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Abstract

This paper investigates the optimality condition of transport network development in a closed city with residents' and absentee land ownerships. We set up an urban land use model in which, taking prices and characteristics of transport network as given, households that are identical in their preferences and endowments maximize utility by choosing residential location, lot size, and travel modes. Social planner then optimizes with respect to the characteristics of transportation network so as to maximize the level of utility in the spatial equilibrium. The key findings of this paper include that under resident landlord case the general optimality condition of the transport network improvement is such that the marginal cost of improvement is equal to the marginal increase in the aggregated differential land rent evaluated at current level of land rent.

Keywords: urban land use, transportation network development, land ownership

JEL codes: R14, R21, R42

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1 Introduction

Transport network improvement is a very old but important issue for urban planners in most of the cities. There are a bunch of literature in this field. Some literature is limited on the analysis of purely transport models ignoring its effect on the city characteristics which, in turn, may influence the transport investment. One the other hand there are some literature that considers the transport network improvement within the city framework and therefore draws more attention then the previous case. However, most of the existing literature of monocentric city models deals with the shape, size of a city along with the locational and modal choice, transport network and urban land use pattern under the existence of radial highway/train transport. Literature is limited on the analysis of transport network improvement within a city model under general settings. Given this background, in this paper we try to find the social optimal requirements that is to be considered for the improvement of transport network under different land ownerships in a more general setup.

The urban land use theory using monocentric city models are pioneered by Alonso (1964), and further analyzed by Mills and de Ferranti (1971), Solow (1973), and then by Brueckner (1987). Mills and de Ferranti found the location-varying optimal capacity of city streets while Solow analyzed the land use pattern and locational choice in an equilibrium. These models are mainly monocentric and consider only dense city street as the means of travel to the CBD. In contrast, Anas and Moses (1977) take into account the radial highways explicitly along with the dense city streets, and explain the shape of a city depending on various generalized transport cost structures. Unfortunately, all of these class of literature consider specific type of transport mode e.g. dense city street, radial highway or railway network. Thus the existing literature still lags of having optimality condition of transport network improvement within a city model that can be equally applicable for any type of transport network improvement, any number of transport mode options, and irrespective of the number of the city center and shape of the city.

Given this limitation of the existing literature, this paper formulate a general model of city with a transport network that can be of any form e.g. dense city streets, radial or circumferential highway, railway, subway etc. The purpose of this paper is to determine the optimality conditions of the transport network improvement in a closed city model under resident and absentee land ownership. The key findings of this paper is that when land is owned by households, the general optimality condition of the transport network improvement is such that the marginal cost of improvement is equal to the marginal increase in the aggregated differential land rent evaluated at current level of land rent.

The capitalization hypothesis says that any investment in public projects is fully capitalized into the land values. Thus, for example, the capital investment in a transport project will be reflected in the land value.

Kanemoto (1984) works on the optimal pricing and investment policy of transport infrastructure in a monocentric city. Considering radial railway network in a circular city, this paper finds that the maximum length of the railway is such that the cost of extending the railway should be equal to the total differential land rent. Another finding of this paper, that lies in line with the capitalization hypothesis, is that the reduction in net income due to the increase transport fair is fully capitalized in the decline in land rent.

Under general equilibrium settings in another paper Kanemoto (2011) tries to calculate the benefit of a transport investment project. Defining benefit as the change in utility he shows that the benefit can be determined by the increase in real national income when the change in transport capacity is small. Using quasi-linear utility function with marginal utility of income to be unity he finds that for a downward slopping general equilibrium demand curve the gross consumer surplus is higher than the increase in consumption multiplied by the price before the investment and is lower than that multiplied by the price after the investment. Thus the real national income determined by using pre-investment price gives the upper limit of the benefit and the real national income determined by using post-investment price gives the lower limit of the benefit. Kanemoto. (2011) also shows that under general equilibrium settings when there is no market failure, the benefit of a public investment project can be determined by the reduction in transport cost only. Thus one do not need to consider the the change in benefits in all other sectors. Since in the first best case under general equilibrium, price is set equal to marginal cost, the benefits and costs of all other sectors become equal and thus cancels each other. The change in social surplus is the change in the increase in consumer surplus over the project cost.

