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Any comments on this paper are welcomed.

A Strange Case of the Cost and Allocative Efficiency in DEA

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Abstract

We will point out shortcomings of the cost and allocative efficiencies as used in the DEA literature, and propose a new approach to cost efficiency evaluation.
Keywords: DEA, cost (overall) efficiency, allocative efficiency, returns to cost, revenue efficiency

1 Introduction

The cost (overall) and allocative efficiency was first introduced by Farrell (1957), and then developed by Färe, Grosskopf and Lovell (1985) by using linear programming technologies.

In this paper, we will show that, in a single input case, the conventional cost efficiency is equal to the technical efficiency, and there is no room for adopting cost factors into cost efficiency evaluation. Then, in a more general case, we will demonstrate that, if two DMUs have the same amount of inputs and outputs and one has unit-cost for inputs twice the other, then the two DMUs have the same cost (overall) and allocative efficiencies. After pointing out the irrationality of these efficiencies, we will propose a new scheme that is free from the above shortcomings and has several favorable properties.

The rest of the paper is organized as follows. Sections 2 and 3 reveal shortcomings of cost efficiency evaluation used thus far. Sections 4 and 5 propose

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a new scheme and comment the difference between the two approaches. Section 6 demonstrates the validity of the new model. Returns to cost issue will be discussed in Section 7, and then an empirical study is presented in Section 8. The scheme is extended in Section 9 and finally some concluding remarks follow in Section 10.

2 Single Input Case

In this section, we deal with n Decision Making Units (DMUs) with a single input $x(>0)$ to produce s outputs $\boldsymbol{y}=(y_1, y_2, \ldots, y_s)$. For a DMU$_o$ $(o = 1, \ldots, n)$, let the input and outputs be $x_o(>0)$ and $\boldsymbol{y}_o = (y_{1o}, \ldots, y_{so})$, respectively, and the unit cost of input x_o be $c_o(>0)$. Then, the *cost* (overall) efficiency γ^* of DMU$_o$ is defined as:

$$\gamma^* = c_o x_o^* / c_o x_o, \tag{1}$$

where x_o^* is an optimal solution of the following linear programming problem.

$$[\text{Cost}] \quad \min \quad c_o x \tag{2}$$

$$\text{subject to} \quad x \geq \sum_{j=1}^{n} x_j \lambda_j \tag{3}$$

$$y_{ro} \leq \sum_{j=1}^{n} y_{rj} \lambda_j \;\; (r = 1, \ldots, s) \tag{4}$$

$$\lambda_j \geq 0. \; (\forall j) \tag{5}$$

On this definition of cost efficiency, refer to Färe *et al.* (1994, p. 78), Coelli *et al.* (1998, pp. 162-166), Byrnes and Valdmans (1994) and Cooper *et al.* (1999, pp. 236-237) among others.

The *technical* efficiency θ^*, called "CCR-efficiency" (Charnes *et al.*(1978)), is defined as the optimal θ of the linear programming problem below:

$$[\text{CCR}] \quad \min \; \theta \tag{6}$$

$$\text{subject to} \quad \theta x_o \geq \sum_{j=1}^{n} x_j \lambda_j \tag{7}$$

$$y_{ro} \leq \sum_{j=1}^{n} y_{rj} \lambda_j \;\; (r = 1, \ldots, s) \tag{8}$$

$$\lambda_j \geq 0. \; (\forall j) \tag{9}$$

Then, we have the following theorem.

