider the following market economy. Financial intermediaries freely trade the good with other intermediaries at a gross interest rate R, and post vacancies in the labor market by paying the flow cost k per vacancy. When a financial intermediary and a worker are matched and are not immediately hit by an exogenous separation shock, they negotiate, before the realization of y^1, on the worker's

is not the source of the declining consumption path in the Rawlsian efficient allocation.

expected value of being newly employed, $\bar{V}_1^e$. The value of $\bar{V}_1^e$ is determined according to Nash bargaining, where $\eta \in (0, 1)$ is the worker's bargaining power. During the entire course of the match, the financial intermediary receives output and provides consumption to the worker in a way consistent with $\bar{V}_1^e$. In any period, the worker can walk away from the match with a fraction of the current output and become unemployed with a blank employment history, just as in the planner's problem. In equilibrium, the expected profit from posting a vacancy is zero, and R equals the value that clears the consumption loan market. This setup can be considered a hybrid of the setups of Krueger (2000) and Rudanko (2009).[23]

Note that for any $R > (1 - s)^{-1}$,[24] the financial intermediary's optimal contracting problem, which is to maximize the expected profit from a contract, given initial output y^1 and the value promised to the worker $V_1(y^1)$, can be analyzed using the standard recursive approach. In particular, the worker's marginal utility of consumption grows at rate $(\beta R)^{-1} - 1$ whenever the participation constraint is slack. Clearly, no value of R generates the rich consumption dynamics of the Benthamite case, so the Benthamite efficient allocation cannot be obtained as an equilibrium of this market economy.

Decentralization of the Rawlsian efficient allocation appears more promising, given its simple consumption dynamics. This turns out to be the case, as shown below.

Proposition 5 *If the Rawlsian efficient allocation satisfies $-\theta q'(\theta)/q(\theta) = \eta$, it can be supported as an equilibrium of the market economy described above; in this equilibrium, the gross interest rate $R = 1$.*

The condition $-\theta q'(\theta)/q(\theta) = \eta$ in Proposition 5 is the Hosios (1990) condition, which ensures the efficiency of equilibrium in a wide range of Mortensen–Pissarides-type models.[25] This condition turns out to play a critical role in the decentralization of the efficient allocation in our environment as well.

[23]Rudanko (2009) presents a directed search model in which financial intermediaries post long-term contracts and shows that, under the Hosios (1990) condition, the model is equivalent to a random matching model in which the worker's expected value of being newly employed is determined by Nash bargaining. If we fix R and do not require market clearing in the consumption loan market, the market economy described here becomes similar to that of Rudanko (2009). It may thus be possible to decentralize the efficient allocation in our environment using the directed search framework, but here we present a model with random matching and Nash bargaining, which is easier to describe.

[24]When there is no exogenous destruction of the principal–agent relationship, as in Krueger (2000), $R > 1$ is necessary to make the problem well defined. Note that since goods are perishable, negative interest rates ($R < 1$) are not inconsistent with equilibrium; Huggett (1993)'s model of self-insurance, which features a similar consumption loan market as the one here, exhibits negative interest rates under some parameters.

[25]As is well known, with a Cobb–Douglas matching function $M(u, v) = \mu u^{\kappa} v^{1-\kappa}$, this condition becomes $\kappa = \eta$, which is independent of θ.

5 Conclusion

In this paper, we examine efficient risk sharing in an environment featuring limited commitment and search frictions. Introducing the creation and destruction of the principal–agent relationship drastically changes the model's properties, and enables richer welfare analysis by providing room to introduce various social welfare functions. Most notably, we find striking consumption dynamics in the Benthamite case, summarized as follows.

Consumption in a match is initially low and then rises toward a certain level, c_∞, until the participation constraint binds for the first time. When the participation constraint binds, consumption jumps up; the resulting consumption level may exceed c_∞, in which case consumption subsequently falls toward c_∞ so long as the participation constraint is slack. After a sufficiently long employment spell, consumption exceeds c_∞ with probability one. Such a consumption dynamics contrasts starkly with the dynamics observed in Krueger (2000) and Atkeson and Lucas (1995).

While we introduce search frictions into a model of limited commitment, it would also be interesting to introduce search frictions into a model of private information. A variant of the variational approach adopted in this paper could also be useful in such an environment. We leave such analyses to future research.

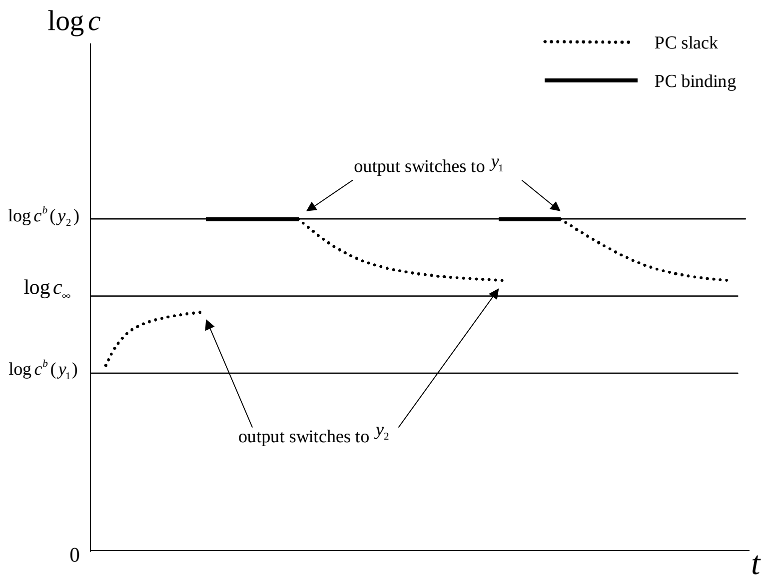

Figure 1: Consumption path under the Benthamite planner

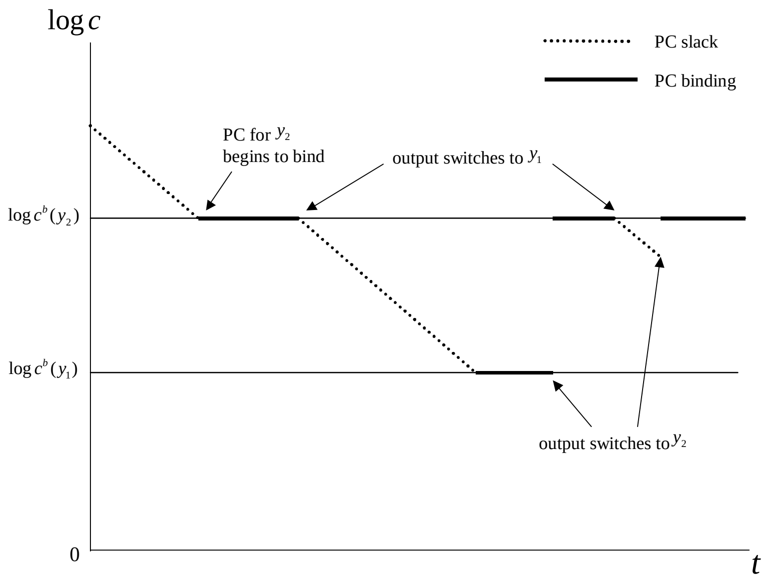

Figure 2: Consumption path under the Rawlsian planner

Appendix A: Sequential Forms of V^B and V^R

Appendix A derives the sequential expressions for V^B and V^R, given by (15) and (25), respectively.

To obtain (15), note from (2) and (4) that

$$
\begin{aligned}
& V^u + p(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) V_t^e\left(y^t\right) \\
&= U(b) + \beta p(\theta)(1-s) \sum_{y^1} \pi\left(y^1\right) V_1^e\left(y^1\right) + \beta\left[1 - p(\theta)(1-s)\right] V^u \\
&+ p(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) \left[U\left(c_t\left(y^t\right)\right) + \beta(1-s) \sum_{y^{t+1}|y^t} \frac{\pi\left(y^{t+1}\right)}{\pi\left(y^t\right)} V_{t+1}^e\left(y^{t+1}\right) + \beta s V^u\right] \\
&= U(b) + p(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) U\left(c_t\left(y^t\right)\right) \\
&+ \beta p(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) V_t^e\left(y^t\right) + \beta\left[1 - p(\theta)(1-s)\right] V^u + \beta p(\theta)(1-s) V^u.
\end{aligned}
$$

Rearranging and dividing by $1-\beta$, we obtain

$$
\begin{aligned}
& V^u + p(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) V_t^e\left(y^t\right) \\
&= \frac{1}{1-\beta}\left[U(b) + p(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) U\left(c_t\left(y^t\right)\right)\right], \qquad (35)
\end{aligned}
$$

which yields (15).

To obtain (25), rearrange (3) as

$$
\left[1 + \frac{\beta p(\theta)(1-s)}{1-\beta+\beta s}\right](1-\beta) V^u = U(b) + p(\theta) \sum_{t=1}^{\infty} \beta^t (1-s)^t \sum_{y^t} \pi\left(y^t\right) U\left(c_t\left(y^t\right)\right). \qquad (36)
$$

Then, (25) follows immediately from (36).

Appendix B: Lemmas Relevant for Both Cases

Appendix B collects lemmas that hold for the benchmark Benthamite efficient allocation, as well as for the Rawlsian efficient allocation discussed in Section 4. These lemmas will be used in the proofs of the propositions in Appendix C.

In what follows, let $dc_t(y^t)$ denote an infinitesimal perturbation in $c_t(y^t)$ and similarly for $d\theta$. Further, let $d_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to \mathcal{R}_+$ be the distance function on $\mathcal{D}$, where, for any $x, x' \in \mathcal{D}$,

$$d_{\mathcal{D}}(x, x') = \max \left\{ |\theta - \theta'|, \sup_{y^t} \left\| c_t\left(y^t\right) - c_t'\left(y^t\right) \right\| \right\}. \tag{37}$$

Then, from (5), (7), (15), and (25), clearly ED, $V_t^e(y^t)$, V^B, and $V^R = V^u$ are continuous functions from a metric space $(\mathcal{D}, d_{\mathcal{D}})$ to $\mathcal{R}$. Thus, the changes in these variables from an infinitesimal perturbation, denoted, for example, as $d(ED)$, are also infinitesimal.

