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On the Consistency of Slacks-based Measure-Max Model and Super-Slacks-based Measure Model

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Abstract. Slacks-based measure (SBM) (Tone (2001), Pastor et al. (1999)) has been widely utilized as a representative non-radial DEA model. However, this model, called *SBM-Min* here, evaluates the efficiency of an inefficient DMU referring to the furthest frontier point within a range. In contrast, the *SBM-Max* model looks for the nearest frontier point and hence its score is generally greater than the SBM-Min score. The *Super-SBM* model (Tone (2002)) evaluates the efficiency of an efficient DMU referring to the nearest point on the frontier except itself. We can foresee a close connection between SBM-Max and Super-SBM models, because the motivations behind the two models are same. In this paper we demonstrate this consistency using a real dataset.

Keywords: Data Envelopment Analysis, Slacks-based Measure, SBM-Max, Super-SBM

1. Introduction

There are two types of models in DEA (Data Envelopment Analysis); radial and non-radial. Radial models are represented by the CCR (Charnes-Cooper-Rhodes) model (Charnes et al. (1978)). Basically they deal with proportional changes of inputs or outputs. As such, the CCR score reflects the proportional maximum input (output) reduction (expansion) rate which is common to all inputs (outputs). However, in real world businesses, not all inputs (outputs) behave in the proportional way. For example, if we employ labor, materials and capital as inputs, some of them are substitutional and do not change proportionally. Another shortcoming of the radial models is the neglect of slacks in reporting the efficiency score. In many cases, we find a lot of remaining non-radial slacks. So, if these slacks have an important role in evaluating managerial efficiency, the radial approaches may mislead the decision when we utilize the efficiency score as the only index for evaluating performance of DMUs.

In contrast, the non-radial SBM models put aside the assumption of proportionate changes in inputs and outputs, and deal with slacks directly. This may discard varying proportions of original inputs and outputs. The SBM models are designed to meet the following two conditions.

- (1) Units invariant: The measure should be invariant with respect to the units of data
- (2) Monotone: The measure should be monotone decreasing in each slack in input and output.

The original SBM (SBM-Min) model evaluates efficiency of DMUs referring to the furthest frontier point within a range. This results in the worst score for the objective DMU and the projection may go to a remote point on the efficient frontier which may be inappropriate as the reference.

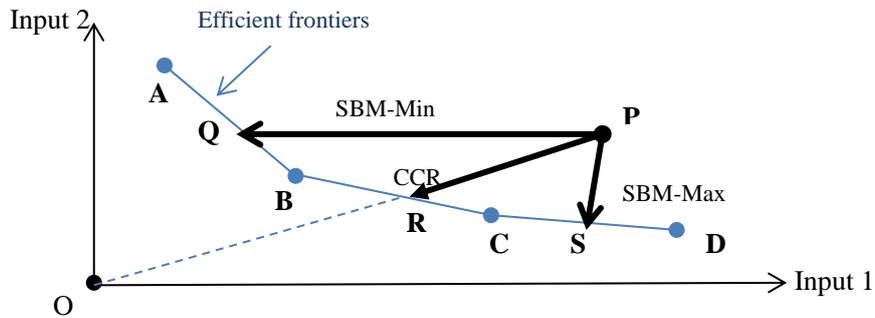


Figure 1: Comparisons of SBM-Min, CCR and SBM-Max models

We depict the relationship among the ordinary SBM (SBM-Min), CCR and SBM-Max models by Fig. 1. Inefficient DMU P's projections are Q, R and S respectively by SBM-Min, CCR and SBM-Max. Mathematically, finding S belongs to a NP-hard problem, because it is a maximization problem of a convex function over a non-convex region. However, the projected point S indicates that we can attain an efficient status with less input reductions and less output expansions than the ordinary SBM (Min) models. We can say that the projection by the SBM-Max model represents a practical "Kaizen" (improvement) by DEA.