Theorem 1 *For the single input case, the technical efficiency θ^* is equal to the cost efficiency γ^*.*

Proof: Let us denote x as θx_o in [Cost] and change the variable from x to θ. Then, noting $x_o > 0$ and $c_o > 0$, [Cost] becomes:

$$\min \ c_o x_o \theta \tag{10}$$

$$\text{subject to} \quad \theta x_o \geq \sum_{j=1}^{n} x_j \lambda_j \tag{11}$$

$$y_{ro} \leq \sum_{j=1}^{n} y_{rj} \lambda_j \ (r = 1, \ldots, s) \tag{12}$$

$$\lambda_j \geq 0. \ (\forall j) \tag{13}$$

This program is equivalent to [CCR] and its optimal objective value is $\theta^* c_o x_o$. Thus we have $\gamma^* = \theta^* c_o x_o / c_o x_o = \theta^*$ □

Definition 1 (Allocative efficiency) *The allocative efficiency α^* of DMU_o is defined as the ratio: cost vs. technical efficiencies, i.e., $\alpha^* = \gamma^*/\theta^*$.*

The allocative efficiency α^* is less than or equal to one, and DMU_o is called *allocatively* efficient when $\alpha^* = 1$ holds.

Corollary 1 *In the single input case, the allocative efficiency is always one for every DMU.*

This sounds very strange, since, in this case, the input cost has nothing to do with the allocative efficiency.

3 General Case

Here we observe a more general case where we have m inputs $(x_1, \ldots, x_m)$. Suppose that DMUs A and B have the same inputs and outputs, i.e., $\boldsymbol{x}_A = \boldsymbol{x}_B$ and $\boldsymbol{y}_A = \boldsymbol{y}_B$. Assume further that the unit cost of A is twice that of B for each input, i.e., $\boldsymbol{c}_A = 2\boldsymbol{c}_B$. Under these assumptions, we have:

Theorem 2 *Both A and B have the same cost (overall) and allocative efficiencies.*

Proof: Since A and B have the same inputs and outputs, they have the same technical efficiency, i.e., $\theta_A^* = \theta_B^*$. The cost efficiency of A (or B) can be obtained by solving the following LP:

$$\min \ \boldsymbol{c}_A \boldsymbol{x} (= 2\boldsymbol{c}_B \boldsymbol{x}) \tag{14}$$

$$\text{subject to} \quad x_i \geq \sum_{j=1}^{n} x_{ij} \lambda_j \ \ (i = 1, \ldots, m) \tag{15}$$

$$y_{rA} (= y_{rB}) \leq \sum_{j=1}^{n} y_{rj} \lambda_j \ \ (r = 1, \ldots, s) \tag{16}$$

$$\lambda_j \geq 0. \ (\forall j) \tag{17}$$

Apparently, DMUs A and B have the same optimal solution (inputs) $\boldsymbol{x}_A^* = \boldsymbol{x}_B^*$, and the same cost efficiency, since we have:

$$\gamma_A^* = \boldsymbol{c}_A \boldsymbol{x}_A^* / \boldsymbol{c}_A \boldsymbol{x}_A = 2\boldsymbol{c}_B \boldsymbol{x}_B^* / 2\boldsymbol{c}_B \boldsymbol{x}_B = \boldsymbol{c}_B \boldsymbol{x}_B^* / \boldsymbol{c}_B \boldsymbol{x}_B = \gamma_B^*.$$

They also have the same allocative efficiency by definition. □

This also sounds very strange, since A and B have the same value of the cost and allocative efficiencies even though the cost of B is half that of A.

4 A New Scheme

The previous two sections revealed shortcomings and irrationality of the cost and allocative efficiencies proposed thus far. These shortcomings are caused by the structure of the supposed production possibility set P as defined by:

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0\}. \tag{18}$$

P is defined only by using technical factors $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in R^{m \times n}$ and $Y = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n) \in R^{s \times n}$, but has no concern with the input cost $C = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n)$.

Let us define another cost-based production possibility set P_c as:

$$P_c = \left\{(\bar{\boldsymbol{x}}, \boldsymbol{y}) | \bar{\boldsymbol{x}} \geq \bar{X}\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0\right\}, \tag{19}$$

where $\bar{X} = (\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_n)$ with $\bar{\boldsymbol{x}}_j = (c_{1j}x_{1j}, \ldots, c_{mj}x_{mj})^T$.

In this section, we assume that all inputs are associated with cost, although we will discuss the inclusion of non-cost inputs later. Also we assume that $\bar{x}_{ij} = (c_{ij}x_{ij})$ $(\forall(i,j))$ has a common unit of cost, e.g., dollars, so that adding the elements of $\bar{x}_{ij}$ has meaning.

