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Dealing with Desirable Inputs in Data Envelopment Analysis: A Slacks-based Measure Approach

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Abstract: In Data Envelopment Analysis (DEA) the situation of inputs vs. outputs is positioned as cause and effect. Effects include desirable (ordinary) outputs and undesirable outputs, e.g. pollutants. This situation is well studied and many applications have been published. In this paper, we introduce a new type of inputs, called Good (Desirable) Inputs. As explained in Introduction, we find several examples of such inputs, e.g. Electric car, Women in office and Test takers of vaccine. We handle this by means of SBM (Slacks-based Measure). Usually, efficiency values of DEA models are in the range $(0, 1]$, while in this model a negative efficiency value may be assigned to inefficient DMUs (decision making units). This is caused by shortages of Good Input values. As an example, we refer to "Women's Rights Movements" in a country where women's right is not fully guaranteed. Suppose local governments where men and women are serving as officers. They are inputs to office, while Women are Desirable input and Men are Ordinary input. As outputs, we assume Service as Ordinary output and Claim as Undesirable output. Several extensions of this model are introduced. (a) Variable returns to scale, (b) Weight restrictions, (c) Super-efficiency issue and (d) SBM_Max model.

Keywords: DEA, SBM, Desirable Inputs

1. Introduction

Generally speaking, in DEA, inputs (I) indicate input resources and the smaller the better, while outputs (O) correspond to productions induced by inputs (I), which are the larger the better. Among outputs, there exist undesirable outputs (OBad) incidental to outputs, e.g. CO_2, which are the smaller the better. Tone [8] formulated these situations in the framework of SBM (slacks-based measure) and this model is utilized all over the world. See Bai et al. [3], Ding et al. [4], Chen et al. [2], Liu et al. [5], and Wu et al. [14] for recent applications.

In this paper, a new input style, called Desirable inputs (IGood), is proposed, which are the larger the better. We show four potential examples of (IGood).

(a) In accordance with the increasing concern on environmental pollution, spread of Electric cars is worldwide required. We compare the energy consumption issue of countries. We consider "Number of Electric Cars" as a Good Input (IGood). Other IO items are (I) Total consumption of Energy, (O) GDP and (OBad) Pollutants (e.g. CO_2). DMUs are Country 1 to Country N.

(b) Consider the case that there are many new projects in a company and the manager wishes to evaluate the efficiency of projects. For this purpose, a certain number of evaluators is assigned to each project and they report Cost (Input), Return (Output), Risk (Undesirable Output) for the assigned project. In this case, we consider the number of evaluators is a Desirable input (IGood) and the larger the better.

(c) "Women's Rights Movement" is a matter of urgency in a country where women's right is not fully guaranteed. In such a country, suppose local governments where men and women are serving as officers. They are inputs to office, while women are (IGood) and men are (I). As outputs, we assume (O) Service and (OBad) Claims.

(d) Suppose that, inspired by COVID-19, many medical enterprises are eager to develop vaccine. In this case, (IGood) Number of test takers, (I) Cost, (O) Recovery, (OBad) Failure. As DMUs, we assume potential Vaccine 1 to Vaccine N.

We reasoned that such situations occur in many fields of enterprise.

We formulate this situation in the framework of SBM.

2. An SBM Model with Desirable Inputs

Suppose that there are n DMUs (decision making units) each having four factors: inputs, good (desirable) inputs, outputs and bad (undesirable) outputs, as represented by four vectors $x \in R^{m_1}, x^g \in R^{m_2}, y \in R^{s_1}, y^b \in R^{s_2}$, respectively. We define the matrices X, X^g, Y and Y^b as follows.

$X = [x_1, \ldots, x_n] \in R^{m_1 \times n}, X^g = [x_1^g, \ldots, x_n^g] \in R^{m_2 \times n}, Y = [y_1, \ldots, y_n] \in R^{s_1 \times n}$ and $Y^b = [y_1^b, \ldots, y_n^b] \in R^{s_2 \times n}$. We assume $X > 0, X^g > 0, Y > 0$ and $Y^b > 0$.