The findings of this paper is different from that of the capitalization hypothesis, measurement of benefit based on national income approach, and evaluation of public project under general equilibrium demand function. All of these approaches mainly explain how to measure the benefits of a public project. For the resident landlord case our paper finds that the marginal cost of network improvement is equal to the marginal increase in the aggregated differential land rent (caused by change in land area) evaluated at pre-project land price. Ont the other hand, for absentee landlord case the the marginal cost of network improvement is equal to the marginal increase in the aggregated differential land rent (caused by change in land price) evaluated in post-project land price. So, in both cases our optimality conditions does not require to consider the total change of the aggregate differential land rent, we just need to consider the partial change of it. To the best of our knowledge our optimality condition is the unique one in urban transport model under a very general settings.

The flow of the paper is as follows. In section 2, we set up a model with general settings and solved for the general optimality conditions under resident landlord case. The optimality condition for absentee landlord has been presented in section 3. Section 4 concludes our paper. All proofs relevant derivations were summarized in the appendix.

2 Optimal Transportation Network in a City with Resident Landlord

In this section we investigate the condition for designing the optimal transport network in a city where the land is owned by the households living within the city. We first set up a model with a closed city where a fixed number of households that are identical in their preferences and endowments reside, commute, and work. Households maximize utility by choosing the amount of lot, other goods, leisure time, as well as residential location and mode of commuting. Alternative to urban land use is agriculture. Agricultural land market is perfectly competitive and thus needs to pay only the agricultural land rent. There is a benevolent city government who runs transport sector, collects tax from the city residents to raise fund and uses this fund to construct transport network. We assume full employment: one member per household is commuting to CBD where production takes place.[1] This assumption leads to the fact that the demand for transport service is inelastic. Improving the transport network facilitates commuting by reducing the generalized transport cost and therefore increases the utility of households. While the land allocated for the transport network is ignored, there is an alternative of agricultural use of land with uniform opportunity cost.

2.1 Generalized transport Costs

In this city there are multiple travel modes, combining one or more of which consists a route of travel from a given location to CBD. For example, modes can be auto, rail, and walking. Alternatively, one can consider modes to be highway and city streets, and commuters can choose a route on highways, city streets, or both to get to CBD.

The size and therefore the transport cost within CBD are assumed to be negligible. Let (r, θ) represent the location in the city in polar coordinates. Let $T_h(r, \theta)$ and $C_h(r, \theta)$ be the time and monetary costs of commuting from (r, θ) to CBD via route h.[2] Let us define G_h as the generalized

[1] We begin by considering a very general model in this section. The number of CBD does not have to be one nor located at the center of the city. Moreover the shape of the city does not have to be circular nor symmetric, and it can be as large as a country.

[2] Just as the CBD is not necessarily unique, its location does not have to be at $(0, 0)$ to derive the main results. Similarly, the dimension of the city can be more, or less than two to derive the main results. However in what follows,

transport cost of commuting from (r,θ) to CBD via route h such that

$$G_h = wT_h(r,\theta) + C_h(r,\theta). \tag{1}$$

Thus the generalized transport cost captures both time cost and monetary cost.

Residents at each location choose the route that incurs the lowest generalized transport cost to CBD. We define $G(r,\theta)$ as the minimum generalized transport cost of all routes at location (r,θ), that is

$$G(r,\theta) = \min_h G_h(r,\theta). \tag{2}$$

2.2 Household's Utility Maximization Problem

Each household consumes three kinds of goods: residential land Q, all other non-residential goods termed as composite goods Z, and leisure time L. Each household has a fixed endowment of time, say H that she allocates into leisure, working, and commuting time. Thus the residents living far away from the city center will have greater commuting time and lower leisure time.