Lemma A1 confirms a standard result in the literature.[26]

Lemma A1 *Let $\{R_t(y^t)\}_{t=1}^{\infty}$ be the sequence of history-contingent intertemporal relative prices, and recursively define $\tilde{R}_{t+1}(y^t, y_{t+1}) = \tilde{R}_t(y^t) R_t(y^t)$, where, for all y^1, $\tilde{R}_1(y^1) = R$ for some $R > (1-s)^{-1}$. Consider a component planner who takes V^u and $\{R_t(y^t)\}_{t=1}^{\infty}$ as given and chooses $\{c_{t+\tau}(y^{t+\tau})\}_{\tau=0}^{\infty}$ to minimize the expected resource cost $Q(V_t^e(y^t); y^t) = \sum_{\tau=0}^{\infty} (1-s)^{\tau} \sum_{y^{t+\tau}|y^t} \frac{\tilde{R}_t(y^t)}{\tilde{R}_{t+\tau}(y^{t+\tau})} \frac{\pi(y^{t+\tau})}{\pi(y^t)} c_{t+\tau}(y^{t+\tau})$ of providing $V_t^e(y^t)$ to a single worker with history y^t. Then, for any y^t, $Q(V_t^e(y^t); y^t)$ is increasing and strictly convex in $V_t^e(y^t)$ and, if $\{c_{t+\tau}(y^{t+\tau})\}_{\tau=0}^{\infty}$ achieves $Q(V_t^e(y^t); y^t)$, $Q'(V_t^e(y^t); y^t) = 1/U'(c_t(y^t))$.*

Proof. Take any y^t. That $Q(V_t^e(y^t); y^t)$ is increasing in $V_t^e(y^t)$ is immediate. To show that $Q\left(\hat{V}_t^e(y^t); y^t\right)$ is strictly convex in $V_t^e(y^t)$, suppose $\{c_{t+\tau}(y^{t+\tau})\}_{\tau=0}^{\infty}$ achieves $Q(V_t^e(y^t); y^t)$, and $\{\hat{c}_{t+\tau}(y^{t+\tau})\}_{\tau=0}^{\infty}$ achieves $Q\left(\hat{V}_t^e(y^t); y^t\right)$. Noting (5), the participation constraint implies that, at any $y^{t+\tau}$,

$$\sum_{\tau'=0}^{\infty} [\beta(1-s)]^{\tau'} \sum_{y^{t+\tau+\tau'}|y^{t+\tau}} \frac{\pi\left(y^{t+\tau+\tau'}\right)}{\pi\left(y^{t+\tau}\right)} U\left(c_{t+\tau+\tau'}\left(y^{t+\tau+\tau'}\right)\right) + \frac{\beta s}{1-\beta(1-s)} V^u \geq V^o(y_{t+\tau}), \tag{38}$$

$$\sum_{\tau'=0}^{\infty} [\beta(1-s)]^{\tau'} \sum_{y^{t+\tau+\tau'}|y^{t+\tau}} \frac{\pi\left(y^{t+\tau+\tau'}\right)}{\pi\left(y^{t+\tau}\right)} U\left(\hat{c}_{t+\tau+\tau'}\left(y^{t+\tau+\tau'}\right)\right) + \frac{\beta s}{1-\beta(1-s)} V^u \geq V^o(y_{t+\tau}), \tag{39}$$

[26] Below, we suppress the dependence of Q on V^u and $\{R_t(y^t)\}_{t=1}^{\infty}$ to avoid notational clutter.

where V^u and thus also $V^o(y_{t+\tau})$ are given.

Now, take any $\lambda \in (0,1)$, and let $V_t^{\lambda,e}(y^t) \equiv (1-\lambda)V_t^e(y^t)+\lambda\hat{V}_t^e(y^t)$ and $c_{t+\tau}^{\lambda}(y^{t+\tau}) \equiv (1-\lambda)c_{t+\tau}(y^{t+\tau})+\lambda\hat{c}_{t+\tau}(y^{t+\tau})$. From the strict concavity of U, $\left\{c_{t+\tau}^{\lambda}(y^{t+\tau})\right\}_{\tau=0}^{\infty}$ provides the worker strictly greater value than $V_t^{\lambda,e}(y^t)$; further, from (38) and (39),

$$\sum_{\tau'=0}^{\infty}[\beta(1-s)]^{\tau'}\sum_{y^{t+\tau+\tau'}|y^{t+\tau}}\frac{\pi\left(y^{t+\tau+\tau'}\right)}{\pi\left(y^{t+\tau}\right)}U\left(c_{t+\tau+\tau'}^{\lambda}\left(y^{t+\tau+\tau'}\right)\right)+\frac{\beta s}{1-\beta(1-s)}V^u > V^o(y_{t+\tau}), \tag{40}$$

which implies that $\left\{c_{t+\tau}^{\lambda}(y^{t+\tau})\right\}_{\tau=0}^{\infty}$ makes the participation constraint slack at any $y^{t+\tau}$. Thus, the planner can provide $V_t^{\lambda,e}(y^t)$ with fewer resources than $\left\{c_{t+\tau}^{\lambda}(y^{t+\tau})\right\}_{\tau=0}^{\infty}$, which implies

$$\begin{aligned}Q\left(V_t^{\lambda,e}\left(y^t\right);y^t\right) &< \sum_{\tau=0}^{\infty}(1-s)^{\tau}\sum_{y^{t+\tau}|y^t}\frac{\tilde{R}_t\left(y^t\right)}{\tilde{R}_{t+\tau}\left(y^{t+\tau}\right)}\frac{\pi\left(y^{t+\tau}\right)}{\pi\left(y^t\right)}c_{t+\tau}^{\lambda}\left(y^{t+\tau}\right)\\ &= (1-\lambda)Q\left(V_t^e\left(y^t\right);y^t\right)+\lambda Q\left(\hat{V}_t^e\left(y^t\right);y^t\right),\end{aligned}$$

where the equality follows from the definitions of $\{c_{t+\tau}(y^{t+\tau})\}_{\tau=0}^{\infty}$, $\{\hat{c}_{t+\tau}(y^{t+\tau})\}_{\tau=0}^{\infty}$, and $\left\{c_{t+\tau}^{\lambda}(y^{t+\tau})\right\}_{\tau=0}^{\infty}$. Therefore, Q is strictly convex.

To show that $Q'(V_t^e(y^t);y^t) = 1/U'(c_t(y^t))$,[27] take $\epsilon > 0$ sufficiently small such that there exist $c_-, c_+ \in (0,\infty)$ satisfying $U(c_-) = U(c_t(y^t)) - \epsilon$ and $U(c_+) = U(c_t(y^t)) + \epsilon$. Then, $\{c_-, c_{t+\tau}(y^{t+\tau})\}_{\tau=1}^{\infty}$ is an incentive-feasible but not necessarily optimal allocation that provides $V_t^e(y^t) - \epsilon$, while $\{c_+, c_{t+\tau}(y^{t+\tau})\}_{\tau=1}^{\infty}$ is an incentive-feasible but not necessarily optimal allocation that provides $V_t^e(y^t) + \epsilon$, and thus

$$Q\left(V_t^e\left(y^t\right)+\epsilon;y^t\right)-Q\left(V_t^e\left(y^t\right);y^t\right) \le c_+ - c_t\left(y^t\right), \tag{41}$$

$$Q\left(V_t^e\left(y^t\right);y^t\right)-Q\left(V_t^e\left(y^t\right)-\epsilon;y^t\right) \ge c_t\left(y^t\right) - c_-. \tag{42}$$

Further, $Q(V_t^e(y^t);y^t) < (1/2)Q(V_t^e(y^t)-\epsilon;y^t)+(1/2)Q(V_t^e(y^t)+\epsilon;y^t)$ from the strict convexity of Q, and thus

$$\frac{Q\left(V_t^e(y^t)+\epsilon;y^t\right)-Q\left(V_t^e(y^t);y^t\right)}{\epsilon} > \frac{Q\left(V_t^e(y^t);y^t\right)-Q\left(V_t^e(y^t)-\epsilon;y^t\right)}{\epsilon}. \tag{43}$$

[27] This part of the proof builds on the argument in Oyama (2013).

From (41)–(43),

$$\begin{aligned}\frac{c_+ - c_t\left(y^t\right)}{\epsilon} &\geq \frac{Q\left(V_t^e\left(y^t\right)+\epsilon; y^t\right) - Q\left(V_t^e\left(y^t\right); y^t\right)}{\epsilon} \\ &> \frac{Q\left(V_t^e\left(y^t\right); y^t\right) - Q\left(V_t^e\left(y^t\right)-\epsilon; y^t\right)}{\epsilon} \\ &\geq \frac{c_t\left(y^t\right) - c_-}{\epsilon}.\end{aligned}$$

Noting $dc_-/d\epsilon = -1/U'(c_-)$ and $dc_+/d\epsilon = 1/U'(c_+)$ and invoking the l'Hopital's rule, $\lim_{\epsilon\to 0}\left(c_+ - c_t\left(y^t\right)\right)/\epsilon = \lim_{\epsilon\to 0}\left(dc_+/d\epsilon\right) = 1/U'\left(c_t\left(y^t\right)\right)$, and $\lim_{\epsilon\to 0}\left(c_t\left(y^t\right) - c_-\right)/\epsilon = -\lim_{\epsilon\to 0}\left(dc_-/d\epsilon\right) = 1/U'\left(c_t\left(y^t\right)\right)$. The squeeze theorem thus implies Q is differentiable at $V_t^e\left(y^t\right)$ and $Q'\left(V_t^e\left(y^t\right); y^t\right) = 1/U'\left(c_t\left(y^t\right)\right)$ and, since this result holds for any $V_t^e\left(y^t\right)$, the claim follows. ■

Lemma A2 states that the resource constraint always binds in the planner's problems we analyze. This result is not obvious, since distributing extra resources may, by raising the outside option values, lead to the violation of the participation constraints. As proved below, however, there are always incentive-feasible ways to distribute extra resources.

Lemma A2 *In both the Benthamite and Rawlsian efficient allocations,* $ED = 0$.

Proof. Suppose $ED < 0$ for an efficient allocation x. Perturb x by $dc_t\left(y^t\right) = \Delta/U'\left(c_t\left(y^t\right)\right) > 0$ for all y^t. Clearly, the perturbed allocation satisfies the resource constraint since $dc_t\left(y^t\right)$ is infinitesimal and, for all y^t, raises $V_t^e\left(y^t\right)$ by an equal amount, which we denote by dV^e. Thus, from (2) and (4),

$$dV^u = \beta p(\theta)(1-s)\,dV^e + \beta\left[1 - p(\theta)(1-s)\right]dV^u, \tag{44}$$

$$dV^e = \Delta + \beta(1-s)\,dV^e + \beta s dV^u. \tag{45}$$

Subtracting (44) from (45), we obtain

$$\begin{aligned}dV^e - dV^u &= \Delta + \beta(1-s)(1-p(\theta))\left(dV^e - dV^u\right) \\ &= \frac{1}{1-\beta(1-s)(1-p(\theta))}\Delta \\ &> 0\end{aligned}$$

hence, $dV^e > dV^u$. Therefore, the perturbation raises, for all y^t, $V_t^e\left(y^t\right)$ by more than V^u (and thus V^o); hence, it does not violate any participation constraint. Further, since the perturbation raises consumption for all employed workers without altering θ, it increases the value of any social welfare function that respects the Pareto principle, including V^B and V^R. This contradicts the fact that x is an efficient allocation, so $ED = 0$. ■

An immediate corollary of Lemma A2 is that if x is the Benthamite (Rawlsian) efficient allocation, then there cannot be a feasible allocation x' that achieves the same V^B (V^R) as x with $ED < 0$. Such an x' would also be an efficient allocation, contradicting Lemma A2.

Lemma A3 states how, given an incentive-feasible allocation, a new incentive-feasible allocation can be obtained by perturbing consumption at some history y^τ and $(y^\tau, y_{\tau+1})$.