The production possibility set (P) is defined by

$$P = \{(x, x^g, y, y^b) \mid x \geq X\lambda, x^g \leq X^g\lambda, y \leq Y\lambda, y^b \geq Y^b\lambda, \lambda \geq 0\}, \quad (1)$$

where $\lambda \in R^n$ is the intensity vector. Notice that the above definition corresponds to the constant returns to scale technology. We discuss the other return to scale cases in Section 4.

Definition 1 (*Efficient DMU*) A DMU0 (x_0, x_0^g, y_0, y_0^b) is efficient in the presence of desirable inputs and undesirable outputs if there is no vector $(x, x^g, y, y^b) \in P$ such that $x_0 \geq x, x_0^g \leq x^g, y_0 \leq y$ and $y_0^b \geq y^b$ with at least one strict inequality.

In accordance with this definition, we modify the SBM in Tone [6] as follows.

$$[SMB]\ \rho^* = min \frac{1 - \frac{1}{m_1+m_2}\left(\sum_{i=1}^{m_1} \frac{s_i^-}{x_{i0}} + \sum_{i=1}^{m_2} \frac{s_i^{g-}}{x_i^g}\right)}{1 + \frac{1}{s_1+s_2}\left(\sum_{r=1}^{s_1} \frac{s_r}{y_{r0}} + \sum_{r=1}^{s_2} \frac{s_r^b}{y_{r0}^b}\right)} \quad (2)$$

$$\text{Subject to } x_0 = X\lambda + s^- \quad (3)$$

$$x_0^g = X^g\lambda - s^{g-} \quad (4)$$

$$y_0 = Y\lambda - s \quad (5)$$

$$y_0^b = Y^b\lambda + s^b \quad (6)$$

$$s^- \geq 0, s^{g-} \geq 0, s \geq 0, s^b \geq 0, \lambda \geq 0.$$

The vectors $s^- \in R^{m_1}$ and $s^b \in R^{s_2}$ correspond to excesses in inputs and bad outputs, respectively, while $s^{g-} \in R^{m_2}$ represents shortages in good inputs and $s \in R^{s_1}$ expresses shortages in good outputs. The objective function (2) strictly decreases with respect to $s_i^-(\forall i)$, $s_i^{g-}(\forall i)$, $s_r^g(\forall r)$ and $s_r^b(\forall r)$. Let an optimal solution of the above program be $(\lambda^*, s^{-*}, s^{g-*}, s^*, s^{b*})$. Then, we have:

Theorem 1 The DM U0 is efficient in the presence of desirable inputs and undesirable outputs if and only if $\rho^* = 1, i.e., s^{-*} = 0, s^{g-*} = 0, s^* = 0$ and $s^{b*} = 0$.

If the DMU0 is inefficient, i.e., $\rho^* < 1$, it can be improved and become efficient by deleting the excesses in inputs and bad outputs, and augmenting the shortfalls in good inputs and good outputs via the following SBM-projection:

$$x_0 \leftarrow x_0 - s^{-*} \quad (7)$$

$$x_0^g \leftarrow x_0^g + s^{g-*} \quad (8)$$

$$y_0 \leftarrow y_0 + s^* \quad (9)$$

$$y_0^b \leftarrow y_0^b - s^{b*} \quad (10)$$

Using the transformation by Charnes and Cooper, [1], we arrive at an equivalent linear program in $t, \Lambda, S^-, S^{g-}, S$ and S^b as displayed below.

$$[LP]\ \tau^* = min\ t - \frac{1}{m_1+m_2}\left(\sum_{i=1}^{m_1} \frac{S_i^-}{x_{i0}} + \sum_{i=1}^{m_2} \frac{S_i^{g-}}{x_i^g}\right) \quad (11)$$

subject to

$$1 = t + \frac{1}{s_1+s_2}\left(\sum_{r=1}^{s_1} \frac{S_r}{y_{r0}} + \sum_{r=1}^{s_2} \frac{S_r^b}{y_{r0}^b}\right) \quad (12)$$

$$x_0 t = X\Lambda + S^- \quad (13)$$

$$x_0^g t = X^g\Lambda - S^{g-} \quad (14)$$

$$y_0 t = Y\Lambda - S^g \quad (15)$$

$$y_0^b t = Y^b\Lambda + S^b \quad (16)$$

$$S^- \geq 0, S^{g-} \geq 0, S^g \geq 0, S^b \geq 0, \Lambda \geq 0, t > 0. \quad (17)$$