Based on this new production possibility set, a new technical efficiency $\bar{\theta}^*$ is defined by

$$\begin{aligned} &[\text{NTec}] && \bar{\theta}^* = \min\ \bar{\theta} && (20)\\ &\text{subject to} && \bar{\theta}\bar{\boldsymbol{x}}_o \geq \bar{X}\boldsymbol{\lambda} && (21)\\ &&& \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} && (22)\\ &&& \boldsymbol{\lambda} \geq \boldsymbol{0}. && (23)\end{aligned}$$

The new cost efficiency $\bar{\gamma}^*$ is defined as

$$\bar{\gamma}^* = \boldsymbol{e}\bar{\boldsymbol{x}}_o^*/\boldsymbol{e}\bar{\boldsymbol{x}}_o, \tag{24}$$

where $\boldsymbol{e} \in R^m$ is a row vector with all elements equal to 1 and $\bar{\boldsymbol{x}}_o^*$ is the optimal solution for the LP below:

$$\begin{aligned} &[\text{NCost}] && \min\ \boldsymbol{e}\bar{\boldsymbol{x}} && (25)\\ &\text{subject to} && \bar{\boldsymbol{x}} \geq \bar{X}\boldsymbol{\lambda} && (26)\\ &&& \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} && (27)\\ &&& \boldsymbol{\lambda} \geq \boldsymbol{0}. && (28)\end{aligned}$$

Theorem 3 *The new cost efficiency $\bar{\gamma}^*$ is not greater than the new technical efficiency $\bar{\theta}^*$.*

Proof : Let an optimal solution for (20)-(23) be $(\bar{\theta}^*, \boldsymbol{\lambda}^*)$. Then, $(\bar{\theta}^*\bar{\boldsymbol{x}}_o, \boldsymbol{\lambda}^*)$ is feasible for (25)-(28). Hence, it holds $\boldsymbol{e}\bar{\theta}^*\bar{\boldsymbol{x}}_o \geq \boldsymbol{e}\bar{\boldsymbol{x}}_o^*$. This leads to $\bar{\theta}^* \geq \boldsymbol{e}\bar{\boldsymbol{x}}_o^*/\boldsymbol{e}\bar{\boldsymbol{x}}_o = \bar{\gamma}^*$. □

The new *allocative* efficiency $\bar{\alpha}^*$ is defined as the ratio $\bar{\gamma}^*$ vs. $\bar{\theta}^*$:

$$\bar{\alpha}^* = \bar{\gamma}^*/\bar{\theta}^*. \tag{29}$$

We note that the new efficiency measures $\bar{\theta}^*$, $\bar{\gamma}^*$ and $\bar{\alpha}^*$ are units invariant so long as $\bar{X}$ has a common unit of cost, e.g., dollar, yen or pound.

5 Comments

Here we will comment on the difference between the traditional and new models.

In the traditional model, the unit cost of DMU$_o$ is fixed at $\boldsymbol{c}_o$ and find the optimal input mix $\boldsymbol{x}^*$ that produces the output $\boldsymbol{y}_o$. As we observed in Section 3, this model does not pay attention to possible choices of other unit cost.

In the new model, we search for the optimal input mix $\bar{\boldsymbol{x}}^*$ for producing $\boldsymbol{y}_o$ (or more). More concretely, the optimal mix is described as:

$$\bar{x}_i^* = \sum_{j=1}^{n} c_{ij} x_{ij} \lambda_j^*. \ (i = 1, \ldots, m) \tag{30}$$

Hence, it is assumed that, for a given output $\boldsymbol{y}_o$, the optimal input mix can be found (and realized) independently of the current unit cost $\boldsymbol{c}_o$ of DMU$_o$.

These points are the fundamental differences between the two models. Using the traditional one we cannot recognize the existence of other cheaper input mix, as we demonstrated in Section 3.