Lemma A3 *Let x be an incentive-feasible allocation. Take any history y^τ and $(y^\tau, y_{\tau+1})$, and perturb x by $(dc_\tau(y^\tau), dc_{\tau+1}(y^\tau, y_{\tau+1}))$, keeping V^u unchanged. In the perturbed allocation, the participation constraint holds at all histories, except possibly at $(y^\tau, y_{\tau+1})$; further, if the participation constraint is initially slack at $(y^\tau, y_{\tau+1})$, or if $dc_{\tau+1}(y^\tau, y_{\tau+1}) > 0$, the participation constraint holds at all histories.*

Proof. Let x and $(dc_\tau(y^\tau), dc_{\tau+1}(y^\tau, y_{\tau+1}))$ be as specified. Then, from (3),

$$\begin{aligned} 0 &= dV^u \\ &= p(\theta)\left[\beta^\tau (1-s)^\tau \pi(y^\tau) U'(c_\tau(y^\tau))\, dc_\tau(y^\tau)\right. \\ &\left.+\beta^{\tau+1}(1-s)^{\tau+1}\pi(y^\tau, y_{\tau+1}) U'(c_{\tau+1}(y^\tau, y_{\tau+1}))\, dc_{\tau+1}(y^\tau, y_{\tau+1})\right] \end{aligned} \tag{46}$$

and thus $dc_\tau(y^\tau)$ and $dc_{\tau+1}(y^\tau, y_{\tau+1})$ satisfy

$$dc_\tau(y^\tau) = -\beta(1-s)\frac{\pi(y^\tau, y_{\tau+1})}{\pi(y^\tau)}\frac{U'(c_{\tau+1}(y^\tau, y_{\tau+1}))}{U'(c_\tau(y^\tau))} dc_{\tau+1}(y^\tau, y_{\tau+1}). \tag{47}$$

Since V^u is unchanged in the perturbed allocation, (6) implies that neither is $V^o(y_t)$ for any $y_t \in Y$. Let us now consider how the perturbation affects $V_t^e(y^t)$ for different y^t. First, from (5) and (47), $V_\tau^e(y^\tau)$ is unchanged so the participation constraint still holds at y^τ, and the same applies to any history preceding y^τ. Next, this perturbation raises $V_{\tau+1}^e(y^\tau, y_{\tau+1})$ by $U'(c_{\tau+1}(y^\tau, y_{\tau+1}))\, dc_{\tau+1}(y^\tau, y_{\tau+1})$, whose sign coincides with that of $dc_{\tau+1}(y^\tau, y_{\tau+1})$. Thus, if $dc_{\tau+1}(y^\tau, y_{\tau+1}) > 0$, the participation constraint still holds at $(y^\tau, y_{\tau+1})$. In contrast, if $dc_{\tau+1}(y^\tau, y_{\tau+1}) < 0$, the participation constraint can be violated at $(y^\tau, y_{\tau+1})$; however, if the participation constraint is initially slack at $(y^\tau, y_{\tau+1})$, then it remains thus for infinitesimal $dc_{\tau+1}(y^\tau, y_{\tau+1})$. Finally, at all other history y^t, $V_t^e(y^t)$ remains constant given that V^u is unchanged, so the participation constraint continues to hold. ■

Appendix C: Proofs of Propositions

Appendix C provides the proofs of all the propositions.

Proof of Proposition1

The proof proceeds through a series of lemmas. For now, we take as given the existence of the Benthamite efficient allocation and provide its proof in Lemma A9.

Lemma A4 characterizes the efficient consumption path when the participation constraint is slack.

Lemma A4 *In the Benthamite efficient allocation, for any history y^τ and $(y^\tau, y_{\tau+1})$ such that the participation constraint is slack at $(y^\tau, y_{\tau+1})$, $1/U'(c_{\tau+1}(y^\tau, y_{\tau+1})) = \alpha + \beta/U'(c_\tau(y^\tau))$, where $\alpha > 0$ is the marginal cost of V^B.*

Proof. Let x be the Benthamite efficient allocation. Take any $(y^\tau, y_{\tau+1})$ at which the participation constraint is slack, and perturb x by $(dc_\tau(y^\tau), dc_{\tau+1}(y^\tau, y_{\tau+1}))$ while satisfying (47) or, equivalently, keeping V^u unchanged. Then, from Lemma A3, the perturbed allocation is incentive feasible.

From (7), (15), and (47), we obtain

$$d(ED) = \left(1 - \beta\frac{U'(c_{\tau+1}(y^\tau, y_{\tau+1}))}{U'(c_\tau(y^\tau))}\right) e_{\tau+1}(y^\tau, y_{\tau+1})\, dc_{\tau+1}(y^\tau, y_{\tau+1}), \tag{48}$$

$$dV^B = U'(c_{\tau+1}(y^\tau, y_{\tau+1}))\, e_{\tau+1}(y^\tau, y_{\tau+1})\, dc_{\tau+1}(y^\tau, y_{\tau+1}). \tag{49}$$

Dividing (48) by (49) reveals that the planner's marginal cost of increasing V^B through this particular perturbation is

$$\hat{\alpha}(y^\tau, y_{\tau+1}) \equiv \frac{1}{U'(c_{\tau+1}(y^\tau, y_{\tau+1}))} - \beta\frac{1}{U'(c_\tau(y^\tau))}. \tag{50}$$

Let $\alpha > 0$ be the marginal cost of V^B. Clearly $\hat{\alpha}(y^\tau, y_{\tau+1}) \geq \alpha$, because, in increasing V^B, the planner can do no worse than the perturbation above. Further, if $\hat{\alpha}(y^\tau, y_{\tau+1}) > \alpha$, then the marginal cost reduction from decreasing V^B through the reverse perturbation $(-dc_\tau(y^\tau), -dc_{\tau+1}(y^\tau, y_{\tau+1}))$ exceeds α. Then, combining the perturbation $(-dc_\tau(y^\tau), -dc_{\tau+1}(y^\tau, y_{\tau+1}))$ and the perturbation that increases V^B with marginal cost α yields a feasible allocation that achieves the same V^B as x with $ED < 0$. This contradicts the fact that x is the Benthamite efficient allocation, so $\hat{\alpha}(y^\tau, y_{\tau+1}) = \alpha$ and

$$\frac{1}{U'(c_{\tau+1}(y^\tau, y_{\tau+1}))} = \alpha + \beta\frac{1}{U'(c_\tau(y^\tau))}, \tag{51}$$

as was to be shown. ■

Lemma A5 provides the economic interpretation of $\lambda_t(y^t)$.

Lemma A5 *The direct marginal cost of* $V_t^e(y^t)$ *for the Benthamite planner is increasing in* $V_t^e(y^t)$ *and equals* $\lambda_t(y^t) = 1/U'(c_t(y^t))$.

Proof. Recall the component planner in Lemma A1, who takes V^u and $\{R_t(y^t)\}_{t=1}^{\infty}$ as given and minimizes the expected resource cost of providing $V_t^e(y^t)$ to a single worker with history y^t. The first-order conditions imply that such a component planner sets $U'(c_t(y^t)) = \beta R_t(y^t) U'(c_{t+1}(y^t, y_{t+1}))$ when the participation constraint is slack at (y^t, y_{t+1}), and $V_{t+1}^e(y^t, y_{t+1}) = V^o(y_{t+1})$ otherwise. This result and Lemma A4 imply that the consumption profile chosen by the Benthamite planner to provide $\{V_t^e(y^t)\}_{t=1}^{\infty}$, given V^u, will also be chosen by relevant component planners who face $R_t(y^t) = \alpha U'(c_t(y^t))/\beta + 1$ for all y^t. This, in turn, implies that the Benthamite planner values any $\{c_t(y^t)\}_{t=1}^{\infty}$ using prices $\{R_t(y^t)\}_{t=1}^{\infty}$, where $R_t(y^t) = \alpha U'(c_t(y^t))/\beta + 1$ for all y^t. By definition, the direct cost of $V_t^e(y^t)$ for the Benthamite planner is the minimized expected value, computed using such $\{R_t(y^t)\}_{t=1}^{\infty}$ and taking V^u as given, of the consumption profile that provides $V_t^e(y^t)$ to a single worker with history y^t; thus, it coincides with cost $Q(V_t^e(y^t); y^t)$ for the component planner who faces the same $\{R_t(y^t)\}_{t=1}^{\infty}$. The claim thus follows from Lemma A1. ■

The proof above reveals that the Benthamite planner's expected resource cost of providing any $V_t^e(y^t)$, taking V^u as given, to M workers is simply M times that for a single worker. Thus, as a normalization, we consider a single worker (or a measure one of workers) in defining the direct cost and direct marginal cost of $V_t^e(y^t)$.

Lemma A6 confirms the standard result in the limited commitment literature, namely, that a binding participation constraint raises consumption. Lemma A6 also shows that initial consumption is equalized across states in which the participation constraint is slack.

Lemma A6 *In the Benthamite efficient allocation, (i) for any history* $(y^\tau, y_{\tau+1})$ *and* $(y^\tau, \hat{y}_{\tau+1})$ *such that the participation constraint is slack at* $(y^\tau, y_{\tau+1})$ *and binds at* $(y^\tau, \hat{y}_{\tau+1})$, $c_{\tau+1}(y^\tau, y_{\tau+1}) < c_{\tau+1}(y^\tau, \hat{y}_{\tau+1})$ *and (ii) for any history* y^1 *and* $\hat{y}^1$ *such that the participation constraint is slack at* y^1 *and binds at* $\hat{y}^1$, $c_1(y^1) = \bar{c}_1 < c_1(\hat{y}^1)$, *where* $\bar{c}_1 > 0$

satisfies

$$-ku\frac{\theta q'(\theta)}{q(\theta)} = \frac{1}{\beta}\left(\frac{1}{U'(\bar{c}_1)} - \alpha\right) u\beta p'(\theta)(1-s)\left(\sum_{y^1}\pi\left(y^1\right)V_1^e\left(y^1\right) - V^u\right) \tag{52}$$
$$+ \alpha u p'(\theta)\sum_{t=1}^{\infty}(1-s)^t\sum_{y^t}\pi\left(y^t\right)V_t^e\left(y^t\right)$$
$$- \alpha u(1-u)\frac{p'(\theta)}{p(\theta)}\left[V^u + p(\theta)\sum_{t=1}^{\infty}(1-s)^t\sum_{y^t}\pi\left(y^t\right)V_t^e\left(y^t\right)\right].$$

Proof. Let x be the Benthamite efficient allocation. To prove that consumption is greater under a binding participation constraint ((i) and the inequality part of (ii)), note that when the participation constraint binds at y^t, the planner sets $V_t^e(y^t) = V^o(y_t)$, which exceeds the value chosen in the absence of a binding participation constraint. The proof is then immediate from Lemma A5.

To prove that consumption is equalized across y^1 at which the participation constraint is slack (the part $c_1(y^1) = \bar{c}_1$ of (ii)), take any y^1 and $\hat{y}^1$ at which the participation constraint is slack, and perturb x by $(dc_1(y^1), dc_1(\hat{y}^1))$, keeping V^u unchanged. Clearly, the perturbed allocation is incentive feasible. From (3), (8), and $dV^u = 0$, we obtain

$$-e_1\left(y^1\right)U'\left(c_1\left(y^1\right)\right)dc_1\left(y^1\right) = e_1\left(\hat{y}^1\right)U'\left(c_1\left(\hat{y}^1\right)\right)dc_1\left(\hat{y}^1\right), \tag{53}$$

which implies $dV^B = 0$ from (15). On the other hand, (7) and (53) imply

$$d(ED) = e_1\left(y^1\right)dc_1\left(y^1\right) + e_1\left(\hat{y}^1\right)dc_1\left(\hat{y}^1\right) = e_1\left(y^1\right)\left(1 - \frac{U'\left(c_1\left(y^1\right)\right)}{U'\left(c_1\left(\hat{y}^1\right)\right)}\right)dc_1\left(y^1\right). \tag{54}$$

Note from (54) that unless $c_1(y^1) = c_1(\hat{y}^1)$, $dc_1(y^1)$ can be set positive or negative to yield $d(ED) < 0$, resulting in a feasible allocation that achieves the same V^B as x with $ED < 0$. This contradicts the fact that x is the Benthamite efficient allocation, hence, $c_1(y^1) = c_1(\hat{y}^1) = \bar{c}_1$ for some $\bar{c}_1$.