Let an optimal solution of [LP] be $(t^*, \Lambda^*, S^{-*}, S^{g-*}, S^{g*}, S^{b*})$. Then we have an optimal solution of [SBM] as defined by

$$\rho^* = \tau^*, \lambda^* = \Lambda^*/t^*, s^{-*} = S^{-*}/t^*, s^{g-*} = S^{g-*}/t^*, s^{g*} = S^{g*}/t^*, s^{b*} = S^{b*}/t^*. \quad (18)$$

(See Tone (2001) for detail). The existence of $(t^*, \Lambda^*, S^{-*}, S^{g-*}, S^{g*}, S^{b*})$ with $t^* > 0$ is guaranteed by [LP].

3. Why We Employ (IGood) and (I) as Inputs and (O) and (OBad) as Outputs

The situation of inputs vs. outputs is positioned as cause and effect. Use of (I) and (OBad) as inputs and (O) and (IGood) as outputs seems plausible, because the former is the less the better and the latter is the larger the better. However, this standpoint is just a pretense and neglects the cause and effect relationship. Our objective function (2) expresses orthodoxly cause and effect relationship.

4. An Illustrative Example

Table 1 shows a fictional data on 14 cities in which inputs are Women and Men with resulting outputs Service and Claim. From the point of view of "Women's Right Movements", we assume Women as (IGood) while Men as an ordinary input. As outputs we take Service as an ordinary output while Claim as an Undesirable Output (OBad).

Table 1. *City Data.*

DMU	(IGood)Women	(I)Men	(O)Service	(OBad)Claim
City1	10	100	1000	20
City2	5	100	1000	20
City3	30	200	1500	10
City4	30	200	1500	15
City5	20	200	1500	30
City6	25	400	2500	20
City7	30	500	3500	40
City8	40	600	4000	50
City9	45	800	5000	25
City10	50	800	4000	50
City11	55	1000	4500	55
City12	60	1100	4100	50
City13	25	1150	4200	55
City14	70	1150	5000	40

We solved this data using the program (2) to (6) under the constant returns to scale (CRS) assumption. Table 2 exhibits the efficiency score. It is remarkable that City 13 has a negative efficiency score (-0.3505). This indicates that City 13 has a large slack (59) in (IGood) Women compared with its current value (25). See Table 3. Hence, its numerator of the objective function (2) comes to be negative. City 13 is the worst in efficiency.

Table 2. *Efficiency Score.*

DMU	Score	Rank
City1	1	1
City2	0.5	6
City3	1	1
City4	0.8571	4
City5	0.5625	5
City6	0.2727	11
City7	0.2483	12
City8	0.297	10
City9	1	1
City10	0.4324	8
City11	0.3926	9
City12	0.1446	13
City13	-0.3505	14
City14	0.4461	7

Table 3. *Slacks.*

			Slack	Slack	Slack	Slack
DMU	Score	Rank	(IGood)Women	(I)Men	(O)Service	(OBad)Claim
City1	1	1	0	0	0	0
City2	0.5	6	5	0	0	0
City3	1	1	0	0	0	0
City4	0.8571	4	0	0	0	5
City5	0.5625	5	10	0	0	20
City6	0.2727	11	35	0	500	0
City7	0.2483	12	40	33.333	0	16.667
City8	0.297	10	50	0	500	20
City9	1	1	0	0	0	0
City10	0.4324	8	30	266.667	0	23.333
City11	0.3926	9	35	400	0	25
City12	0.1446	13	90	100	3400	0
City13	-0.3505	14	59	590	0	27
City14	0.4461	7	50	350	1000	0
	Score		(IGood)Women	(I)Men	(O)Service	(OBad)Claim
Average	0.4859		28.8571	124.2857	385.7143	9.7857
Max	1		90	590	3400	27
Min	-0.3505		0	0	0	0
St Dev	0.3827		27.7318	195.4414	918.91	11.3021

Table 4 exhibits projections to efficient status.