We can divide household's decision problem into three stages. In the first stage, the household chooses the location (r,θ). In the second stage, they choose the route h; and at the third stage they choose the amount of consumption of composite good Z, land lot Q, and the leisure time L. We solve this problem backward.

In the third stage, each household maximizes utility given the location (r,θ) and route h, given the budget and time constraints. Wage rate w is exogenously given and the same for all residents. In addition to wage income, each household receives an equal share of aggregated differential land rent denotedy by Φ and pays the lump-sum tax D that will finance the highway construction costs. This yields the indirect utility function conditional on the location and the route h. In the second stage, the household chooses the route h to maximize this conditional indirect utility. This problem is identical to the route-choice problem of minimizing the generalized transport cost. In the first stage household maximizes this indirect utility by choosing the location.

We describe the dual of the above. Denoting the land rent at location (r,θ) by $\widetilde{R}(r,\theta)$ household's expenditure minimization problem becomes

$$\begin{aligned} E\left(\widetilde{R}(r,\theta), U\right) &= \min_{Z,Q,L} Z + \widetilde{R}(r,\theta)Q + wL \\ & \quad s.t.\ U = U(Z,Q,L) \end{aligned}$$

where E is the expenditure function. Solving this yields the compensated demand functions

$$\begin{aligned} Q^c &= Q^c\left(\widetilde{R}(r,\theta), U\right) \\ Z^c &= Z^c\left(\widetilde{R}(r,\theta), U\right) \\ L^c &= L^c\left(\widetilde{R}(r,\theta), U\right). \end{aligned}$$

By the envelope theorem we have

$$\begin{aligned} \frac{\partial E}{\partial \widetilde{R}} &= Z_R^c + Q^c + \widetilde{R}(r,\theta)Q_R^c + wL_R^c \\ &= Q^c \end{aligned}$$

or equivalently

$$Z_R^c + \widetilde{R}(r,\theta)Q_R^c + wL_R^c = 0$$

our expostition postulates a classic two-dimentional monocentric city where CBD is at its center.

where subscript R represents the partial derivatives with respect to the land rent[3].

The bid rent function $\widetilde{R}_h(r,\theta)$ gives the maximum amount that household commuting via route h is willing to pay for the land lot provided that it achieves a given utility level:

$$\widetilde{R}_h(r,\theta) = \frac{1}{Q^c}\left[wH - D + \frac{\Phi}{N} - G_h(r,\theta) - Z^c - wL^c\right]. \tag{3}$$

Since the land goes to the highest bidder the land rent is the maximum of the bid rents:

$$\widetilde{R}(r,\theta) = \max_h \widetilde{R}_h(r,\theta).$$

The argument of the maximum here coincides with that of the minimum of generalized transport cost given in (2).

2.3 Spatial Equilibrium

Since the city is closed, while the population is fixed the equilibrium utility level as well as the city boundary are endogenously determined. Homogeneity of the households implies that they achieve the same level of utility regardless of the location, which we refer to as the spatial equilibrium condition. Define the utility level that all households achieve in the spatial equilibrium as $\overline{U}$. The land rent varies across location to let all households achieve the same utility level, and at the city boundary the residential land rent equals to the agricultural land rent that is exogenously given.

Let N be the number of households and $\mathfrak{R}$ be the set of all locations in the city such that

$$\mathfrak{R} \equiv \left\{(r,\theta)\,|\widetilde{R}(r,\theta) \geq R_A\right\}$$

where R_A is an agricultural rent as opportunity cost of the land. Then we have

$$N = \iint_{(r,\theta)\in\mathfrak{R}} \frac{r dr d\theta}{Q^c\left(\widetilde{R}(r,\theta), \overline{U}\right)}. \tag{4}$$

At the city boundary the residential bid rent must be equal to the agricultural land rent:

$$\widetilde{R}\left(\overline{r},\overline{\theta}\right) = R_A \tag{5}$$

where $\left(\overline{r},\overline{\theta}\right)$ represent the location at the city boundary. The aggregate differential land rent Φ is such that

$$\Phi = \iint_{(r,\theta)\in\mathfrak{R}} \left[\widetilde{R}(r,\theta) - R_A\right] r dr d\theta. \tag{6}$$

Solving equations (4), (5) and (6) we get $\overline{U}$, $\left(\overline{r},\overline{\theta}\right)$ and Φ in terms of all exogenous variables as well as the lump-sum tax D.