The rest of this paper is organized as follows. Section 2 introduces the ordinary SBM-Min model briefly. Section 3 presents the SBM-Max model, while Section 4 describes the Super-SBM model. A numerical example is reported in Section 5. Section 6 concludes this paper. Although we present the model in non-oriented model, we can treat input- and output-oriented model as well. As to returns-to-scale characteristics, we

present the constant returns-to-scale (CRS) case. However we can deal with the variable returns-to-scale (VRS) model as well.

2. The SBM-Min Model

The SBM model was introduced by Tone (2001) (see also Pastor et al. (1999)). It has three variations, i.e. input-, output- and non-oriented. The non-oriented model indicates both input- and output-oriented.

Let the set of DMUs be $J = \{1, 2, \dots, n\}$, each DMU having m inputs and s outputs. We denote the vectors of inputs and outputs for DMU $_j$ by $\mathbf{x}_j = (x_{1j}, x_{2j}, \dots, x_{mj})^T$ and $\mathbf{y}_j = (y_{1j}, y_{2j}, \dots, y_{sj})^T$, respectively. We define input and output matrices \mathbf{X} and \mathbf{Y} by

$$\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \in R^{m \times n} \text{ and } \mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n) \in R^{s \times n}. \quad (1)$$

We assume that all data are positive i.e. $\mathbf{X} > \mathbf{0}$ and $\mathbf{Y} > \mathbf{0}$.

2.1 Production Possibility Set

The production possibility set is defined using the non-negative combination of the DMUs in the set J as:

$$P = \left\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \geq \sum_{j=1}^n \lambda_j \mathbf{x}_j, \mathbf{0} \leq \mathbf{y} \leq \sum_{j=1}^n \lambda_j \mathbf{y}_j, \boldsymbol{\lambda} \geq \mathbf{0} \right\}. \quad (2)$$

$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ is called the intensity vector.

The inequalities in (2) can be transformed into equalities by introducing slacks as follows:

$$\begin{aligned} \mathbf{x} &= \sum_{j=1}^n \lambda_j \mathbf{x}_j + \mathbf{s}^- \\ \mathbf{y} &= \sum_{j=1}^n \lambda_j \mathbf{y}_j - \mathbf{s}^+ \\ \mathbf{s}^- &\geq \mathbf{0}, \mathbf{s}^+ \geq \mathbf{0}, \end{aligned} \quad (3)$$

where $\mathbf{s}^- = (s_1^-, s_2^-, \dots, s_m^-)^T \in R^m$ and $\mathbf{s}^+ = (s_1^+, s_2^+, \dots, s_s^+)^T \in R^s$ are respectively called input and output slacks.

2.2 Non-oriented SBM

Non-oriented or both-oriented SBM efficiency $\rho^{\min}(\mathbf{x}_o, \mathbf{y}_o)$ is defined by

$$[\text{SBM-Min}] \quad \rho_o^{\min}(\mathbf{x}_o, \mathbf{y}_o) = \min_{\lambda, \mathbf{s}^-, \mathbf{s}^+} \frac{1 - \frac{1}{m} \sum_{i=1}^m \frac{S_i^-}{x_{io}}}{1 + \frac{1}{s} \sum_{r=1}^s \frac{S_r^+}{y_{ro}}}$$

subject to

$$x_{io} = \sum_{j=1}^n x_{ij} \lambda_j + s_i^- \quad (i = 1, \dots, m) \quad (4)$$

$$y_{ro} = \sum_{j=1}^n y_{rj} \lambda_j - s_r^+ \quad (r = 1, \dots, s)$$

$$\lambda_j \geq 0 \quad (\forall j), \quad s_i^- \geq 0 \quad (\forall i), \quad s_r^+ \geq 0 \quad (\forall r).$$

[Definition 1] (SBM-efficient) A DMU $_o = (\mathbf{x}_o, \mathbf{y}_o)$ is called SBM-efficient if

$\rho^{\min}(\mathbf{x}_o, \mathbf{y}_o) = 1$ holds.