Since the worldwide globalization of production has become a current trend, we should be able to find the optimal input mix or, at least, to notify the existence of cheaper one through the cost efficiency evaluation.

For comparisons of the two models, we observe a simple example composed of three DMUs A, B and C each with two inputs (x_1, x_2) and one output (y) along with input costs (c_1, c_2) as exhibited in Table 1.

For DMUs A and B, the traditional model gives the same CCR (θ^*), Cost (γ^*) and Allocative (α^*) scores, as expected from Theorem 2. DMU C is the only one best performer in this framework.

On the other hand, the new scheme distinguishes A and B by assigning $(\bar{\theta}_A^* = 0.1,\ \bar{\theta}_B^* = 1)$ and $(\bar{\gamma}_A^* = 0.085,\ \bar{\gamma}_B^* = 0.85)$. This is caused by the difference in their unit costs. Moreover, DMU B is judged as technically efficient and its cost efficiency rises from $0.35(\gamma_B^*)$ to $0.85(\bar{\gamma}_B^*)$, although DMU A drops it sharply from $0.35(\gamma_A^*)$ to $0.085(\bar{\gamma}_A^*)$. This drop reflects its high cost structure.

Table 1: A Simple Example

	Traditional							
	x_1	c_1	x_2	c_2	y	θ^*	γ^*	α^*
A	10	10	10	10	1	0.5	0.35	0.7
B	10	1	10	1	1	0.5	0.35	0.7
C	5	1	2	6	1	1	1	1
	New Scheme							
	$\bar{x}_1$	$\bar{c}_1$	$\bar{x}_2$	$\bar{c}_2$	y	$\bar{\theta}^*$	$\bar{\gamma}^*$	$\bar{\alpha}^*$
A	100	1	100	1	1	0.1	0.085	0.85
B	10	1	10	1	1	1	0.85	0.85
C	5	1	12	1	1	1	1	1

6 Rationale of the New Scheme

Firstly, we point out that, if the unit cost for inputs, $\boldsymbol{c} = (c_1, \ldots, c_m)$, is the same among all DMUs, the proposed new efficiencies are the same as the traditional ones as described in Sections 2 and 3, and no strange phenomena occur in this case. However, it is quite usual that unit costs, e.g., labor, material and capital, differ from one DMU to another. We will investigate several characteristics of the new measures.

6.1 On the Monotonousness of New Measures with Respect to Cost

Theorem 4 *If $\boldsymbol{x}_A = \boldsymbol{x}_B$, $\boldsymbol{y}_A = \boldsymbol{y}_B$ and $\boldsymbol{c}_A \geq \boldsymbol{c}_B$, then we have inequalities $\bar{\theta}_A^* \leq \bar{\theta}_B^*$ and $\bar{\gamma}_A^* \leq \bar{\gamma}_B^*$. Furthermore, strict inequalities hold if $\boldsymbol{c}_A > \boldsymbol{c}_B$.*

Proof: Since $\bar{\boldsymbol{x}}_A \geq \bar{\boldsymbol{x}}_B$ and $\boldsymbol{y}_A = \boldsymbol{y}_B$, the new technical measure $\bar{\theta}_A^*$ is less than $\bar{\theta}_B^*$ and a strict inequality holds if $\boldsymbol{c}_A > \boldsymbol{c}_B$. Regarding the new cost efficiency, we note that the optimal solution of [NCost] depends only on $\boldsymbol{y}_o$. Hence, DMUs $(\bar{\boldsymbol{x}}_A, \boldsymbol{y}_A)$ and $(\bar{\boldsymbol{x}}_B, \boldsymbol{y}_B)$ with $\boldsymbol{y}_A = \boldsymbol{y}_B$ have a common optimal solution $\bar{\boldsymbol{x}}^*$. Therefore, we have $\bar{\gamma}_A^* = \boldsymbol{e}\bar{\boldsymbol{x}}^*/\boldsymbol{e}\bar{\boldsymbol{x}}_A \leq \boldsymbol{e}\bar{\boldsymbol{x}}^*/\boldsymbol{e}\bar{\boldsymbol{x}}_B = \bar{\gamma}_B^*$ and a strict inequality holds if $\boldsymbol{c}_A > \boldsymbol{c}_B$. □

Thus, the new measure helps to secure against the strange phenomenon observed in Section 3.