To solve this problem we use isomorphism. Let us now define the land rent R as a function of G such that

$$R(G(r,\theta)) = \widetilde{R}(r,\theta)$$

and rewrite the consumption of lot, composite good, and leisure in terms of G:

$$Q^*(G) = Q^c\left(R(G), \overline{U}\right) \tag{7}$$
$$Z^*(G) = Z^c\left(R(G), \overline{U}\right) \tag{8}$$
$$L^*(G) = L^c\left(R(G), \overline{U}\right) \tag{9}$$

[3] See Kanemoto (1977)

where $\overline{U}$ is the spatial equilibrium level of utility as obtained above. Using these, $R(G)$ satisfies the following:

$$R(G) \quad = \quad \frac{1}{Q^*(G)}\left[y^d(G) - Z^*(G) - wL^*(G)\right] \tag{10}$$

where y^d is the disposable income such that

$$y^d(G) \quad = \quad wH - D + \frac{\Phi}{N} - G.$$

Define $\xi(G)\,dG$ as the land area where the generalized transport cost is equal to G. That is, by letting $\Xi(G)$ be the area of the city where the generalized transport cost is less than or equal to G, we have

$$\xi(G) = \frac{d\Xi(G)}{dG} \tag{11}$$

for $G \geq 0$. Then the spatial equilibrium can now be rewritten in terms of G:

$$N \quad = \quad \int_0^{\overline{G}} \frac{\xi(G)\,dG}{Q^*(G)} \tag{12}$$

$$R\left(\overline{G}\right) \quad = \quad \frac{wH - D + \frac{\Phi}{N} - \overline{G} - Z^*\left(\overline{G}\right) - wL^*\left(\overline{G}\right)}{Q^*\left(\overline{G}\right)} = R_A \tag{13}$$

$$\Phi \quad = \quad \int_0^{\overline{G}} \left(R(G) - R_A\right)\xi(G)\,dG \tag{14}$$

where $\overline{G}$ is the generalized transport cost at the city boundary, i.e., $\overline{G} = G\left(\overline{r}, \overline{\theta}\right)$. Equilibrium utility $\overline{U}$, generalized transport cost at the city boundary $\overline{G}$, and aggregated differential land rent Φ satisfy these three equations, which are expressed as

$$\overline{U} \quad = \quad \overline{U}\left(D, \Xi(G), n, w, H, N, R_A\right) \tag{15}$$

$$\overline{G} \quad = \quad \overline{G}\left(D, \Xi(G), n, w, H, N, R_A\right) \tag{16}$$

$$\Phi \quad = \quad \Phi\left(D, \Xi(G), n, w, H, N, R_A\right). \tag{17}$$

2.4 Optimization of Transportation Network

Construction cost of the transport network depends on the characteristics of network improvement, for example, the number and the length of the radial highways, the location and capacity of a circumferential highway, and so on. We denote the construction cost and these characteristics of the transport network by K and a vector η respectively, i.e.,

$$K = K(\eta). \tag{18}$$

Transport authority maintains a balanced budget; therefore, construction cost K must be financed by lump-sum tax D collected equally from the households:

$$K = DN. \tag{19}$$

The transport authority will optimize the transport network in the city so as to maximize the equilibrium utility level. Transport authority's problem is thus

$$\max_{(\eta)} \overline{U} \quad = \quad \overline{U}\left(D, \Xi(G), n, w, H, N, R_A\right)$$

subject to the budget constraint (19).