This means $\mathbf{s}^- = \mathbf{0}$ and $\mathbf{s}^{+*} = \mathbf{0}$, i.e. all input and output slacks are zero.

[SBM-Min] can be transformed into a linear program using the Charnes-Cooper transformation as follows:

$$[\text{SBM-Min-LP}] \quad \tau^* = \min_{t, \Lambda, \mathbf{S}^-, \mathbf{S}^+} t - \frac{1}{m} \sum_{i=1}^m \frac{S_i^-}{x_{io}}$$

subject to

$$1 = t + \frac{1}{s} \sum_{r=1}^s \frac{S_r^+}{y_{ro}} \quad (5)$$

$$tx_{io} = \sum_{j=1}^n x_{ij} \Lambda_j + S_i^- \quad (i = 1, \dots, m)$$

$$ty_{ro} = \sum_{j=1}^n y_{rj} \Lambda_j - S_r^+ \quad (r = 1, \dots, s)$$

$$\Lambda_j \geq 0 \quad (\forall j), \quad S_i^- \geq 0 \quad (\forall i), \quad S_r^+ \geq 0 \quad (\forall r), \quad t > 0.$$

Let an optimal solution be $(\tau^*, t^*, \Lambda^*, \mathbf{S}^{-*}, \mathbf{S}^{+*})$. Then, we have an optimal solution of [SBM-Min] as defined by

$$\rho^{\min}(\mathbf{x}_o, \mathbf{y}_o) = \tau^*, \quad \boldsymbol{\lambda}^* = \Lambda^* / t^*, \quad \mathbf{s}^{-*} = \mathbf{S}^{-*} / t^*, \quad \mathbf{s}^{+*} = \mathbf{S}^{+*} / t^*. \quad (6)$$

3. The Frontier of Production Possibility Set and the SBM-Max Model

We define the frontier F of the production possibility set P as follows:

$$\begin{aligned}
F &= \{(\mathbf{x}_f, \mathbf{y}_f)\} \\
&\text{such that} \\
\theta^{\min}(\mathbf{x}_f, \mathbf{y}_f) &= 1 \\
\mathbf{x}_f &\geq \mathbf{X}\boldsymbol{\lambda}, \mathbf{y}_f \leq \mathbf{Y}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}.
\end{aligned} \tag{7}$$

In Figure 1, the set of line segments (AB, BC and CD) is the frontier which is non-convex.

For an inefficient DMU $(\mathbf{x}_o, \mathbf{y}_o)$, we define the SBM-Max score as follows:

$$\begin{aligned}
\theta^{\max}(\mathbf{x}_o, \mathbf{y}_o) &= \max \frac{1 - \frac{1}{m} \sum_{i=1}^m \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s} \sum_{r=1}^s \frac{s_r^+}{y_{ro}}} \\
&\text{subject to} \\
(\mathbf{x}_o - \mathbf{s}^-, \mathbf{y}_o + \mathbf{s}^+) &\in F.
\end{aligned} \tag{8}$$

Referring these variations, several authors published new solution methods. Among them, we introduce three papers.

Fukuyama et al. (2014) developed a least distance efficiency measure with the strong/weak monotonicity of the ratio form measure under several norms including 1-norm, 2-norm and ∞ -norm. This model utilizes mixed-integer linear programming (MILP) to identify efficiency frontiers and hence a computational difficulty arises for large-scale problems.

Hadi-Vencheh et al. (2015) developed a new SBM model to find the nearest point on the efficient frontiers. They utilize the multiplier form model to find all supporting hyperplanes. It also utilizes software which uses fractional coefficients (high precision arithmetic) to avoid loss data. Hence, computational time increases for large-scale problems.