To prove that $\bar{c}_1$ satisfies (52), take any $y^1 = y_1 = \bar{y}_m$ and (y^1, y_2) at which the participation constraint is slack[28], and perturb x by $(dc_1(y^1), dc_2(y^1, y_2), d\theta)$, keeping V^u and ED unchanged. Clearly, the perturbed allocation is feasible for infinitesimal

[28] If the participation constraint is slack at some $y^1 = y_1 = \bar{y}_m$, it is again slack at some (y^1, y_2), as explained below. As we show in Lemma A7 below, $U'(c_1(y^1)) \geq (1-\beta)/\alpha$. Thus, given Lemma A4, $c_2(y^1, y_2) \geq c_1(y^1)$ for (y^1, y_2) at which the participation constraint is slack. Therefore, the participation constraint is slack, for example, at $(y^1, y_2) = (\bar{y}_m, \bar{y}_m)$.

$(dc_1(y^1), dc_2(y^1, y_2), d\theta)$, so the efficiency of x requires the change in V^B to be zero. The proof proceeds by obtaining the expression for this condition.

From (15) and $dV^B = 0$, we obtain, after using (8) and multiplying by $1-\beta$,

$$
\begin{aligned}
0 &= e_1\left(y^1\right) U'\left(c_1\left(y^1\right)\right) dc_1\left(y^1\right) + e_2\left(y^1, y_2\right) U'\left(c_2\left(y^1, y_2\right)\right) dc_2\left(y^1, y_2\right) \\
&\quad + up'(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) U\left(c_t\left(y^t\right)\right) d\theta \\
&\quad - u(1-u) \frac{p'(\theta)}{p(\theta)} \left[U(b) + p(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) U\left(c_t\left(y^t\right)\right)\right] d\theta,
\end{aligned} \tag{55}
$$

where the third line on the RHS uses

$$
\frac{\partial u(\theta, s)}{d\theta} = -\frac{sp'(\theta)(1-s)}{\left[s + p(\theta)(1-s)\right]^2} = -u(1-u)\frac{p'(\theta)}{p(\theta)}, \tag{56}
$$

which follows from (9). Let us now express $dc_1(y^1)$ and $dc_2(y^1, y_2)$ in (55) using $d\theta$.

From (3) and $dV^u = 0$, we have

$$
\begin{aligned}
0 &= \beta p(\theta)(1-s)\pi\left(y^1\right) U'\left(c_1\left(y^1\right)\right) dc_1\left(y^1\right) \\
&\quad + \beta^2 p(\theta)(1-s)^2 \pi\left(y^1, y_2\right) U'\left(c_2\left(y^1, y_2\right)\right) dc_2\left(y^1, y_2\right) - \beta V^u p'(\theta) d\theta \\
&\quad + \beta p'(\theta) \sum_{t=1}^{\infty} \beta^{t-1} (1-s)^{t-1} \left[(1-s) \sum_{y^t} \pi\left(y^t\right) U\left(c_t\left(y^t\right)\right) + sV^u\right] d\theta,
\end{aligned}
$$

which, by multiplying by u and rearranging using (3) and (8), becomes

$$
\begin{aligned}
&\beta e_1\left(y^1\right) U'\left(c_1\left(y^1\right)\right) dc_1\left(y^1\right) + \beta^2 e_2\left(y^1, y_2\right) U'\left(c_2\left(y^1, y_2\right)\right) dc_2\left(y^1, y_2\right) \\
&= -u\frac{p'(\theta)}{p(\theta)} \left[(1-\beta) V^u - U(b)\right] d\theta.
\end{aligned} \tag{57}
$$

From (7) and $d(ED) = 0$, we have

$$
\begin{aligned}
0 &= e_1\left(y^1\right) dc_1\left(y^1\right) + e_2\left(y^1, y_2\right) dc_2\left(y^1, y_2\right) \\
&\quad + u\left[k - p'(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right) \left(y_t - c_t\left(y^t\right)\right)\right] d\theta,
\end{aligned} \tag{58}
$$

which, using (13), becomes

$$dc_1\left(y^1\right)=\frac{ku\frac{\theta q'(\theta)}{q(\theta)}}{e_1\left(y^1\right)}d\theta-\frac{e_2\left(y^1,y_2\right)}{e_1\left(y^1\right)}dc_2\left(y^1,y_2\right). \tag{59}$$

From (57) and (59), we obtain

$$dc_1\left(y^1\right)=-u\frac{\beta^2U'\left(c_2\left(y^1,y_2\right)\right)k\frac{\theta q'(\theta)}{q(\theta)}+\frac{p'(\theta)}{p(\theta)}\left[\left(1-\beta\right)V^u-U\left(b\right)\right]}{\beta e_1\left(y^1\right)\left(U'\left(c_1\left(y^1\right)\right)-\beta U'\left(c_2\left(y^1,y_2\right)\right)\right)}d\theta, \tag{60}$$

$$dc_2\left(y^1,y_2\right)=u\frac{\beta U'\left(c_1\left(y^1\right)\right)k\frac{\theta q'(\theta)}{q(\theta)}+\frac{p'(\theta)}{p(\theta)}\left[\left(1-\beta\right)V^u-U\left(b\right)\right]}{\beta e_2\left(y^1,y_2\right)\left(U'\left(c_1\left(y^1\right)\right)-\beta U'\left(c_2\left(y^1,y_2\right)\right)\right)}d\theta. \tag{61}$$

Further, rewriting (14) as

$$V^B=u\left[V^u+p\left(\theta\right)\sum_{t=1}^{\infty}\left(1-s\right)^t\sum_{y^t}\pi\left(y^t\right)V_t^e\left(y^t\right)\right] \tag{62}$$

and combining with (15) yields

$$\begin{aligned}&p\left(\theta\right)\sum_{t=1}^{\infty}\left(1-s\right)^t\sum_{y^t}\pi\left(y^t\right)U\left(c_t\left(y^t\right)\right)\\&=\left(1-\beta\right)V^u-U\left(b\right)+\left(1-\beta\right)p\left(\theta\right)\sum_{t=1}^{\infty}\left(1-s\right)^t\sum_{y^t}\pi\left(y^t\right)V_t^e\left(y^t\right).\end{aligned} \tag{63}$$

Substituting (60) and (61) into (55) and using (63), we obtain

$$\begin{aligned}0=&\frac{U'\left(c_1\left(y^1\right)\right)U'\left(c_2\left(y^1,y_2\right)\right)}{U'\left(c_1\left(y^1\right)\right)-\beta U'\left(c_2\left(y^1,y_2\right)\right)}ku\frac{\theta q'\left(\theta\right)}{q\left(\theta\right)}d\theta\\&+\frac{1}{1-\beta}u\left(1+\frac{1}{\beta}\frac{U'\left(c_2\left(y^1,y_2\right)\right)-U'\left(c_1\left(y^1\right)\right)}{U'\left(c_1\left(y^1\right)\right)-\beta U'\left(c_2\left(y^1,y_2\right)\right)}\right)\frac{p'\left(\theta\right)}{p\left(\theta\right)}\left[\left(1-\beta\right)V^u-U\left(b\right)\right]d\theta\\&+up'\left(\theta\right)\sum_{t=1}^{\infty}\left(1-s\right)^t\sum_{y^t}\pi\left(y^t\right)V_t^e\left(y^t\right)d\theta\\&-u\left(1-u\right)\frac{p'\left(\theta\right)}{p\left(\theta\right)}\left[V^u+p\left(\theta\right)\sum_{t=1}^{\infty}\left(1-s\right)^t\sum_{y^t}\pi\left(y^t\right)V_t^e\left(y^t\right)\right]d\theta.\end{aligned} \tag{64}$$

The participation constraint is slack at (y^1, y_2) by assumption, so (51) holds with $c_1(y^1)$

and $c_2(y^1, y_2)$ replacing $c_\tau(y^\tau)$ and $c_{\tau+1}(y^\tau, y_{\tau+1})$, respectively. Thus,

$$\frac{U'(c_1(y^1))\,U'(c_2(y^1,y_2))}{U'(c_1(y^1)) - \beta U'(c_2(y^1,y_2))} = \frac{1}{\frac{1}{U'(c_2(y^1,y_2))} - \frac{\beta}{U'(c_1(y^1))}} = \alpha^{-1}, \tag{65}$$

$$\frac{U'(c_2(y^1,y_2)) - U'(c_1(y^1))}{U'(c_1(y^1)) - \beta U'(c_2(y^1,y_2))} = \frac{\frac{1}{U'(c_1(y^1))} - \frac{1}{U'(c_2(y^1,y_2))}}{\frac{1}{U'(c_2(y^1,y_2))} - \frac{\beta}{U'(c_1(y^1))}} = \frac{1-\beta}{\alpha U'(c_1(y^1))} - 1. \tag{66}$$

Substituting (65) and (66) into (64), setting $c_1(y^1) = \bar{c}_1$, noting (2), and rearranging terms yields (52). ■

Lemma A7 links $\bar{c}_1$ with the direct marginal cost of V^u.

Lemma A7 *Let $\gamma > 0$ be the direct marginal cost of V^u for the Benthamite planner. Then, (i) $\gamma = (1/U'(\bar{c}_1) - \alpha)/\beta$, (ii) $\gamma \leq \alpha/(1-\beta)$, with strict inequality if there is any binding participation constraint.*

Proof. Let x be the Benthamite efficient allocation, and let $\gamma > 0$ be the direct marginal cost of V^u for the Benthamite planner. To prove (i), take any y^1 at which the participation constraint is slack, and perturb x by $dV_1^e(y^1) \neq 0$ while keeping V^u unchanged. Since the direct marginal cost of $V_1^e(y^1)$ is $\lambda_1(y^1)$, the planner's cost of providing $V_1^e(y^1)$ to measure $e_1(y^1)$ of workers with history y^1 rises by $e_1(y^1)\lambda_1(y^1)\,dV_1^e(y^1)$. As observed from (14), however, the perturbation directly raises V^B by $e_1(y^1)\,dV_1^e(y^1)$, whose value in resource units is $\alpha e_1(y^1)\,dV_1^e(y^1)$. Also, as observed from (2), the rise in $V_1^e(y^1)$ increases, taking V^u in future periods as given, V^u by $\beta p(\theta)(1-s)\pi_1(y^1)\,dV_1^e(y^1)$; hence, the cost of providing the same V^u as before to measure u of agents falls by $\gamma u \beta p(\theta)(1-s)\pi_1(y^1)\,dV_1^e(y^1) = \gamma\beta e_1(y^1)\,dV_1^e(y^1)$. Since the participation constraint is slack at y^1, both positive and negative $dV_1^e(y^1)$ are consistent with incentive feasibility; thus, net gains from such a perturbation must be zero, or

$$e_1(y^1)\lambda_1(y^1)\,dV_1^e(y^1) = \alpha e_1(y^1)\,dV_1^e(y^1) + \gamma\beta e_1(y^1)\,dV_1^e(y^1). \tag{67}$$

Dividing by $dV_1^e(y^1)$ and noting $\lambda_1(y^1) = 1/U'(\bar{c}_1)$ yields $\gamma = (1/U'(\bar{c}_1) - \alpha)/\beta$.