6.2 Uses of the Two Technical Efficiency Measures

We have two technical efficiencies θ^* and $\bar{\theta}^*$ for each DMU. The former is determined based only on purely technical input factors, while the latter by input and cost factors. If, for a DMU, θ^* is low and $\bar{\theta}^*$ is high, this suggests the need for input reduction. On the other hand, if θ^* is high and $\bar{\theta}^*$ is low, the DMU will need improvement in cost factors, i.e., cost reduction. Thus, both efficiency measures will be utilized for characterizing the DMU and at the same time suggest directions for improvement. Uses of these two measures prevent the strange case described in Section 2.

Several papers, e.g., Athanassopoulos (1998) and Athanassopoulos *et al.*(1999), adopt this direction of research by separating cost and production efficiency. However, so far as the author knows, none of them have pointed out the difference between the two approaches explicitly.

6.3 On the New Cost Efficiency

Let an optimal solution for the new cost efficiency model (25)-(28) be $(\widehat{\boldsymbol{x}}_o^*, \widehat{\boldsymbol{\lambda}}^*)$. Given the production possibility set P_c, $\widehat{\boldsymbol{x}}_o^*$ is determined dependent only on the output $\boldsymbol{y}_o$ and is not directly related to the input $\bar{\boldsymbol{x}}_o$ of DMU$_o$. Then we have:

Theorem 5 *The new technical efficiency of the activity* $(\widehat{\boldsymbol{x}}_o^*, \boldsymbol{y}_o)$ *is 1.*

Proof: Since $(\widehat{\boldsymbol{x}}_o^*, \boldsymbol{y}_o) \in P_c$, we can evaluate the new technical efficiency $\widehat{\theta}^*$ of $(\widehat{\boldsymbol{x}}_o^*, \boldsymbol{y}_o)$ with respect to P_c by the program below:

$$\widehat{\theta}^* = \min\ \widehat{\theta} \tag{31}$$

$$\text{subject to} \quad \widehat{\theta}\widehat{\boldsymbol{x}}_o^* \geq \bar{X}\boldsymbol{\lambda} \tag{32}$$

$$\boldsymbol{y}_o \leq Y\boldsymbol{\lambda} \tag{33}$$

$$\boldsymbol{\lambda} \geq 0. \tag{34}$$

Suppose that $\widehat{\theta}^* < 1$. Then, from (21), we have $(\widehat{\theta}_o^*\widehat{\boldsymbol{x}}_o^*, \boldsymbol{y}_o) \in P_c$ with an accompanying optimal $\widehat{\boldsymbol{\lambda}}^*$, and $(\widehat{\theta}_o^*\widehat{\boldsymbol{x}}_o^*, \widehat{\boldsymbol{\lambda}}^*)$ is feasible for (26), (27) and (28). The corresponding objective value in (25) is $e\widehat{\theta}_o^*\widehat{\boldsymbol{x}}_o^*$ which is less than $e\widehat{\boldsymbol{x}}_o^*$ since $\widehat{\theta}^* < 1$. This leads to a contradiction. □

Corollary 2 *The activity* $(\widehat{\boldsymbol{x}}_o^*, \boldsymbol{y}_o)$ *is both technical and cost efficient.*

The activity $(\widehat{\boldsymbol{x}}_o^*, \boldsymbol{y}_o)$ is on the boundary of P_c and $\widehat{\boldsymbol{x}}_o^*$ the optimal inputs for the given output $\boldsymbol{y}_o$ with respect to both technical and cost efficiencies.