Total differentiation of the equations (7) through (9) as well as (10) yields that, for a given G,

$$\begin{aligned} dQ^* &= \frac{Q_R^c}{Q^*(G)}\left(-dD+\frac{d\Phi}{N}\right)+\left[Q_U^c-\frac{\lambda(G)\,Q_R^c}{Q^*(G)}\right]d\overline{U} \\ dZ^* &= \frac{Z_R^c}{Q^*(G)}\left(-dD+\frac{d\Phi}{N}\right)+\left[Z_U^c-\frac{\lambda(G)\,Z_R^c}{Q^*(G)}\right]d\overline{U} \\ dL^* &= \frac{L_R^c}{Q^*(G)}\left(-dD+\frac{d\Phi}{N}\right)+\left[L_U^c-\frac{\lambda(G)\,L_R^c}{Q^*(G)}\right]d\overline{U} \end{aligned} \tag{20}$$

where subscripts R and U represent the partial derivatives and $\lambda(G)\equiv \partial E/\partial U = Z_U^c + R(G)\,Q_U^c + wL_U^c$.[4] By using these, total differentiation of the equations (12) through (14) as well as (18) and (19) with respect to network characteristics η gives

$$\begin{aligned} 0 &= \int_0^{\overline{G}} \frac{1}{Q^*(G)}\left[-\frac{\xi(G)}{Q^*(G)}dQ^*+\frac{\partial\xi(G)}{\partial\eta}d\eta\right]dG+\frac{\xi(\overline{G})}{Q^*(\overline{G})}d\overline{G} \\ 0 &= -dD+\frac{d\Phi}{N}-d\overline{G}-dZ^*|_{\overline{G}}-w\;dL^*|_{\overline{G}}-R_A\;dQ^*|_{\overline{G}} \\ d\Phi &= \int_0^{\overline{G}}\left[\left(\frac{-dD+\frac{d\Phi}{N}-\lambda(G)\,d\overline{U}}{Q^*(G)}\right)\xi(G)+(R(G)-R_A)\frac{\partial\xi(G)}{\partial\eta}d\eta\right]dG \\ dK &= \frac{\partial K}{\partial\eta}d\eta \\ dD &= \frac{dK}{N}. \end{aligned} \tag{21}$$

These reduce to

$$\begin{aligned} &\left[\int_0^{\overline{G}}\frac{\xi(G)}{(Q^*(G))^2}\left(Q_U^c-\frac{\lambda(G)-\lambda(\overline{G})}{Q^*(G)}Q_R^c\right)dG\right]d\overline{U} \\ &-\left[\frac{\xi(\overline{G})}{Q^*(\overline{G})}-\int_0^{\overline{G}}\frac{\xi(G)}{(Q^*(G))^3}Q_R^c dG\right]d\overline{G} \\ =&\left[\int_0^{\overline{G}}\frac{1}{Q^*(G)}\frac{\partial\xi(G)}{\partial\eta}dG\right]d\eta \end{aligned} \tag{22}$$

where

$$d\overline{U} = \left[\int_0^{\overline{G}}\lambda(G)\frac{\xi(G)}{Q^*(G)}dG\right]^{-1}\left[\int_0^{\overline{G}}(R(G)-R_A)\frac{\partial\xi(G)}{\partial\eta}dG-\frac{\partial K}{\partial\eta}\right]d\eta. \tag{23}$$

These eventually become

$$\frac{d\overline{G}}{d\eta} = -\frac{\int_0^{\overline{G}}\frac{1}{Q^*(G)}\frac{\partial\xi(G)}{\partial\eta}dG}{\frac{\xi(\overline{G})}{Q^*(\overline{G})}-\int_0^{\overline{G}}\frac{\xi(G)}{(Q^*(G))^3}Q_R^c dG} \tag{24}$$

where

$$\frac{\partial K}{\partial\eta} = \int_0^{\overline{G}}(R(G)-R_A)\frac{\partial\xi(G)}{\partial\eta}dG. \tag{25}$$

[4]See appendix for derivation.