To prove (ii), again take any y^1 at which the participation constraint is slack, and perturb x by $(dc_1(y^1), dc_2(y^1, y_2))$, where the perturbation sustains $ED = 0$ and $dc_2(y^1, y_2) > 0$. From (7) and $d(ED) = 0$, we have

$$dc_1(y^1) = -\frac{e_2(y^1,y_2)}{e_1(y^1)}dc_2(y^1,y_2) = -(1-s)\frac{\pi(y^1,y_2)}{\pi(y^1)}dc_2(y^1,y_2). \tag{68}$$

To see that the perturbed allocation is incentive feasible, note from (25) that the perturbation raises V^u by

$$
\begin{aligned}
& dV^u \\
& = \frac{p(\theta)}{(1-\beta)\left[1+\frac{\beta p(\theta)(1-s)}{1-\beta+\beta s}\right]} \beta (1-s) \pi \left(y^1\right) U'\left(c_1\left(y^1\right)\right) dc_1\left(y^1\right) \\
& + \frac{p(\theta)}{(1-\beta)\left[1+\frac{\beta p(\theta)(1-s)}{1-\beta+\beta s}\right]} \beta^2 (1-s)^2 \pi\left(y^1, y_2\right) U'\left(c_2\left(y^1, y_2\right)\right) dc_2\left(y^1, y_2\right) \\
& = -\frac{p(\theta)}{(1-\beta)\left[1+\frac{\beta p(\theta)(1-s)}{1-\beta+\beta s}\right]} \beta (1-s)^2 \pi\left(y^1, y_2\right)\left(U'\left(c_1\left(y^1\right)\right) - \beta U'\left(c_2\left(y^1, y_2\right)\right)\right) dc_2\left(y^1, y_2\right),
\end{aligned}
$$

where the second equality uses (68). Since $U'\left(c_1\left(y^1\right)\right) > \beta U'\left(c_2\left(y^1, y_2\right)\right)$ from Lemmas A4 and A6, and since $dc_2\left(y^1, y_2\right) > 0$ by assumption, $dV^u < 0$. Further, (5) implies that for all y^t except this particular y^1, $V_t^e\left(y^t\right)$ falls by $\beta s/\left(1-\beta+\beta s\right) < 1$ times the fall in V^u; thus, $V_t^e\left(y^t\right)$ falls by less than V^o, so the participation constraint still holds. The participation constraint also holds at $\left(y^1, y_2\right)$, at which current consumption is increased. Finally, the participation constraint is initially slack at y^1, so it remains thus for an infinitesimal perturbation.

Thus, the perturbed allocation is feasible since it sustains $ED = 0$ and is incentive feasible. Therefore, the efficiency of x requires the change in V^B to be such that $dV^B \leq 0$. From (35) and (68), we obtain

$$
\begin{aligned}
dV^B &= \frac{1}{1-\beta}\left(e_1\left(y^1\right) U'\left(c_1\left(y^1\right)\right) dc_1\left(y^1\right) + e_2\left(y^1, y_2\right) U'\left(c_2\left(y^1, y_2\right)\right) dc_2\left(y^1, y_2\right)\right) \\
&= -\frac{1}{1-\beta} e_2\left(y^1, y_2\right)\left(U'\left(c_1\left(y^1\right)\right) - U'\left(c_2\left(y^1, y_2\right)\right)\right) dc_2\left(y^1, y_2\right). \qquad (69)
\end{aligned}
$$

Since $dV^B \leq 0$ and $dc_2\left(y^1, y_2\right) > 0$, we have $U'\left(c_1\left(y^1\right)\right) \geq U'\left(c_2\left(y^1, y_2\right)\right)$. In particular, if we take $\left(y^1, y_2\right)$ at which the participation constraint is slack, (51) holds with $c_1\left(y^1\right) = \bar{c}_1$ and $c_2\left(y^1, y_2\right)$ replacing $c_\tau\left(y^\tau\right)$ and $c_{\tau+1}\left(y^\tau, y_{\tau+1}\right)$, respectively. Therefore,

$$
\alpha + \beta \frac{1}{U'\left(c_1\left(y^1\right)\right)} = \frac{1}{U'\left(c_2\left(y^1, y_2\right)\right)} \geq \frac{1}{U'\left(c_1\left(y^1\right)\right)}, \qquad (70)
$$

hence $1/U'\left(c_1\left(y^1\right)\right) = 1/U'\left(\bar{c}_1\right) \leq \alpha/\left(1-\beta\right)$, and thus $\gamma = \left(1/U'\left(\bar{c}_1\right) - \alpha\right)/\beta \leq \alpha/\left(1-\beta\right)$.

Finally, suppose the participation constraint binds at some y^t. Then, if $dV^B = 0$, the perturbation sustains $ED = 0$ and keeps V^B unchanged, while lowering V^u and thus V^o. The resulting relaxation of the participation constraint enables achieving a greater value of V^B; this contradicts the fact that x is the Benthamite efficient allocation, so $dV^B < 0$. Thus, $U'(c_1(y^1)) > U'(c_2(y^1, y_2))$, such that the inequality in (70) becomes strict; hence, $\gamma < \alpha/(1-\beta)$. ■

Lemma A8 *Let $\psi_t(y^t)$ be as defined by (21) and (22). Then, $\psi_t(y^t)$ equals the shadow cost of the participation constraint at y^t for the Benthamite planner.*

Proof. Note from (21) that $\psi_{t+1}(y^t, y_{t+1})$ is positive if the participation constraint binds at y^t and zero otherwise. Now, if the participation constraint at (y^t, y_{t+1}) is slack, its shadow cost is zero. If it binds, its shadow cost is $1/U'\left(c^b(y_{t+1})\right) - (\alpha + \beta\lambda_t(y^t))$. To see this, note that as the binding participation constraint at (y^t, y_{t+1}) is tightened by one infinitesimal unit, $V^e_{t+1}(y^t, y_{t+1})$ must be increased by one infinitesimal unit. Thus, the shadow cost rises one to one with $1/U'\left(c^b(y_{t+1})\right)$, the direct marginal cost of $V^e_{t+1}(y^t, y_{t+1}) = V^o(y_{t+1})$, and vanishes as $\alpha + \beta\lambda_t(y^t)$ approaches $1/U'\left(c^b(y_{t+1})\right)$, where the participation constraint turns slack. This proves the claim for $\psi_{t+1}(y^t, y_{t+1})$, and a similar argument proves the claim for $\psi_1(y^1)$. ■

We now prove the existence of the Benthamite efficient allocation by invoking the Weierstrass theorem. Given the aggregate resource constraint, $\sup_{x\in D'} V^B$ is clearly finite, while it is not immediate that $\max_{x\in D'} V^B$ is well defined. The challenge here lies in showing the compactness of the set of feasible allocations, which is infinite dimensional. We achieve this by using the consumption rule above to restrict attention to a subset of feasible allocations, which is shown to have a homeomorphism to a finite-dimensional set.

Lemma A9 *The Benthamite efficient allocation exists.*

Proof. Let $\mathcal{D}'' \subset D'$ be the set of feasible allocations $x = (\theta, \{c_t(y^t)\}_{t=1}^{\infty})$ such that, for some $\hat{c}_1 \in C \equiv [\underline{c}^*, \bar{c}^*] \subset (0,\infty)$, $\underline{c}(\bar{y}_n) \in C$, $n = 1, 2, \ldots, N$, and $\hat{\alpha} \in [\underline{\alpha}^*, \bar{\alpha}^*]$, (i) $\theta \in [\underline{\theta}, \bar{\theta}] \subset (0,\infty)$, (ii) $c_1(y^1) = \max\{\hat{c}_1, \underline{c}(y_1)\}$ for all $y^1 = y_1$, and (iii) $c_{t+1}(y^t, y_{t+1}) = \max\left\{(U')^{-1}\left(\frac{1}{\hat{\alpha}+\beta/U'(c_t(y^t))}\right), \underline{c}(y_{t+1})\right\}$ for all (y^t, y_{t+1}). Here, $\underline{\theta}$ $(\bar{\theta})$ is a sufficiently small (large) constant such that the Benthamite planner never optimally chooses $\theta < \underline{\theta}$ $(\theta > \bar{\theta})$ and similarly for $\underline{c}^*$ $(\bar{c}^*)$, while $\underline{\alpha}^* \equiv 1/U'(\underline{c}^*) - \beta/U'(\bar{c}^*)$ and $\bar{\alpha}^* \equiv (1-\beta)/U'(\bar{c}^*)$.[29]

In words, $\mathcal{D}''$ is a set of feasible allocations with consumption rules that are compatible with those described in Lemmas A4–A6. Thus, if the Benthamite efficient allocation

[29] The restriction $\hat{\alpha} \geq \underline{\alpha}^*$ does not bind because, for any $\hat{\alpha} < \underline{\alpha}^*$ and $c_t(y^t) \leq \bar{c}^*$, we have $(U')^{-1}\left(\frac{1}{\hat{\alpha}+\beta/U'(c_t(y^t))}\right) \leq \underline{c}^* \leq \underline{c}(y_{t+1})$ and thus $c_{t+1}(y^t, y_{t+1}) = \underline{c}(y_{t+1})$, just as for $\hat{\alpha} = \underline{\alpha}^*$. In contrast, $\hat{\alpha} \leq \bar{\alpha}^*$ implies that if $c_t(y^t) \leq \bar{c}^*$, then $(U')^{-1}\left(\frac{1}{\hat{\alpha}+\beta/U'(c_t(y^t))}\right) \leq \bar{c}^*$ and thus ensures that consumption never exceeds $\bar{c}^*$; this is also not a binding restriction if $\bar{c}^*$ is set sufficiently large.

exists, it must belong to $\mathcal{D}''$. Importantly, despite being infinite dimensional, allocations in $\mathcal{D}''$ can be characterized by a finite number of variables, $\left(\theta, \hat{c}_1, \hat{\alpha}, \{\underline{c}(\bar{y}_n)\}_{n=1}^N\right)$. To see that $\mathcal{D}''$ is nonempty, consider allocations in which $c_t(y^t) = b$ for all y^t. Such allocations satisfy conditions (ii) and (iii) above for $\hat{c}_1 = b$, $\underline{c}(\bar{y}_n) = b, n \in \{1, 2, \ldots, N\}$, and $\hat{\alpha} = (1-\beta)/U'(b)$, and are incentive feasible since they yield $V_t^e(y^t) = V^u = U(b)/(1-\beta)$ for any y^t. Further, given $\bar{y}_n > b$ for all $\bar{y}_n \in Y$, such allocations with sufficiently small θ satisfy (11) and are thus also resource feasible; hence, $\mathcal{D}''$ is nonempty.

Below, we first prove that V^B attains its maximum in D'' by establishing the compactness of $(\mathcal{D}'', d_{\mathcal{D}''})$. For any $c \in C$, let $G(c) \equiv 1/U'(c)$ and $g(c) \equiv G'(c)$. Further, let $\underline{g} \equiv \min_{c \in C} g(c)$ and $\bar{g} \equiv \max_{c \in C} g(c)$. Clearly, $0 < \underline{g} < \bar{g} < \infty$. Then, for any $c, c' \in C$, the intermediate value theorem implies

$$G(c) - G(c') = g\left(c^\lambda\right)(c - c') \tag{71}$$

for some c^λ between c and c', and thus

$$\underline{g}\,|c - c'| \leq |G(c) - G(c')| \leq \bar{g}\,|c - c'|. \tag{72}$$

Now, define $f : \mathcal{D}'' \to \mathcal{R}^{N+3}$ by $f(x) = \left(\theta, \hat{c}_1, \hat{\alpha}, \{\underline{c}(\bar{y}_n)\}_{n=1}^N\right)$ for $x \in \mathcal{D}''$, where, for any $x \in \mathcal{D}''$ in which consumption is constant at some c, $\underline{c}(\bar{y}_n) \equiv c$ for all $n \in \{1, 2, \ldots, N\}$.[30] Further, let $\tilde{\mathcal{D}}'' \equiv f(\mathcal{D}'') \subset \mathcal{R}^{N+3}$. Now, define $d_{\mathcal{D}''} : \mathcal{D}'' \times \mathcal{D}'' \to \mathcal{R}_+$ by

$$d_{\mathcal{D}''}(x, x') = \max\left\{|\theta - \theta'|, \sup_{y^t} \left\|c_t\left(y^t\right) - c'_t\left(y^t\right)\right\|\right\} \tag{73}$$

for x and x' in $\mathcal{D}''$, and $d_{\tilde{\mathcal{D}}''} : \tilde{\mathcal{D}}'' \times \tilde{\mathcal{D}}'' \to \mathcal{R}_+$ by

$$d_{\tilde{\mathcal{D}}''}(z, z') = \max\left\{|\theta - \theta'|, |\hat{c}_1 - \hat{c}'_1|, [(1-\beta)\bar{g}]^{-1}|\hat{\alpha} - \hat{\alpha}'|, \max_{n \in \{1,2,\ldots,N\}} |\underline{c}(\bar{y}_n) - \underline{c}'(\bar{y}_n)|\right\} \tag{74}$$

for $z = \left(\theta, \hat{c}_1, \hat{\alpha}, \{\underline{c}(\bar{y}_n)\}_{n=1}^N\right)$ and $z' = \left(\theta', \hat{c}'_1, \hat{\alpha}', \{\underline{c}'(\bar{y}_n)\}_{n=1}^N\right)$ in $\tilde{\mathcal{D}}''$. Then, $d_{\mathcal{D}''}$ and $d_{\tilde{\mathcal{D}}''}$ are distance functions in $\mathcal{D}''$ and $\tilde{\mathcal{D}}''$, respectively.