If the cost efficiency $\gamma^* = \boldsymbol{e}\widehat{\boldsymbol{x}}_o^*/\boldsymbol{e}\bar{\boldsymbol{x}}_o$ is low for the DMU$_o$, the input improvement is given by $\widehat{\boldsymbol{x}}_o^*$. This will be realized by changing $c_{io}x_{io}$ to $\widehat{x}_{io}^*$ for $i = 1, \ldots, m$. There may be several ways, i.e., by changing the input x_{io} to $\widehat{x}_{io}^*/c_{io}$, by changing the cost c_{io} to $\widehat{x}_{io}^*/x_{io}$, or by applying both policies together.

6.4 On the New Allocative Efficiency

The allocative efficiency $\bar{\alpha}^*$ is defined by (29) and is less than or equal to one by Theorem 3. Now, we will investigate the case $\bar{\alpha}^* = 1$. In this case, since it holds $\bar{\theta}^* = \bar{\gamma}^* = \boldsymbol{e}\bar{\boldsymbol{x}}_o^*/\boldsymbol{e}\bar{\boldsymbol{x}}_o$ with $\bar{\boldsymbol{x}}_o^*$ being optimal for [NCost], $\bar{\theta}^*\bar{\boldsymbol{x}}_o$ with it's accompanying optimal $\boldsymbol{\lambda}^*$ is feasible for [NCost] and its objective function value is $\boldsymbol{e}\bar{\theta}^*\bar{\boldsymbol{x}}_o = \boldsymbol{e}\bar{\boldsymbol{x}}_o^*$. Hence, $\bar{\theta}^*\bar{\boldsymbol{x}}_o$ is optimal for [NCost]. This means that the current (cost-based) input mix $\bar{\boldsymbol{x}}_o$ is proportional to the optimal $\bar{\boldsymbol{x}}_o^*$ with the reduction rate $\bar{\theta}^*$. Hence, no mismatch exists between the input mix. The lesser the allocative efficiency is, the more mismatch among input mix is observed.

Thus we have a decomposition of the cost efficiency into the product of the technical and allocative efficiencies:

$$\bar{\gamma}^* = \bar{\theta}^* \times \bar{\alpha}^*. \tag{35}$$

7 Estimation of Returns to Cost

Up to now, we have dealt with the new efficiency measure issues under the constant returns to scale environment. However, we can extend our results to the variable returns to scale case as well by imposing a constraint on $\boldsymbol{\lambda}$:

$$\boldsymbol{e}\boldsymbol{\lambda} = 1. \tag{36}$$

The concept of variable returns to scale leads to identify the behavior of average productivity for each DMUs, i.e., *increasing, constant* or *decreasing.* See Banker and Thrall (1992) and Cooper *et al.* (1999) among others for the detailed process of identification.

However, in our case, the term "returns to cost" will be appropriate, since we are dealing with the production possibility set P_c that is defined as a relationship between input costs and outputs.

8 An Empirical Study

In this section, we apply our new method to a set of hospital data and observe the results. Table 2 records behavior of 12 hospitals in terms of two inputs, number of doctors and nurses, and two outputs identified as number of outpatients and inpatients (each in units of 100 person/month). Relative unit costs of doctors and nurses for each hospital are also recorded.

Table 2: Data for 12 Hospitals

		Inputs				Outputs	
		Doctor		Nurse		Outpat.	Inpat.
No.	DMU	Number	Cost	Number	Cost	Number	Number
1	A	20	500	151	100	100	90
2	B	19	350	131	80	150	50
3	C	25	450	160	90	160	55
4	D	27	600	168	120	180	72
5	E	22	300	158	70	94	66
6	F	55	450	255	80	230	90
7	G	33	500	235	100	220	88
8	H	31	450	206	85	152	80
9	I	30	380	244	76	190	100
10	J	50	410	268	75	250	100
11	K	53	440	306	80	260	147
12	L	38	400	284	70	250	120

By multiplying the number and unit cost of doctors and nurses, respectively, we obtained the new data set $(\bar{X}, Y)$ as exhibited in Table 3. The results of efficiency scores: CCR(θ^*), New technical ($\bar{\theta}^*$), New cost ($\bar{\gamma}^*$) and New allocative ($\bar{\alpha}^*$), are also recorded.