Proposition 1. *At the optimum marginal increase in the construction cost of the circumferential highway and the aggregate differential land rent, evaluated at the current level of land rent, due to a marginal change in the network design must be equal.*

We also have,

$$\begin{aligned} \frac{d\Phi}{d\eta} &= \frac{\partial K}{\partial \eta} + N \frac{d\overline{G}}{d\eta} \\ &= N \frac{d\overline{G}}{d\eta} + \int_0^{\overline{G}} \left(R(G) - R_A\right) \frac{\partial \xi(G)}{\partial \eta} dG. \end{aligned}$$

Proposition 2. *At the optimum the marginal change in aggregate differential land rent due to a change in network design is equal to the sum of marginal change in construction cost and marginal change in generalized transport cost at the city multiplied by the total number of household.*

This further implies that the disposable income of the household living at the city boundary is unchanged by optimal network improvement.

Beside equation (24) we can express $\frac{d\overline{G}}{d\eta}$ in a more useful form. We have,

$$\frac{d\Phi}{d\eta} = \int_0^{\overline{G}} \left(R(G) - R_A\right) \frac{\partial \xi(G)}{\partial \eta} dG + \int_0^{\overline{G}} \xi(G) \frac{\partial R(G)}{\partial \eta} dG$$

which after using proposition 1 and 2 yields

$$\frac{d\overline{G}}{d\eta} = \frac{1}{N} \int_0^{\overline{G}} \xi(G) \frac{\partial R(G)}{\partial \eta} dG.$$

Proposition 3. *At the optimum, the marginal change in generalized transport cost at the city boundary is equal to the per head marginal change in land value due to the change in land rent.*

It is to be noted that proposition 2 is independent of proposition 1 and 3. Proposition 1 is independent of proposition 3; however depends on proposition 2. Thus even if proposition 3 does not hold we have proposition 1.

The optimization outcome in proposition 1 is quite interesting. It says that when the land is owned by the households, for any type of transport projects, we do not need to consider the post-project land price to determine the optimum investment. Equating the marginal construction with the marginal aggregate differential land rent evaluated in terms of the pre-project land price is sufficient to fix the optimization.

3 Optimal Transportation Network Under Absentee Landlord

In this section we derive an optimality condition under absentee landlord assumption. The basic model setting is the same as that of section 2 in all respects except that the households do not receive the redistribution of land rent revenue. In the absentee landlord case landlords live out of the city and therefore they are not not considered as a city dweller, and the full portion of aggregate differential land rent goes to the absentee landlords. The transport authority maximizes the utility of the city dwellers, not considering the welfare of the absentee landlords.

The disposable income of the city dweller now becomes

$$y^d(G) = wH - D - G$$

and the residential land rent is

$$R(G) = \frac{1}{Q^*(G)}\left[y^d(G) - Z^*(G) - wL^*(G)\right]. \tag{26}$$

Thus at spatial equilibrium we have

$$N = \int_0^{\overline{G}} \frac{\xi(G)\,dG}{Q^*(G)} \tag{27}$$

$$R\left(\overline{G}\right) = \frac{wH - D - \overline{G} - Z^*\left(\overline{G}\right) - wL^*\left(\overline{G}\right)}{Q^*\left(\overline{G}\right)} = R_A \tag{28}$$

$$\Phi = \int_0^{\overline{G}} \left(R(G) - R_A\right)\xi(G)\,dG \tag{29}$$

Optimization of transport Network

Again, transport authority maximizes the equilibrium utility level given the cost function K and the budget constraint. Taking total differentiation of the equation (27), (28) as well as (18) and (19) with respect to network design η yields