Clearly, f is a continuous function from a metric space $(\mathcal{D}'', d_{\mathcal{D}''})$ to $\left(\tilde{\mathcal{D}}'', d_{\tilde{\mathcal{D}}''}\right)$; further, f is a bijection between $\mathcal{D}''$ to $\tilde{\mathcal{D}}''$, so f^{-1} exists. As shown below, f^{-1} is also a continuous

[30]For $x \in \mathcal{D}''$ in which consumption is constant at some c, $\{\underline{c}(\bar{y}_n)\}_{n=1}^N$ is not uniquely identified from condition (iii) above, since any $\{\underline{c}(\bar{y}_n)\}_{n=1}^N$ with $\underline{c}(\bar{y}_n) \leq c$, $n \in \{1, 2, \ldots, N\}$, is consistent with (iii). The assumption here on $\{\underline{c}(\bar{y}_n)\}_{n=1}^N$ is made simply to ensure that f is a function, not a correspondence, even in such a case.

function from $\left(\tilde{\mathcal{D}}'', d_{\tilde{\mathcal{D}}''}\right)$ to $(\mathcal{D}'', d_{\mathcal{D}''})$. Take any $\epsilon > 0$. Then, take $z, z' \in \tilde{\mathcal{D}}''$ such that $d_{\tilde{\mathcal{D}}''}(z, z') < \delta \equiv \left(\underline{g}/\bar{g}\right)\epsilon$, which implies $|\theta - \theta'| < \delta$, $|\hat{c}_1 - \hat{c}_1'| < \delta$, $|\hat{\alpha} - \hat{\alpha}'| < (1-\beta)\,\bar{g}\delta$, and $|\underline{c}(\bar{y}_n) - \underline{c}'(\bar{y}_n)| < \delta$, $n \in \{1, 2, \ldots, N\}$. The last inequality and (72) then imply

$$|G(\underline{c}(\bar{y}_n)) - G(\underline{c}'(\bar{y}_n))| < \bar{g}\delta,\ n \in \{1, 2, \ldots, N\}. \tag{75}$$

Now, let $x \equiv f^{-1}(z)$ and $x' \equiv f^{-1}(z')$. Since $|\theta - \theta'| < \delta < \epsilon$, if we can show $|c_t(y^t) - c_t'(y^t)| < \epsilon$ for all y^t, then $d_{\mathcal{D}''}(x, x') < \epsilon$ and thus f^{-1} is continuous. We show this by induction. First, for any y^1,

$$\begin{aligned} c_1\left(y^1\right) - c_1'\left(y^1\right) &= \max\{\hat{c}_1, \underline{c}(y_1)\} - \max\{\hat{c}_1', \underline{c}'(y_1)\} \\ &\le \max\{\hat{c}_1 - \hat{c}_1', \underline{c}(y_1) - \underline{c}'(y_1)\} \\ &< \delta. \end{aligned}$$

By symmetry, $c_1'(y^1) - c_1(y^1) < \delta$, so $|c_1(y^1) - c_1'(y^1)| < \delta$. Therefore, (72) implies $|G(c_1(y^1)) - G(c_1'(y^1))| < \bar{g}\delta$.

Next, take any y^t, and suppose $|G(c_t(y^t)) - G(c_t'(y^t))| < \bar{g}\delta$. Let $\hat{c}_{t+1} \equiv (U')^{-1}\left(\frac{1}{\hat{\alpha}+\beta/U'(c_t(y^t))}\right)$ and $\hat{c}_{t+1}' \equiv (U')^{-1}\left(\frac{1}{\hat{\alpha}'+\beta/U'(c_t'(y^t))}\right)$. Then, $c_{t+1}(y^t, y_{t+1}) = \max\{\hat{c}_{t+1}, \underline{c}(y_{t+1})\}$ and $c_{t+1}'(y^t, y_{t+1}) = \max\left\{\hat{c}_{t+1}', \underline{c}'(y_{t+1})\right\}$ for any $y_{t+1} \in Y$. Since G is an increasing function, these expressions can be rewritten as $G(c_{t+1}(y^t, y_{t+1})) = \max\{G(\hat{c}_{t+1}), G(\underline{c}(y_{t+1}))\}$ and $G\left(c_{t+1}'(y^t, y_{t+1})\right) = \max\left\{G\left(\hat{c}_{t+1}'\right), G(\underline{c}'(y_{t+1}))\right\}$. Thus,

$$\begin{aligned} &G\left(c_{t+1}\left(y^t, y_{t+1}\right)\right) - G\left(c_{t+1}'\left(y^t, y_{t+1}\right)\right) \\ &= \max\{G(\hat{c}_{t+1}), G(\underline{c}(y_{t+1}))\} - \max\left\{G\left(\hat{c}_{t+1}'\right), G(\underline{c}'(y_{t+1}))\right\} \\ &\le \max\left\{G(\hat{c}_{t+1}) - G\left(\hat{c}_{t+1}'\right), G(\underline{c}(y_{t+1}) - G(\underline{c}'(y_{t+1})))\right\} \\ &< \bar{g}\delta. \end{aligned}$$

Here, the last inequality follows since the definitions of $\hat{c}_{t+1}$ and $\hat{c}_{t+1}'$ imply $G(\hat{c}_{t+1}) = \hat{\alpha} + \beta G(c_t(y^t))$ and $G\left(\hat{c}_{t+1}'\right) = \hat{\alpha}' + \beta G(c_t'(y^t))$, which, combined with $|\hat{\alpha} - \hat{\alpha}'| < (1-\beta)\,\bar{g}\delta$ and $|G(c_t(y^t)) - G(c_t'(y^t))| \le \bar{g}\delta$, yield

$$\begin{aligned} G(\hat{c}_{t+1}) - G\left(\hat{c}_{t+1}'\right) &= \hat{\alpha} - \hat{\alpha}' + \beta\left(G\left(c_t\left(y^t\right)\right) - G\left(c_t'\left(y^t\right)\right)\right) \\ &< (1-\beta)\,\bar{g}\delta + \beta\bar{g}\delta = \bar{g}\delta, \end{aligned}$$

while (75) implies $|G(\underline{c}(y_{t+1}) - G(\underline{c}'(y_{t+1})))| < \bar{g}\delta$. By symmetry, $G\left(c_{t+1}'(y^t, y_{t+1})\right) - G(c_{t+1}(y^t, y_{t+1})) < \bar{g}\delta$ and thus $\left|G(c_{t+1}(y^t, y_{t+1})) - G\left(c_{t+1}'(y^t, y_{t+1})\right)\right| < \bar{g}\delta$.

Therefore, by induction, $|G(c_t(y^t)) - G(c'_t(y^t))| < \bar{g}\delta$ for any y^t. Combining this result with the first inequality in (72) yields

$$\left|c_t\left(y^t\right) - c'_t\left(y^t\right)\right| < \frac{\bar{g}}{\underline{g}}\delta = \epsilon, \ \forall y^t.$$

Thus, $d_{\mathcal{D}''}(x, x') < \epsilon$ for any $z, z' \in \tilde{\mathcal{D}}''$ with $d_{\tilde{\mathcal{D}}''}(z, z') < \delta = \left(\underline{g}/\bar{g}\right)\epsilon$, so f^{-1} is continuous.

Now, $\left(\tilde{\mathcal{D}}'', d_{\tilde{\mathcal{D}}''}\right)$ is bounded by assumption and is also closed, as explained below. Take any convergent sequence $\{x_k\}_{k=1}^{\infty}$ such that $x_k \in \mathcal{D}''$ for all k, and suppose $x_\infty \notin \mathcal{D}''$. Then, x_∞ must violate at least one of the participation constraints or the resource constraint, because as a limit of $\{x_k\}_{k=1}^{\infty}$, x_∞ clearly satisfies conditions (i)–(iii) above. However, since the expressions in these constraints are continuous functions from $(\mathcal{D}'', d_{\mathcal{D}''})$ to $\mathcal{R}$,[31] the constraint violated at x_∞ must also be violated at x_k for k sufficiently large. This contradicts $x_k \in \mathcal{D}''$, so $x_\infty \in \mathcal{D}''$ and thus $\mathcal{D}''$ is closed. Then, since f^{-1} is continuous, $\tilde{\mathcal{D}}'' = f(\mathcal{D}'')$ is also closed. Therefore, $\left(\tilde{\mathcal{D}}'', d_{\tilde{\mathcal{D}}''}\right)$ is compact, because it is closed and bounded and is a subset of $\mathcal{R}^{N+3}$ equipped with a metric corresponding to $d_{\tilde{\mathcal{D}}''}$. Then, $(\mathcal{D}'', d_{\mathcal{D}''})$ is also compact since f^{-1} is continuous and $\mathcal{D}'' = f^{-1}\left(\tilde{\mathcal{D}}''\right)$.[32]

Therefore, since V^B is a continuous function from $(\mathcal{D}'', d_{\mathcal{D}''})$ to $\mathcal{R}$,[33] the Weierstrass theorem implies that V^B attains its maximum in D''. It remains to show that $\max_{x \in D''} V^B = \sup_{x \in D'} V^B$, which in turn implies that $\max_{x \in D'} V^B$ exists and equals $\max_{x \in D''} V^B$. To see this, suppose $\varepsilon \equiv \left(\sup_{x \in D'} V^B - \max_{x \in D''} V^B\right)/2 > 0$, and let $\varepsilon_m \equiv \varepsilon/(m+1)$, $m = 1, 2, \ldots$. From the definition of the supremum, for each m, there exists some $x_m \in D'$ that provides the value of V^B, denoted as V^B_m, with $V^B_m > \sup_{x \in D'} V^B - \varepsilon_m$. In fact, $x_m \in D'/\mathcal{D}''$, since $V^B_m > \max_{x \in D''} V^B$ by construction. Now, for any allocation in $D'/\mathcal{D}''$, the Benthamite planner can provide the same V^B with fewer resources by resorting to the perturbations discussed in the proof of Lemmas A4 and A6. As m increases and ε_m approaches zero, such resources savings from perturbations must also approach zero, because otherwise the planner could use those saved resources to increase V^B_m by more than ε_m and thus provide V^B that exceeds $\sup_{x \in D'} V^B$. This requires, for sufficiently large m, that x_m become arbitrarily close to some $\hat{x}_m \in \mathcal{D}''$. The continuity of V^B then implies that, for sufficiently large m, V^B_m must also become arbitrarily close to $\hat{V}^B_m$, the value of V^B achieved by $\hat{x}_m$. In particular, $\hat{V}^B_m > V^B_m - \varepsilon$, hence combined

[31] To elaborate, the expressions in these constraints are clearly continuous functions from $(\mathcal{D}, d_{\mathcal{D}})$ to $\mathcal{R}$. Also, since $\mathcal{D}'' \subseteq \mathcal{D}$ and $d_{\mathcal{D}''}(x, x') = d_{\mathcal{D}}(x, x')$ for all $x, x' \in \mathcal{D}''$, $(\mathcal{D}'', d_{\mathcal{D}''})$ is a metric subspace of $(\mathcal{D}, d_{\mathcal{D}})$ (and, viewed as a topological space, $(\mathcal{D}'', d_{\mathcal{D}''})$ is a topological subspace of $(\mathcal{D}, d_{\mathcal{D}})$). Thus, the restrictions of these expressions to $(\mathcal{D}'', d_{\mathcal{D}''})$ are also continuous (see Munkres (2000), Theorem 18.2(d)).