From the results, it can be concluded that the best performer is Hospital B with all scores being one. Regarding the cost-based measures, Hospitals

Table 3: New Data Set and Efficiencies

		Data				Efficiency			
		$\bar{X}$		Y		CCR	Tech.	Cost	Alloc.
No.	DM	Doctor	Nurse	Inp.	Outp.	θ^*	$\bar{\theta}^*$	$\bar{\gamma}^*$	$\bar{\alpha}^*$
1	A	10000	15100	100	90	1	.994	.959	.965
2	B	6650	10480	150	50	1	1	1	1
3	C	11250	14400	160	55	.883	.784	.724	.923
4	D	16200	20160	180	72	1	.663	.624	.941
5	E	6600	11060	94	66	.763	1	1	1
6	F	24750	20400	230	90	.835	.831	.634	.764
7	G	16500	23500	220	88	.902	.695	.693	.997
8	H	13950	17510	152	80	.796	.757	.726	.959
9	I	11400	18544	190	100	.960	.968	.953	.984
10	J	20500	20100	250	100	.871	.924	.776	.841
11	K	23320	24480	260	147	.955	.995	.863	.867
12	L	15200	19880	250	120	.958	1	1	1

E and L received full marks. Although E has the worst CCR score (0.763), its lowest unit costs push up the cost-based rank to the top. This hospital still leaves room for input reduction compared with other technically efficient hospitals. Hospital L may be positioned in the best performer group.

On the other hand, Hospital D is ranked as the worst with respect to cost-based measures, although D received full marks in the CCR score. This gap is caused by its high cost structure. D needs cost reduction to attain good cost-based scores.

Hospital F has the worst allocative efficiency and hence needs change in input-mix. This hospital has the current input mix $\bar{x}_F = (24750, 20400)$, while the optimal mix $\bar{x}_F^*$ is $(11697, 16947)$. So, if F sticks to its current costs, it must reduce the number of doctors from 55 to 11697/450=26, and nurses from 255 to 16947/80=212. Or, if F pays attention to the current input numbers, it must reduce the unit cost of doctors from 450 to 11697/55=213, and that of nurses from 80 to 16947/255=66. Of course, there are many other compromise plans.

9 Extensions

We extend the new model to other situations. First, we deal with inclusion of non-cost input factors and then extend the model to a new revenue efficiency measure.

9.1 Non-Cost Inputs

We divide inputs into two parts: cost-related and non-related. Let us denote the former by $\boldsymbol{x}_C$ with the unit cost $\boldsymbol{c}$, and the latter by $\boldsymbol{x}_N$. Then, we have the following two LPs for evaluating the corresponding technical and cost efficiencies.

$$
\begin{aligned}
\text{[LP1]} \quad & \bar{\theta}^* = \min\ \bar{\theta} && (37)\\
\text{subject to} \quad & \bar{\theta}\bar{\boldsymbol{x}}_{Co} \geq \bar{X}_C\boldsymbol{\lambda} && (38)\\
& \bar{\theta}\boldsymbol{x}_{No} \geq X_N\boldsymbol{\lambda} && (39)\\
& \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} && (40)\\
& \boldsymbol{\lambda} \geq \boldsymbol{0}, && (41)
\end{aligned}
$$

where $\bar{x}_{Cij} = c_{ij}x_{Cij}$, $\bar{X}_C = (\bar{\boldsymbol{x}}_{C1}, \ldots, \bar{\boldsymbol{x}}_{Cn})$ and $X_N = (\boldsymbol{x}_{N1}, \ldots, \boldsymbol{x}_{Nn})$.

$$
\begin{aligned}
\text{[LP2]} \quad & \boldsymbol{e}\bar{\boldsymbol{x}}_C^* = \min\ \boldsymbol{e}\bar{\boldsymbol{x}}_C && (42)\\
\text{subject to} \quad & \bar{\boldsymbol{x}}_C \geq \bar{X}_C\boldsymbol{\lambda} && (43)\\
& \boldsymbol{x}_N \geq X_N\boldsymbol{\lambda} && (44)\\
& \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} && (45)\\
& \boldsymbol{\lambda} \geq \boldsymbol{0}. && (46)
\end{aligned}
$$