Table 4. *Projection.*

			(IGood)Women			**(I)Men**		
DMU	Score	Rank	Data	Projection	Diff.(%)	Data	Projection	Diff.(%)
City1	1	1	10	10	0	100	100	0
City2	0.5	6	5	10	100	100	100	0
City3	1	1	30	30	0	200	200	0
City4	0.8571	4	30	30	0	200	200	0
City5	0.5625	5	20	30	50	200	200	0
City6	0.2727	11	25	60	140	400	400	0
City7	0.2483	12	30	70	133.333	500	466.6667	-6.667
City8	0.297	10	40	90	125	600	600	0
City9	1	1	45	45	0	800	800	0
City10	0.4324	8	50	80	60	800	533.3333	-33.333
City11	0.3926	9	55	90	63.636	1000	600	-40
City12	0.1446	13	60	150	150	1100	1000	-9.091
City13	-0.3505	14	25	84	236	1150	560	-51.304
City14	0.4461	7	70	120	71.429	1150	800	-30.435
			(IGood)Women			(I)Men		
	Score		Data	Projection	Diff.(%)	Data	Projection	Diff.(%)
Average	0.4859		35.3571	64.2143	80.6713	592.8571	468.5714	-12.2021
Max	1		70	150	236	1150	1000	0
Min	-0.3505		5	10	0	100	100	-51.304
St Dev	0.3827		18.8582	41.3729	70.8956	404.2358	283.1137	18.2126

Table 4. *Continue.*

			(O)Service			**(OBad)Claim**		
DMU	Score	Rank	Data	Projection	Diff.(%)	Data	Projection	Diff.(%)
City1	1	1	1000	1000	0	20	20	0
City2	0.5	6	1000	1000	0	20	20	0
City3	1	1	1500	1500	0	10	10	0
City4	0.8571	4	1500	1500	0	15	10	-33.333
City5	0.5625	5	1500	1500	0	30	10	-66.667
City6	0.2727	11	2500	3000	20	20	20	0
City7	0.2483	12	3500	3500	0	40	23.33333	-41.667
City8	0.297	10	4000	4500	12.5	50	30	-40
City9	1	1	5000	5000	0	25	25	0
City10	0.4324	8	4000	4000	0	50	26.66667	-46.667
City11	0.3926	9	4500	4500	0	55	30	-45.455
City12	0.1446	13	4100	7500	82.927	50	50	0
City13	-0.3505	14	4200	4200	0	55	28	-49.091
City14	0.4461	7	5000	6000	20	40	40	0
			(O)Service			(OBad)Claim		
	Score		Data	Projection	Diff.(%)	Data	Projection	Diff.(%)
Average	0.4859		3092.857	3478.571	9.6734	34.2857	24.5	-23.0629
Max	1		5000	7500	82.927	55	50	0
Min	-0.3505		1000	1000	0	10	10	-66.667
St Dev	0.3827		1519.85	1999.684	22.4004	16.0357	11.3037	24.9601

5. Extensions

This section discusses extensions to the following issues:

(a) The variable returns to scale issue

We can extend the model in (2) to the variable returns to scale by adding the constraint,

$$e\lambda = 1 \tag{19}$$

(b) Weight restrictions to (IGood) vs (I) and (O) vs (OBad)