$$\begin{aligned}
0 &= \int_0^{\overline{G}} \frac{1}{Q^*(G)}\left[-\frac{\xi(G)}{Q^*(G)}dQ^* + \frac{\partial\xi(G)}{\partial\eta}d\eta\right]dG + \frac{\xi\left(\overline{G}\right)}{Q^*\left(\overline{G}\right)}d\overline{G} \\
0 &= -dD - d\overline{G} - dZ^*|_{\overline{G}} - w\;dL^*|_{\overline{G}} - R_A\;dQ^*|_{\overline{G}} \\
dK &= \frac{\partial K}{\partial\eta}d\eta \\
dD &= \frac{dK}{N}
\end{aligned}$$

where

$$\begin{aligned}
dQ^* &= \frac{Q_R^c}{Q^*(G)}(-dD) + \left[Q_U^c - \frac{\lambda(G)\,Q_R^c}{Q^*(G)}\right]d\overline{U} \\
dZ^* &= \frac{Z_R^c}{Q^*(G)}(-dD) + \left[Z_U^c - \frac{\lambda(G)\,Z_R^c}{Q^*(G)}\right]d\overline{U} \\
dL^* &= \frac{L_R^c}{Q^*(G)}(-dD) + \left[L_U^c - \frac{\lambda(G)\,L_R^c}{Q^*(G)}\right]d\overline{U}.
\end{aligned} \tag{30}$$

Using equation (30) together with the fact that $Z_R^c + RQ_R^c + wL_R^c = 0$ and eliminating dD in the above equation we get

$$\begin{aligned}
0 = -&\left[\int_0^{\overline{G}} \frac{\xi(G)}{\left(Q^*(G)\right)^3}Q_R^c dG\right]\left(d\overline{G} + \lambda\left(\overline{G}\right)d\overline{U}\right) - \left[\int_0^{\overline{G}} \frac{\xi(G)}{\left(Q^*(G)\right)^2}\left(Q_U^c - \frac{\lambda(G)\,Q_R^c}{Q^*(G)}\right)dG\right]d\overline{U} \\
&+ \left[\int_0^{\overline{G}} \frac{1}{Q^*(G)}\frac{\partial\xi(G)}{\partial\eta}dG\right]d\eta + \frac{\xi\left(\overline{G}\right)}{Q^*\left(\overline{G}\right)}d\overline{G}
\end{aligned}$$

and

$$d\overline{G} = -\frac{1}{N}\frac{\partial K}{\partial\eta}d\eta - \lambda\left(\overline{G}\right)d\overline{U}$$

which can be written as

$$0 = -\left[\int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^3} Q_R^c dG\right]\left(\frac{d\overline{G}}{d\eta} + \lambda\left(\overline{G}\right)\frac{d\overline{U}}{d\eta}\right) - \left[\int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^2}\left(Q_U^c - \frac{\lambda(G) Q_R^c}{Q^*(G)}\right) dG\right]\frac{d\overline{U}}{d\eta}$$
$$+\left[\int_0^{\overline{G}} \frac{1}{Q^*(G)}\frac{\partial \xi(G)}{\partial \eta} dG\right] + \frac{\xi\left(\overline{G}\right)}{Q^*\left(\overline{G}\right)}\frac{d\overline{G}}{d\eta}$$

and

$$\frac{d\overline{G}}{d\eta} = -\frac{1}{N}\frac{\partial K}{\partial \eta} - \lambda\left(\overline{G}\right)\frac{d\overline{U}}{d\eta}. \tag{31}$$

Combining these two equations we get

$$\frac{d\overline{U}}{d\eta} = \frac{-\left[\frac{\xi\left(\overline{G}\right)}{Q^*(G)} - \int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^3} Q_R^c dG\right]\frac{1}{N}\frac{\partial K}{\partial \eta} + \left[\int_0^{\overline{G}} \frac{1}{Q^*(G)}\frac{\partial \xi(G)}{\partial \eta} dG\right]}{\left[\int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^2}\left\{Q_U^c - \frac{\lambda(G) Q_R^c}{Q^*(G)}\right\} dG + \lambda\left(\overline{G}\right)\frac{\xi\left(\overline{G}\right)}{Q^*\left(\overline{G}\right)}\right]}$$