Using the optimal objective value $\boldsymbol{e}\bar{\boldsymbol{x}}_C^*$ of [LP2], we define the cost efficiency of DMU$_o$ by

$$\bar{\gamma}^* = \boldsymbol{e}\bar{\boldsymbol{x}}_C^*/\boldsymbol{e}\bar{\boldsymbol{x}}_{Co}. \qquad (47)$$

Since the optimal solution $(\bar{\theta}^*\bar{\boldsymbol{x}}_{Co}, \bar{\theta}^*\boldsymbol{x}_{No}, \boldsymbol{\lambda}^*)$ for [LP1] is feasible for [LP2], it holds $\boldsymbol{e}\bar{\theta}^*\bar{\boldsymbol{x}}_{Co} \geq \boldsymbol{e}\bar{\boldsymbol{x}}_C^*$ and hence we have $\bar{\theta}^* \geq \boldsymbol{e}\bar{\boldsymbol{x}}_C^*/\boldsymbol{e}\bar{\boldsymbol{x}}_{Co} = \bar{\gamma}^*$. Therefore, Theorem 3 is valid under this new model.

9.2 Revenue Efficiency

Given the unit price $\boldsymbol{p}_j$ for each output $\boldsymbol{y}_j$ $(j = 1, \ldots, n)$, the conventional revenue efficiency ρ_o^* of DMU_o is evaluated by $\rho_o^* = \boldsymbol{p}_o\boldsymbol{y}_o/\boldsymbol{p}_o\boldsymbol{y}_o^*$. Here, $\boldsymbol{p}_o\boldsymbol{y}_o^*$ is obtained as the optimal objective value of the following LP:

$$\boldsymbol{p}_o\boldsymbol{y}_o^* = \max \boldsymbol{p}_o\boldsymbol{y} \tag{48}$$

$$\text{subject to} \quad \boldsymbol{x}_o \geq X\boldsymbol{\lambda} \tag{49}$$

$$\boldsymbol{y} \leq Y\boldsymbol{\lambda} \tag{50}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}. \tag{51}$$

This efficiency ρ_o^* suffers from similar phenomena as the traditional cost efficiency measure described in Sections 2 and 3.

We can get rid of such shortcomings by introducing the price-based output $\bar{Y} = (\bar{\boldsymbol{y}}_1, \ldots, \bar{\boldsymbol{y}}_n)$ with $\bar{\boldsymbol{y}}_j = (p_{1j}y_{1j}, \ldots, p_{sj}y_{sj})^T$, and by solving the following LP:

$$\boldsymbol{e}\bar{\boldsymbol{y}}_o^* = \max \boldsymbol{e}\bar{\boldsymbol{y}} \tag{52}$$

$$\text{subject to} \quad \boldsymbol{x}_o \geq X\boldsymbol{\lambda} \tag{53}$$

$$\bar{\boldsymbol{y}} \leq \bar{Y}\boldsymbol{\lambda} \tag{54}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}. \tag{55}$$

The new revenue efficiency measure $\bar{\rho}_o^*$is defined by

$$\bar{\rho}_o^* = \boldsymbol{e}\bar{\boldsymbol{y}}_o/\boldsymbol{e}\bar{\boldsymbol{y}}_o^*. \tag{56}$$

10 Concluding Remarks

Data Envelopment Analysis (DEA), initiated by Charnes, Cooper and Rhodes (1978) has been applied to many problems for evaluating the relative efficiency of Decision Making Units with multiple inputs and outputs. Applications include measurements of both production and cost efficiencies. However, many measurements of cost and allocative efficiencies were carried out by dint of the traditional method cited in this paper and have shortcomings as demonstrated in this paper.

We have proposed a new scheme for evaluating cost efficiency in DEA that is free from such shortcomings.

We hope that the new scheme will contribute to fathom the true level of cost and allocative efficiencies.

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