If putting preference (or importance) on input/output items is required, we can impose weights to the objective function in (2) as follows:

$$[SBM]\ \rho^* = min \frac{1-\frac{1}{m_1+m_2}\left(\sum_{i=1}^{m_1}\frac{w_i^- s_i^-}{x_{i0}}+\sum_{i=1}^{m_2}\frac{w_i^{-g} s_i^{-g}}{x_{i0}^g}\right)}{1+\frac{1}{s_1+s_2}\left(\sum_{r=1}^{s_1}\frac{w_r^g s_r^g}{y_{r0}^g}+\sum_{r=1}^{s_2}\frac{w_r^b s_r^b}{y_{r0}^b}\right)}, \tag{20}$$

where $w_i^-, w_i^{-g}, w_r^g\ and\ w_r^b$ are the weights to the input i, the desirable input i, the desirable output r, and the undesirable output r, respectively, and $\sum_{i=1}^{m_1} w_i^- + \sum_{i=1}^{m_2} w_i^{-g} = m_1 + m_2, w_i^- \geq 0(\forall i),\ w_i^{-g} \geq 0(\forall i), \sum_{r=1}^{s_1} w_r^g + \sum_{r=1}^{s_2} w_r^b = s_1 + s_2, w_r^g \geq 0(\forall r), w_r^b \geq 0(\forall r)$.

(c) Super efficiency issue

For efficient DMUs, we can apply the super efficiency model developed in Tone [7] by adding (IGood) factors in its formulation. Table 5 reports the results.

Table 5. *Super efficiency score.*

DMU	Score	Rank
City3	1.25	1
City9	1.14286	2
City1	1.05556	3

(d) Application of SBM_Max model

For an inefficient DMU, the SBM_Max model attempts to find nearly closest reference point on the efficient frontiers so that slacks are minimized, while the scores are maximized. Inefficient DMUs can be improved to the efficient status with less input-reductions and less output-enlargement. See Appendix A for a brief introduction and Tone [9] for details. We added (IGood) factors to the formulation in Tone [10].

Table 6. *Comparison of Score and Max_Score.*

DMU	Score	Rank	DMU	Max_Score	Rank
City1	1	1	City1	1	1
City2	0.5	6	City2	0.5	12
City3	1	1	City3	1	1
City4	0.8571	4	City4	0.8571	4
City5	0.5625	5	City5	0.825	6
City6	0.2727	11	City6	0.8463	5
City7	0.2483	12	City7	0.4557	13
City8	0.297	10	City8	0.6618	11
City9	1	1	City9	1	1
City10	0.4324	8	City10	0.7282	8
City11	0.3926	9	City11	0.7057	9
City12	0.1446	13	City12	0.6656	10
City13	-0.3505	14	City13	0.4096	14
City14	0.4461	7	City14	0.7916	7
Average	0.4859		Average	0.7462	
Max	1		Max	1	
Min	-0.3505		Min	0.4096	
St Dev	0.3827		St Dev	0.1957	

Table 7 exhibits slacks by the SBM_Max model.

Table 7. *Slacks by SBM_Max model.*

			Slack	**Slack**	**Slack**	**Slack**
DMU	Score	Rank	(IGood)Women	(I)Men	(O)Service	(OBad)Claim
City1	1	1	0	0	0	0
City2	0.5	12	5	0	0	0
City3	1	1	0	0	0	0
City4	0.857143	4	0	0	0	5
City5	0.825	6	3.33333	0	333.3333	0
City6	0.846262	5	0	0	33.33333	7
City7	0.455696	13	26.25	0	0	18.75
City8	0.661765	11	12.5	0	0	27.5
City9	1	1	0	0	0	0
City10	0.728155	8	0	0	1066.667	24
City11	0.705727	9	0	22.22222	1611.111	24.44444
City12	0.665584	10	0	33.33333	2566.667	16.66667
City13	0.409577	14	12.8	478	0	34
City14	0.791557	7	0	0	2258.333	3
	Score		(IGood)Women	(I)Men	(O)Service	(OBad)Claim
Average	0.7462		4.2774	38.1111	562.1032	11.4544
Max	1		26.25	478	2566.667	34
Min	0.4096		0	0	0	0
St Dev	0.1957		7.7911	127.0235	924.4577	12.2828

Table 8 reports projection to efficient frontiers by the SBM_Max model.