Tone (2016) proposed a scheme for solving the SBM-Max problem. This method requires a limited number of additional linear program solutions for each inefficient DMU and needs no mixed-integer linear program code. Although the point thus obtained is not always the nearest point and does not always satisfy Pareto-Koopmans efficiency condition, it is acceptable for practical purpose for solving large scale problems and from the point of computational loads. We utilized this model for solving the numerical data in Section 5.

4. The Super-SBM Model

The SBM-Max model aims at getting to the nearest point on the efficient frontiers. This concept is in line with the super-efficiency SBM model (Tone (2002)) which solves the following program for an efficient DMU $(\mathbf{x}_o, \mathbf{y}_o)$ to measure the minimum ratio-scale distance from the efficient frontier excluding the DMU $(\mathbf{x}_o, \mathbf{y}_o)$.

$$\begin{aligned}
 \text{[Super-SBM]} \quad \delta^* = \min_{\lambda, \mathbf{s}^-, \mathbf{s}^+} & \frac{1 + \frac{1}{m} \sum_{i=1}^m \frac{s_i^-}{x_{io}}}{1 - \frac{1}{s} \sum_{r=1}^s \frac{s_r^+}{y_{ro}}} \\
 \text{subject to} & \\
 \mathbf{x}_o = \sum_{j=1, j \neq o}^n \mathbf{x}_j \lambda_j - \mathbf{s}^- & \\
 \mathbf{y}_o = \sum_{j=1, j \neq o}^n \mathbf{y}_j \lambda_j + \mathbf{s}^+ & \\
 \lambda \geq \mathbf{0}, \mathbf{s}^- \geq \mathbf{0}, \mathbf{s}^+ \geq \mathbf{0}. & \quad (9)
 \end{aligned}$$

We can solve the super-efficiency SBM model by applying LP code just once, because this problem belongs to a convex programming, i.e., minimization of a convex function over a convex region. However, SBM-Max problem cannot be solved in this manner, because it is a maximization of the objective function over a non-convex region. See Fig. 2 where the efficient B is projected to E on the frontier AC with the minimum distance.

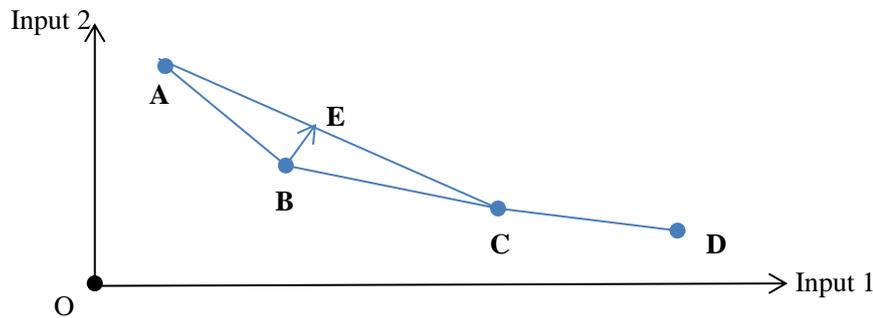


Figure 2: Super-SBM for B

5. A Numerical Example

We compare the two approaches, i.e. (SBM-Min + Super-SBM) and (SBM-Max + Super-SBM), using the data of Japanese municipal hospitals.

5.1 Data

The data were collected from the Annual Databook of Local Public Enterprise published by the Ministry of Internal Affairs and Communications Japanese Government, 2005.

(a) Number of DMUs: 700 hospitals ($n = 700$).

(b) Number of inputs: 5. (1) No. of beds (Bed), (2) Expenses for outsourcing (Outsource), (3) No. of doctors (Doctor), (4) No. of nurses (Nurse) and (5) Expenses for other medical materials (Material). ($m = 5$)

(c) Number of outputs: 4. (1) Revenue from operation per day (Operation), (2) Revenue from first consultation per day (1st time), (3) Revenue from return to clinic per day (Follow-up) and (4) Revenue from hospitalization per day (Hotel). ($s = 4$)

Table 1 exhibits statistics of the dataset.