[32] Note that the preimage of a continuous function on a closed set is closed, and the image of a continuous function of a compact set is compact (see Rudin (1976), Corollary to Theorem 4.8, and Theorem 4.14).

[33] The argument is similar to that in footnote 31.

with $V_m^B > \sup_{x\in D'} V^B - \varepsilon_m$ and $\varepsilon > \varepsilon_m$, we have

$$\hat{V}_m^B > \sup_{x\in D'} V^B - (\varepsilon_m + \varepsilon) > \sup_{x\in D'} V^B - 2\varepsilon = \max_{x\in D''} V^B, \tag{76}$$

which is a contradiction since $\hat{x}_m \in \mathcal{D}''$. Therefore, $\varepsilon = 0$ and thus $\max_{x\in D''} V^B = \max_{x\in D'} V^B = \sup_{x\in D'} V^B$, and the maximizer of V^B in $\mathcal{D}''$, whose existence is proved above, is the Benthamite efficient allocation. ■

Proposition 1 is proved by combining the results above. Let x be the Benthamite efficient allocation. To show (17), take any y^t and (y^t, y_{t+1}). If the participation constraint is slack at (y^t, y_{t+1}), then Lemma A4 and (16) imply $\lambda_{t+1}(y^t, y_{t+1}) = \alpha + \beta\lambda_t(y^t)$. If the participation constraint binds at (y^t, y_{t+1}), then $c_{t+1}(y^t, y_{t+1}) = c^b(y_{t+1})$, hence $\lambda_{t+1}(y^t, y_{t+1}) = 1/U'\left(c^b(y_{t+1})\right)$. Combining these results with Lemma A6 and noting (21) yields (17). Similarly, combining Lemmas A6 and A7 and noting (22) yields (18).

Next, (19) follows from (52) by noting $\gamma = (1/U'(\bar{c}_1) - \alpha)/\beta$ from Lemma A7.

To obtain (20), perturb x by $dV^u \neq 0$ while keeping $V_t^e(y^t)$ unchanged for all y^t. From (2), note that the change in V^u from a given change in some $V_1^e(y^1)$, which takes into account the effect of the change in V^u in future periods on the current V^u, is $1/[(1-\beta) + \beta p(\theta)(1-s)]$ times that when the values of V^u in future periods are taken as given. Accordingly, the increase in the planner's cost of providing V^u to a measure u of unemployed workers, which takes into account the effect of the change in V^u in future periods, is $u\gamma[1-\beta+\beta p(\theta)(1-s)]\,dV^u$. On the other hand, the increase in V^u by dV^u directly raises V^B by udV^u, whose value in terms of resources is αudV^u. Further, as seen from (4), the increase in V^u raises each $V_t^e(y^t)$ by βsdV^u; thus, the cost of providing the same $V_t^e(y^t)$ as before to a measure $e_t(y^t)$ of workers with history y^t falls by $e_t(y^t)\,\beta s\lambda_t(y^t)\,dV^u$. Finally, the increase in V^u tightens the participation constraint at each y^t by dV^u, whose cost is $e_t(y^t)\,\psi_t(y^t)\,dV^u$. Since dV^u can be made positive or negative, the efficiency of x requires the net gain from such a perturbation to be zero, or

$$u\gamma[1-\beta+\beta p(\theta)(1-s)]\,dV^u = \alpha udV^u + \sum_{t=1}^{\infty}\sum_{y^t} e_t\left(y^t\right)\left(\beta s\lambda_t\left(y^t\right) - \psi_t\left(y^t\right)\right)dV^u. \tag{77}$$

Dividing (77) by dV^u yields (20).

Finally, that α, $\lambda_t(y^t)$, γ, and $\psi_t(y^t)$ have the properties described in Proposition 1 is shown in Lemmas A4, A5, A7, and A8. ■

Proof of Proposition 2

To prove (i), note from Lemma A6 that for all y^1 at which the participation constraint is slack, $c_1(y^1) = \bar{c}_1$ for some $\bar{c}_1$. Moreover, by assumption, the participation constraint binds at some y^t, so, from Lemma A7, $\gamma < \alpha/(1-\beta)$. Therefore, $\lambda_1(y^1) = \alpha + \beta\gamma < \alpha/(1-\beta) = 1/U'(c_\infty)$ and thus $\bar{c}_1 < c_\infty$.

To prove (ii), suppose the participation constraint is slack at $(y^\tau, y_{\tau+1})$. From Lemma A4, (51) holds and thus noting $1/U'(c_\infty) = \alpha/(1-\beta)$, it follows that if $c_\tau(y^\tau) < c_\infty$, then $c_{\tau+1}(y^\tau, y_{\tau+1}) \in (c_\tau(y^\tau), c_\infty)$ whereas if $c_\tau(y^\tau) > c_\infty$, then $c_{\tau+1}(y^\tau, y_{\tau+1}) \in (c_\infty, c_\tau(y^\tau))$.

To prove (iii), combine (17) and (18) to obtain

$$\sum_{t=1}^{\infty}(1-s)^t \sum_{y^t} \pi(y^t)\lambda_t(y^t)$$
$$= \frac{1}{1-\beta(1-s)}\left[\frac{1-s}{s}\alpha + (1-s)\gamma\beta + \sum_{t=1}^{\infty}(1-s)^t\sum_{y^t}\pi(y^t)\psi_t(y^t)\right]. \quad (78)$$

Substituting (78) into (20) yields

$$u\gamma + \sum_{t=1}^{\infty}\sum_{y^t} e_t(y^t)\lambda_t(y^t) = \frac{\alpha}{1-\beta}. \quad (79)$$

Since $\gamma < \alpha/(1-\beta)$ and $u + \sum_{t=1}^{\infty}\sum_{y^t} e_t(y^t) = 1$, (79) requires $\lambda_t(y^t) > \alpha/(1-\beta)$ and thus $c_t(y^t) > c_\infty$ for some y^t. From (i) and (ii), however, until the participation constraint binds for the first time, $c_t(y^t) < c_\infty$ and thus $\lambda_t(y^t) < 1/U'(c_\infty) = \alpha/(1-\beta)$. Therefore, there must be at least one state $\bar{y}_n \in Y$ for which a binding participation constraint raises consumption above c_∞, or $c^b(\bar{y}_n) > c_\infty$. ■

Proof of Proposition 3

The proof is simpler than for Proposition 1 because, given $V^R = V^u$, we have one fewer variable for which to examine the effect of a given perturbation. The proof proceeds through Lemmas A10–A15, which parallel Lemmas A4–A9.

Lemma A10 *In the Rawlsian efficient allocation,* $U'(c_\tau(y^\tau)) = \beta U'(c_{\tau+1}(y^\tau, y_{\tau+1}))$ *for any history* y^τ *and* $(y^\tau, y_{\tau+1})$ *such that the participation constraint is slack at* $(y^\tau, y_{\tau+1})$.

Proof. Let x be the Rawlsian efficient allocation. Take any $(y^\tau, y_{\tau+1})$, and perturb x by $(dc_\tau(y^\tau), dc_{\tau+1}(y^\tau, y_{\tau+1}))$, keeping V^u unchanged. If the participation constraint is initially slack at $(y^\tau, y_{\tau+1})$, then from Lemma A3, the perturbed allocation is incentive feasible. Then, the perturbation above must sustain $ED = 0$. This is because if $ED < 0$, the perturbed allocation is feasible and achieves the same $V^R = V^u$ as x; hence, it is also a Rawlsian efficient allocation, contradicting Lemma A2. If $ED > 0$, a similar contradiction follows by perturbing x by $(-dc_\tau(y^\tau), -dc_{\tau+1}(y^\tau, y_{\tau+1}))$. From (7),

$$d(ED) = e_\tau(y^\tau)\, dc_\tau(y^\tau) + e_{\tau+1}(y^\tau, y_{\tau+1})\, dc_{\tau+1}(y^\tau, y_{\tau+1}), \tag{80}$$

so letting $d(ED) = 0$ and using (47) and $e_{\tau+1}(y^\tau, y_{\tau+1}) = e_\tau(y^\tau)(1-s)\,\pi(y^\tau, y_{\tau+1})/\pi(y^\tau)$,

$$\left(1 - \beta \frac{U'(c_{\tau+1}(y^\tau, y_{\tau+1}))}{U'(c_\tau(y^\tau))}\right) e_{\tau+1}(y^\tau, y_{\tau+1})\, dc_{\tau+1}(y^\tau, y_{\tau+1}) = 0. \tag{81}$$

Therefore, $U'(c_\tau(y^\tau)) = \beta U'(c_{\tau+1}(y^\tau, y_{\tau+1}))$, as was to be shown. ■

Lemma A11 *The direct marginal cost of $V_t^e(y^t)$ for the Rawlsian planner is increasing in $V_t^e(y^t)$ and equals $\lambda_t(y^t) = 1/U'(c_t(y^t))$.*

Proof. The proof follows by proceeding as in Lemma A5, and noting from Lemma A10 that the consumption profile chosen by the Rawlsian planner to provide $V_t^e(y^t)$, given V^u, will also be chosen by the relevant component planner who faces $R_t(y^t) = 1$ for all y^t. ■

Lemma A12 *In the Rawlsian efficient allocation, the conclusions of Lemma A6 hold, with $\bar{c}_1$ replaced by some $\tilde{c}_1$ and (52) replaced by*

$$-ku\frac{\theta q'(\theta)}{q(\theta)} = u\frac{1}{\beta U'(\tilde{c}_1)}\beta p'(\theta)(1-s)\left(\sum_{y^1} \pi(y^1)\, V_1^e(y^1) - V^u\right). \tag{82}$$

Proof. That consumption is greater under a binding participation constraint follows by arguing as in the proof of Lemma A6 and noting Lemma A11.

That consumption equals $\tilde{c}_1$ for all y^1 at which the participation constraint is slack also follows from a similar argument as in the proof of Lemma A6.