Table 8. *Projection by SBM_Max model.*

			(IGood)Women			(I)Men		
DMU	Score	Rank	Data	Projection	Diff.(%)	Data	Projection	Diff.(%)
City1	1	1	10	10	0	100	100	0
City2	0.5	12	5	10	100	100	100	0
City3	1	1	30	30	0	200	200	0
City4	0.857143	4	30	30	0	200	200	0
City5	0.825	6	20	23.33333	16.667	200	200	0
City6	0.846262	5	25	25	0	400	400	0

City7	0.455696	13	30	56.25	87.5	500	500	0
City8	0.661765	11	40	52.5	31.25	600	600	0
City9	1	1	45	45	0	800	800	0
City10	0.728155	8	50	50	0	800	800	0
City11	0.705727	9	55	55	0	1000	977.7778	-2.222
City12	0.665584	10	60	60	0	1100	1066.667	-3.03
City13	0.409577	14	25	37.8	51.2	1150	672	-41.565
City14	0.791557	7	70	70	0	1150	1150	0
			(IGood)Women			(I)Men		
	Score		Data	Projection	Diff.(%)	Data	Projection	Diff.(%)
Average	0.7462		35.3571	39.6345	20.4726	592.8571	554.746	-3.3441
Max	1		70	70	100	1150	1150	0
Min	0.4096		5	10	0	100	100	-41.565
St Dev	0.1957		18.8582	18.7618	34.7123	404.2358	366.848	11.0426

Table 8. *Continue.*

			(O)Service			(OBad)Claim		
DMU	Score	Rank	Data	Projection	Diff. (%)	Data	Projection	Diff.(%)
City1	1	1	1000	1000	0	20	20	0
City2	0.5	12	1000	1000	0	20	20	0
City3	1	1	1500	1500	0	10	10	0
City4	0.857143	4	1500	1500	0	15	10	-33.333
City5	0.825	6	1500	1833.333	22.222	30	30	0
City6	0.846262	5	2500	2533.333	1.333	20	13	-35
City7	0.455696	13	3500	3500	0	40	21.25	-46.875
City8	0.661765	11	4000	4000	0	50	22.5	-55
City9	1	1	5000	5000	0	25	25	0
City10	0.728155	8	4000	5066.667	26.667	50	26	-48
City11	0.705727	9	4500	6111.111	35.802	55	30.55556	-44.444
City12	0.665584	10	4100	6666.667	62.602	50	33.33333	-33.333
City13	0.409577	14	4200	4200	0	55	21	-61.818
City14	0.791557	7	5000	7258.333	45.167	40	37	-7.5
			(O)Service			(OBad)Claim		
	Score		Data	Projection	Diff.(%)	Data	Projection	Diff.(%)
Average	0.7462		3092.857	3654.96	13.8424	34.2857	22.8313	-26.0931
Max	1		5000	7258.333	62.602	55	37	0
Min	0.4096		1000	1000	0	10	10	-61.818
St Dev	0.1957		1519.85	2153.552	21.0586	16.0357	8.2498	23.688

6. Conclusion

Probably, this is the first trial to incorporate desirable input (IGood) into DEA research. We believe this model has a reasonable position for complying with the demand of modern society. Especially, the outcome of SBM_Max model is more practically applicable. Future research subjects include applications of this model to Network SBM (Tone and Tsutsui [11]), Dynamic SBM (Tone and Tsutsui [12]) and Dynamic and Network SBM (Tone and Tsutsui [13]).

Appendix

In this appendix, we introduce the non-oriented SBM_Max model briefly.

Step 1. Solve SBM-Min

First, we solve the ordinary SBM (SBM-Min) model as represented by the program (2) for DMU $(\mathrm{x}_o, \mathrm{y}_o)(o=1,\ldots,n)$. Let an optimal solution be $(\lambda^*, \mathrm{s}^{-*}, \mathrm{s}^{+*})$.

Step 2. Define efficient DMUs

We define the set R^{eff} of all efficient DMUs as

$$R^{eff} = \left\{ j \middle| \rho_j^{\min} = 1, j = 1,\ldots,n \right\}. \quad \text{(A1)}$$

We denote these efficient DMUs as $(\mathrm{x}_1^{eff}, \mathrm{y}_1^{eff}), (\mathrm{x}_2^{eff}, \mathrm{y}_2^{eff}), \ldots, (\mathrm{x}_{Neff}^{eff}, \mathrm{y}_{Neff}^{eff})$, wher $Neff$ is the number of efficient DMUs.