Table 1: Statistics of dataset ($n = 700$)

	Bed	Outsource	Doctor	Nurse	Material	Operation	1st time	Follow-up	Hotel
Max	1025	2231247	215.562	955.464	2842350	17341140	1432079	3359160	15445104
Min	25	7767	0.98	11	9197	8979	2706	13636	109650
Average	249.407	300585	32.0519	170.352	469790	2029403	206916	405837	3155864
SD	181.11	310373	31.967	140.106	557030	2322876	206375	306298	2824856

5.2 Model and Method

We applied SBM-Max and SBM-Min models coupled with Super-SBM model, in non-oriented case under the constant-returns-to-scale assumption. For SBM-Max, we employed the method developed in Tone (2016).

5.3 SBM scores

The SBM model found that 66 hospitals among 700 are efficient. Table 2 compares two scores, one SBM-Max coupled with Super-SBM and the other SBM-Min coupled with Super-SBM..

Table 2: Comparisons of two scores

	SBM-Max + Super-SBM	SBM-Min + Super-SBM
Average	0.7935	0.4572
Max	1.4468	1.4468
Min	0.1023	0.0118
St Dev	0.1444	0.2445

Fig. 3 (SBM-Max + Super-SBM) and Fig. 4 (SBM-Min + Super-SBM) exhibit respectively scores of 700 hospitals in ascending order where we can observe big differences. The former shows a smooth transit from inefficient to efficient, while the latter exhibits a non-smooth transit. We can foresee a close connection between SBM-Max and Super-SBM models, because the motivations behind the two models are same.