To prove that $\tilde{c}_1$ satisfies (82), let x be the Rawlsian efficient allocation. Take any y^1 such that the participation constraint is slack, and perturb x by $(d\theta, dc_1(y^1))$, keeping

V^u unchanged. From (3),

$$
\begin{aligned}
0 &= dV^u \\
&= \beta p'(\theta) \sum_{t=1}^{\infty} \beta^{t-1} (1-s)^{t-1} \left[(1-s) \sum_{y^t} \pi\left(y^t\right) U\left(c_t\left(y^t\right)\right) + sV^u \right] d\theta \qquad (83) \\
&+ \beta p(\theta)(1-s)\pi\left(y^1\right) U'\left(c_1\left(y^1\right)\right) dc_1\left(y^1\right) - \beta V^u p'(\theta)\, d\theta,
\end{aligned}
$$

so using (3) and cancelling out terms,

$$
\beta p(\theta)(1-s)\pi\left(y^1\right) U'\left(c_1\left(y^1\right)\right) dc_1\left(y^1\right) = -\frac{p'(\theta)}{p(\theta)} \left[(1-\beta) V^u - U(b)\right] d\theta. \qquad (84)
$$

Since V^u is unchanged in the perturbed allocation, neither is $V^o(y_t)$ for any $y_t \in Y$. From (5), for all y^t except this y^1, clearly $V_t^e(y^t)$ is unchanged, so the participation constraint still holds. On the other hand, $V_1^e(y^1)$ rises by $U'(c_1(y^1))\, dc_1(y^1)$, which is negative if $dc_1(y^1) < 0$; however, since the participation constraint is initially slack at y^1, it remains thus for an infinitesimal $dc_1(y^1)$.

Then, from a similar argument as in the proof of Lemma A10, the perturbation above must sustain $ED = 0$. From (7),

$$
\begin{aligned}
d(ED) &= u \left[k - p'(\theta) \sum_{t=1}^{\infty} (1-s)^t \sum_{y^t} \pi\left(y^t\right)\left(y_t - c_t\left(y^t\right)\right) \right] d\theta + e\left(y^1\right) dc_1\left(y^1\right) \\
&= -u \frac{\theta q'(\theta)}{q(\theta)} k d\theta - u \frac{1}{\beta U'\left(c_1\left(y^1\right)\right)} \beta p'(\theta)(1-s) \left(\sum_{y^1} \pi\left(y^1\right) V_1^e\left(y^1\right) - V^u \right) d\theta,
\end{aligned}
$$

where the second equality uses (2), (13), and (84). Imposing $d(ED) = 0$ and setting $c_1(y^1) = \tilde{c}_1$ yields (82). ■

Lemma A13 *Let $\gamma > 0$ be the direct marginal cost of V^u for the Rawlsian planner. Then, $\gamma = 1/\left(\beta U'(\tilde{c}_1)\right)$.*

Proof. The proof follows by proceeding as in the proof of Lemma A7(i) and noting that, since an increase in $V_t^e(y^t)$ has no direct impact on $V^R = V^u$, (67) is replaced by

$$
e_1\left(y^1\right) \lambda_1\left(y^1\right) dV_1^e\left(y^1\right) = \gamma \beta e_1\left(y^1\right) dV_1^e\left(y^1\right). \qquad (85)
$$

Dividing by $dV_1^e(y^1)$ and noting $\lambda_1(y^1) = 1/U'(\tilde{c}_1)$ yields $\gamma = 1/\left(\beta U'(\tilde{c}_1)\right)$. ■

Lemma A14 *Let $\psi_t(y^t)$ be as defined by (29) and (30). Then, $\psi_t(y^t)$ equals the shadow cost of the participation constraint at y^t for the Rawlsian planner.*

Proof. The proof follows by replacing (21) with (29) and $\alpha + \beta\lambda_t(y^t)$ with $\beta\lambda_t(y^t)$ in the proof of Proposition A8. ■

Lemma A15 *The Rawlsian efficient allocation exists.*

Proof. The proof follows from a similar argument as for Lemma A9, with $\hat{\alpha}$ set to 0. ■

Proposition 3 is proved by combining the results above. Let x be the Rawlsian efficient allocation. Take any y^t and (y^t, y_{t+1}). If the participation constraint is slack at (y^t, y_{t+1}), then Lemma A10 and (16) imply $\lambda_{t+1}(y^t, y_{\tau+1}) = \beta\lambda_t(y^t)$. If the participation constraint binds at (y^t, y_{t+1}), then $c_{t+1}(y^t, y_{\tau+1}) = c^b(y_{t+1})$, hence $\lambda_{t+1}(y^t, y_{\tau+1}) = 1/U'\left(c^b(y_{t+1})\right)$. Combining these results with Lemma A12 and noting (29) yields (26). Similarly, combining Lemmas A12 and A13 and noting (30) yields (27).

Next, (28) follows from (82) by noting $\gamma = 1/(\beta U'(\tilde{c}_1))$ from Lemma A13.

Finally, that $\lambda_t(y^t)$, γ, and $\psi_t(y^t)$ have the properties described in Proposition 3 is shown in Lemmas A11, A13, and A14. ■

Proof of Proposition 4

The proof follows immediately from Lemmas A10 and A12. ■

Proof of Proposition 5

Assume the market economy described in the main text. Let (V^u, θ) be as in the Rawlsian efficient allocation, and let $R = 1$. We proceed in three steps and show that the Rawlsian efficient allocation is consistent with market equilibrium.

Step 1 is to show that if, for all y^1, the values of $V_1(y^1)$ in the market economy coincide with those in the Rawlsian efficient allocation, then so will the consumption profile. This is seen by noting that the optimal contracting problem of the financial intermediary where $R = 1$ is equivalent to the problem of the component planner in Lemma A1 where $R_t(y^t) = 1$ for all y^t. Thus, from the argument in the proof of Lemma A11, if financial intermediaries are to provide the same $V_1(y^1)$ as in the Rawlsian efficient allocation, they will choose the same consumption profile as the Rawlsian planner does. This completes Step 1.

Step 2 is to show that if $\bar{V}_1^e$, the worker's expected value of being newly employed before observing y^1, in the market economy equals the corresponding value in the Rawlsian efficient allocation, then so does $V_1(y^1)$ for all y^1. To see this, let $\Pi(V_t(y^t); y^t)$ be the expected continuation profit for the financial intermediary in a match with the current promised utility $V_t(y^t)$ and history y^t. From the argument in Step 1, $\Pi(V_t(y^t);y^t) = \Phi(y^t) - Q(V_t(y^t);y^t)$, where $\Phi(y^t)$, the expected output from the match, is exogenously given, and Q is the component planner's cost function. Thus, from Lemma A1, $\Pi(V_t(y^t);y^t)$ is decreasing, strictly concave, and $\Pi'(V_t(y^t);y^t) = -1/U'(c_t(y^t))$.

Now, note that given any $\bar{V}_1^e$, the financial intermediary's expected profit from a new match before observing y^1 is given by

$$\bar{\Pi}\left(\bar{V}_1^e\right) = \max_{\{V_1(y^1)\}} \sum_{y^1} \pi\left(y^1\right) \Pi\left(V_1\left(y^1\right);y^1\right) \tag{86}$$

$$\text{s.t.} \sum_{y^1} \pi\left(y^1\right) V_1\left(y^1\right) \geq \bar{V}_1^e, \tag{87}$$

$$V_1\left(y^1\right) \geq V^o\left(y_1\right),\ \forall y^1 = y_1 \in Y. \tag{88}$$

The first-order and envelope conditions imply $\bar{\Pi}'\left(\bar{V}_1^e\right) = \Pi'(V_1(y^1);y^1) = -1/U'(c_1(y^1))$ for any y^1 at which the participation constraint (88) is slack, hence $V_1(y^1)$ leads to the same consumption across such y^1, while $V_1(y^1) = V^o(y_1)$ for y^1 at which (88) binds. If $\bar{V}_1^e$ is set to the value in the Rawlsian efficient allocation, such choices of $V_1(y^1)$ clearly coincide with those of the Rawlsian planner in Propositions 3 and 4. This completes Step 2.

Step 3 is to show that the Rawlsian efficient allocation is consistent with the three equilibrium conditions. First, the Nash bargaining problem is given by

$$\max_{\bar{V}_1^e} \ \left(\bar{V}_1^e - V^u\right)^\eta \left(\bar{\Pi}\left(\bar{V}_1^e\right)\right)^{1-\eta}$$
$$\text{s.t. } \bar{V}_1^e \geq V^u,\ \bar{\Pi}\left(\bar{V}_1^e\right) \geq 0, \tag{89}$$

hence taking the first-order condition yields the Nash bargaining condition,

$$\frac{\eta}{1-\eta}\bar{\Pi}\left(\bar{V}_1^e\right) = -\bar{\Pi}'\left(\bar{V}_1^e\right)\left(\bar{V}_1^e - V^u\right). \tag{90}$$

Second, given $R = 1$, the zero-profit condition for posting a vacancy is given by[34]

$$k = q(\theta)(1-s)\bar{\Pi}\left(\bar{V}_1^e\right). \tag{91}$$

Third, the consumption loan market clearing condition is such that the demand for loans

[34]For a general R, the zero-profit condition for posting a vacancy is expressed as $k = (1/R)\,q(\theta)(1-s)\bar{\Pi}\left(\bar{V}_1^e\right) = q(\theta)\sum_{t=1}^{\infty}\left[(1-s)/R\right]^t \sum_{y^t}\pi(y^t)(y_t - c_t(y^t))$. Note that this condition coincides with the consumption loan market clearing condition below if and only if $R = 1$.

to finance the vacancy cost, kv, equals the supply of loans from existing matches, or

$$kv = \sum_{t=1}^{\infty} \sum_{y^t} e_t\left(y^t\right)\left(y_t - c_t\left(y^t\right)\right). \tag{92}$$

Henceforth, let $\bar{V}_1^e$ be as in the Rawlsian efficient allocation. Given $R = 1$, the results in Steps 1 and 2 imply

$$\bar{\Pi}\left(\bar{V}_1^e\right) = \sum_{t=1}^{\infty} (1-s)^{t-1} \sum_{y^t} \pi\left(y^t\right)\left(y_t - c_t\left(y^t\right)\right), \tag{93}$$

$$\bar{V}_1^e = \sum_{y^1} \pi\left(y^1\right) V_1^e\left(y^1\right), \tag{94}$$

where $\{c_t(y^t)\}_{t=1}^{\infty}$ and $V_1^e(y^1)$ are as in the Rawlsian efficient allocation, and

$$\bar{\Pi}'\left(\bar{V}_1^e\right) = -\frac{1}{U'(\tilde{c}_1)} = -\beta\gamma, \tag{95}$$

where $\tilde{c}_1$ and γ are as defined in Lemmas A12 and A13. Further, since $\eta = -\theta q'(\theta)/q(\theta)$ by assumption and since $p(\theta) = \theta q(\theta)$ implies $p'(\theta) = q(\theta) + \theta q'(\theta)$,

$$\frac{\eta}{1-\eta} = -\frac{\theta q'(\theta)}{q(\theta) + \theta q'(\theta)} = -\frac{\theta q'(\theta)}{p'(\theta)}. \tag{96}$$

Using (93)–(96), we can rewrite the Nash bargaining condition (90) as

$$-\frac{\theta q'(\theta)}{p'(\theta)} \sum_{t=1}^{\infty} (1-s)^{t-1} \sum_{y^t} \pi\left(y^t\right)\left(y_t - c_t\left(y^t\right)\right) = \beta\gamma\left(\sum_{y^1} \pi\left(y^1\right) V_1^e\left(y^1\right) - V^u\right). \tag{97}$$

Since the Rawlsian efficient allocation satisfies (12) and (28), it satisfies (97) and is thus consistent with the Nash bargaining condition. Further, substituting for $\bar{\Pi}\left(\bar{V}_1^e\right)$ in (91) from (93) and noting $v = \theta u$ in (92) reveals that both the zero-profit and the market clearing conditions are implied by (12) and are thus satisfied by the Rawlsian efficient allocation. This concludes Step 3.

Steps 1 to 3 establish that the Rawlsian efficient allocation is supported as a market equilibrium in which $R = 1$. ■

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