Step 3. Local reference set

For an inefficient DMU $(\mathrm{x}_o, \mathrm{y}_o)$, we define the local reference set R_o^{local}, i.e., efficient DMUs set for DMU $(\mathrm{x}_o, \mathrm{y}_o)$, by (A2).

$$R_o^{local} = \left\{ j \middle| \lambda_j^* > 0, j = 1,\ldots,n \right\}. \quad \text{(A2)}$$

Step 4. Pseudo-Max score

For each inefficient DMU, i.e., $\rho_o^{\min} < 1$, we solve the following program.

$$\text{[Pseudo-1]} \quad \max \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_{io}^{-}}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^{+}}{y_{ro}}}$$

subject to

$$\mathrm{x}_o = \sum_{j \in R_o^{local}} \mathrm{x}_j \lambda_j + \mathrm{s}^{-} \quad \text{(A3)}$$

$$\mathrm{y}_o = \sum_{j \in R_o^{local}} \mathrm{y}_j \lambda_j - \mathrm{s}^{+}$$

$$\mathrm{s}^{-}, \mathrm{s}^{+}, \lambda \geq 0.$$

Let an optimal slacks be $(\mathrm{s}^{-*}, \mathrm{s}^{+*})$. We solve the following program with variables $(\lambda, \mathrm{s}^{-}, \mathrm{s}^{+})$.

$$\text{[Pseudo-2]} \quad \min \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_{io}^{-}}{x_{io} - s_i^{-*}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^{+}}{y_{ro} + s_r^{+*}}}$$

subject to

$$\mathrm{x}_o - \mathrm{s}^{-*} = \sum_{j \in R^{eff}} \mathrm{x}_j^{eff} \lambda_j + \mathrm{s}^{-} \quad \text{(A4)}$$

$$\mathrm{y}_o + \mathrm{s}^{+*} = \sum_{j \in R^{eff}} \mathrm{y}_j^{eff} \lambda_j - \mathrm{s}^{+}$$

$$\mathrm{s}^{-}, \mathrm{s}^{+}, \lambda \geq 0.$$

Let the optimal slacks be $(\mathrm{s}^{-**}, \mathrm{s}^{+**})$. We define the Pseudo-Max score $\rho_o^{pseudo\max}$ by

$$\text{[Pseudo-Max]} \quad \rho_o^{pseudo\max} = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_{io}^{-*} + s_{io}^{-**}}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^{+*} + s_r^{+**}}{y_{ro}}}. \quad \text{(A5)}$$

Step 5. Distance and SBM-Max score

For each inefficient DMU $(\mathrm{x}_o, \mathrm{y}_o)$, i.e., $\rho_o^{\min} < 1$, we calculate the distance between $(\mathrm{x}_o, \mathrm{y}_o)$ and $(\mathrm{x}_h^{eff}, \mathrm{y}_h^{eff})(h = 1, \ldots, Neff)$ by

$$\text{[Distance]} \quad d_h = \sum_{i=1}^{m} \frac{\left|x_{ih}^{eff} - x_{io}\right|}{x_{io}} + \sum_{i=1}^{s} \frac{\left|y_{ih}^{eff} - y_{io}\right|}{y_{io}}. \quad \text{(A6)}$$

This distance is units-invariant.