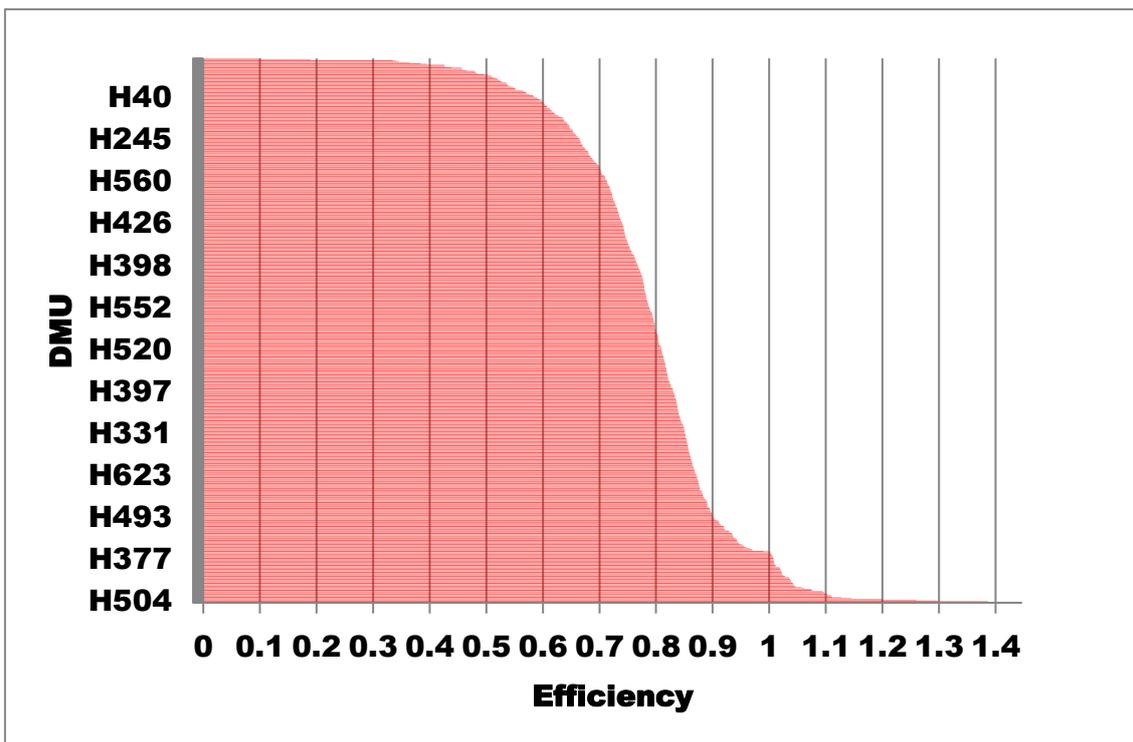


Figure 3: SBM-Max + Super-SBM

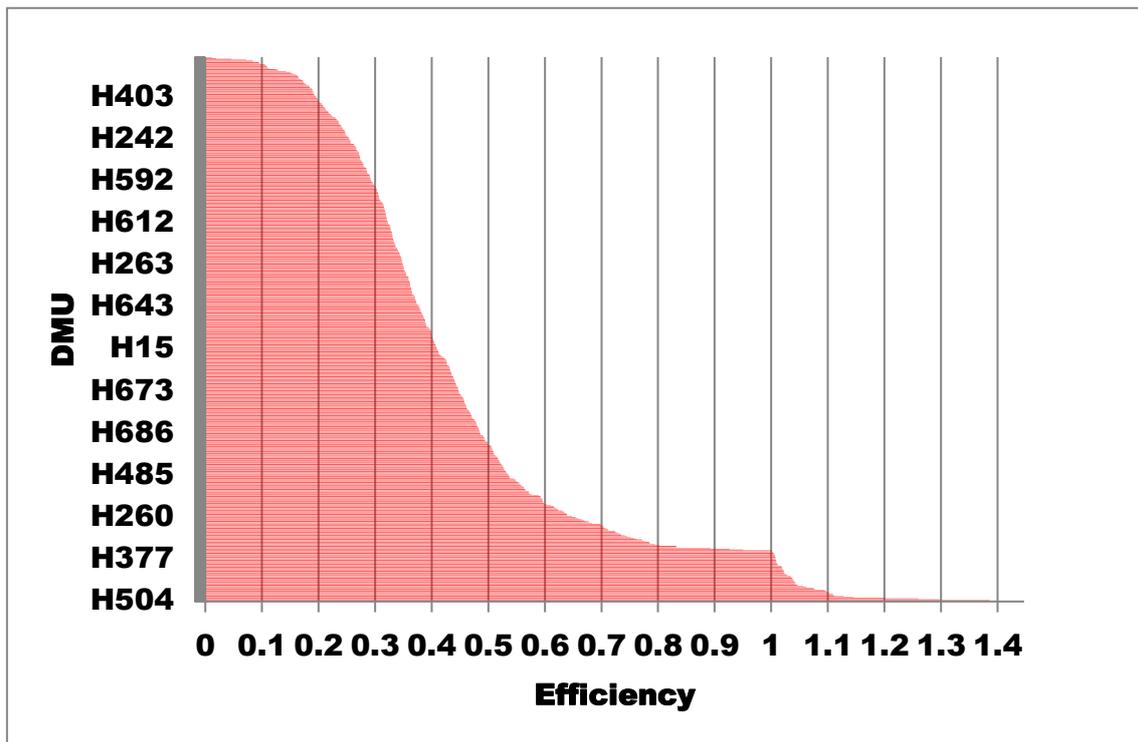


Figure 4: SBM-Min + Super-SBM

6. Conclusions

In this paper, we have compared the SBM-Max and SBM-Min models connected with the Super-SBM model. The finding indicates that the SBM-Max model is smoothly connected with the Super-SBM model, although the former needs 13 times of computation time than the latter in our case (we utilized DEA-Solver Pro: Saitech-Inc).

If one wishes the worst case analysis, the SBM-Min model is the choice. In contrast, if improvement to efficient status is the main concern, the SBM-Max model is qualified better. Inefficient DMUs can be improved to the efficient status with less input-reductions and less output-enlargement. Thus, the SBM-Max model proposes an efficient Kaizen (improvement) tool by DEA.

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