Step 5.1. Reorder the distance

We renumber the efficient DMUs in the ascending order of d_h, so that

$$d_1 \leq d_2 \leq \ldots \leq d_{Neff}. \quad \text{(A7)}$$

We define the set R_h by

$$R_h = \{1, \ldots, h\} \; (h = 1, \ldots, Neff). \quad \text{(A8)}$$

Step 5.2. Find slacks and max-score for the set R_h

We evaluate the efficiency score of the inefficient DMU $(\mathrm{x}_o, \mathrm{y}_o)$ referring to the set R_h by solving the following program.

$$\text{[Max-1]} \quad \max_{\lambda, \mathrm{s}^{-}, \mathrm{s}^{+}} \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_{io}^{-}}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^{+}}{y_{ro}}}$$

subject to

$$\mathrm{x}_o = \sum_{j \in R_h} \mathrm{x}_j^{eff} \lambda_j + \mathrm{s}^{-} \quad \text{(A9)}$$

$$\mathrm{y}_o = \sum_{j \in R_h} \mathrm{y}_j^{eff} \lambda_j - \mathrm{s}^{+}$$

$$\mathrm{s}^{-}, \mathrm{s}^{+}, \lambda \geq 0.$$

If this program is infeasible, we define $\rho_{oh}^{*} = 0$. Otherwise, let an optimal slacks be $(\mathrm{s}^{-*}, \mathrm{s}^{+*})$.

(a) If the optimal objective value is 1, i.e., $\mathrm{s}^{-*} = 0$ and $\mathrm{s}^{+*} = 0$, we define $\rho_{oh}^{*} = 0$. This indicates that DMU $(\mathrm{x}_o, \mathrm{y}_o)$ can be expressed as a non-negative combination of DMUs in R_h and hence, in view of $\rho_o^{\min} < 1$, it is inside the production possibility set.

(b) If the optimal objective value is less than 1, we again solve the following program with the variables $(\lambda, \mathrm{s}^{-}, \mathrm{s}^{+})$.

$$\text{[Max-2]} \quad \min_{\lambda, \mathrm{s}^{-}, \mathrm{s}^{+}} \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_{io}^{-}}{x_{io} - s_i^{-*}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^{+}}{y_{ro} + s_r^{+*}}}$$

subject to

$$\mathrm{x}_o - \mathrm{s}^{-*} = \sum_{j \in R^{eff}} \mathrm{x}_j^{eff} \lambda_j + \mathrm{s}^{-} \quad \text{(A10)}$$

$$\mathrm{y}_o + \mathrm{s}^{+*} = \sum_{j \in R^{eff}} \mathrm{y}_j^{eff} \lambda_j - \mathrm{s}^{+}$$

$$\mathrm{s}^{-}, \mathrm{s}^{+}, \lambda \geq 0.$$

Let the optimal slacks be $(\mathrm{s}^{-**}, \mathrm{s}^{+**})$. We define ρ_{oh}^{*} by

$$[\rho_{oh}^{*}] \quad \rho_{oh}^{*} = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_{io}^{-*} + s_{io}^{-**}}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^{+*} + s_r^{+**}}{y_{ro}}}. \quad \text{(A11)}$$

We assign ρ_{oh}^{*} as the max-score referring to the set R_h.

Step 5.3. SBM_Max and projection

Finally, we define the max-score $\rho_o^{\max}$ of inefficient DMU $(\mathrm{x}_o, \mathrm{y}_o)$ by

$$[\text{SBM-Max}] \quad \rho_o^{\max} = \max\left\{\rho_o^{pseudo\max}, \rho_{o1}^{*}, \cdots, \rho_{oNeff}^{*}\right\}. \text{(A12)}$$

We also hold the slacks $\left(\mathrm{s}^{-**}, \mathrm{s}^{+**}\right)$ corresponding to the maximum $\rho_o^{\max}$. The projection of DMU $(\mathrm{x}_o, \mathrm{y}_o)$ onto efficient frontiers is given by

$$[\text{Projection}] \quad \mathrm{x}_o^{*} = \mathrm{x}_o - \mathrm{s}^{-*} - \mathrm{s}^{-**}, \mathrm{y}_o^{*} = \mathrm{y}_o + \mathrm{s}^{+*} + \mathrm{s}^{+**}. \text{(A13)}$$

The projected point $(\mathrm{x}_o^{*}, \mathrm{y}_o^{*})$ is efficient with respect to the efficient DMU set R^{eff}. However, it does not always satisfy Pareto-Koopmans efficiency condition. This model belongs to polynomial time as for the computational complexity.

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