Therefore at optimum we have

$$-\left[\frac{\xi\left(\overline{G}\right)}{Q^*\left(\overline{G}\right)} - \int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^3} Q_R^c dG\right]\frac{1}{N}\frac{\partial K}{\partial \eta} + \left[\int_0^{\overline{G}} \frac{1}{Q^*(G)}\frac{\partial \xi(G)}{\partial \eta} dG\right] = 0$$

which can be written as

$$\frac{\partial K}{\partial \eta} = N \frac{\int_0^{\overline{G}} \frac{1}{Q^*(G)}\frac{\partial \xi(G)}{\partial \eta} dG}{\frac{\xi\left(\overline{G}\right)}{Q^*\left(\overline{G}\right)} - \int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^3} Q_R^c dG} \tag{32}$$

which can also be expressed by using equation (31) at optimum as

$$\frac{\partial K}{\partial \eta} = -N\frac{d\overline{G}}{d\eta}.$$

Proposition 4. *In a closed city with absentee landlords, at the optimum per head increase in construction cost due to a marginal change in the network design is equal to the decrease in generalized transport cost for the individual living at city boundary.*

From the aggregate differential land rent equation, at optimum we get

$$\frac{d\Phi}{d\eta} = -\frac{\partial K}{\partial \eta} + \int_0^{\overline{G}} \left(R(G) - R_A\right)\frac{\partial \xi(G)}{\partial \eta} dG$$

and by using proposition 5 this can be written as

$$\frac{d\Phi}{d\eta} = N\frac{d\overline{G}}{d\eta} + \int_0^{\overline{G}} \left(R(G) - R_A\right)\frac{\partial \xi(G)}{\partial \eta} dG.$$

Appendix A.5 gives the derivation of these conditions.

Proposition 5. *In a closed city with absentee landlords, at optimum the marginal change in aggregate differential land rent due to a change in network design is equal to the sum of the aggregate differential land rent, evaluated at the current level of land rent, and the marginal change in generalized transport cost at the city multiplied by the total number of household.*

We also have

$$\begin{aligned}\frac{d\overline{G}}{d\eta} &= -\frac{1}{N}\frac{\partial K}{\partial \eta} \\ &= \frac{1}{N}\int_0^{\overline{G}} \xi\left(G\right)\frac{\partial R\left(G\right)}{\partial \eta}dG.\end{aligned}$$

This shows that Proposition 3 still holds in the case of absentee landlords.

4 Conclusions

In this paper we described the optimality condition that can be applicable for any type of transport network development. We termed it as general optimality condition. The things to be underlined here is that in our model we did not specify (1) the functional form of the utility, (2) type of transport network, (3) the number of available transport mode, (4) shape of the city and (5) the number of CBD. Thus we considered a more general urban transport model compared to that of the existing literature. We solved the optimization problem by converting the locational choice problem into generalized transport cost minimizing problem. In doing so we expressed all endogenous variables as functions of generalized transport cost. Thus it enable us to analyze even for the case of polycentric city. The general optimality condition (proposition 1) for resident land rent implies that the marginal change in construction cost due to a marginal change in network design will be equal to the marginal change in aggregate differential land rent when the current land price is considered. However for absentee landlord this is not the case. In case of absentee landlord the optimality condition of proposition 4 in fact says that the marginal change in construction cost is equal to the equal to the marginal change in aggregate differential land rent evaluated at the post project land rent. Thus the optimality condition for resident landlord case we have to equate marginal construction cost with marginal aggregate differential land rent due to increase in city area (keeping land price fixed), and for absentee landlord case we have to equate the same with marginal aggregate differential land rent due to increase in land price (keeping land fixed). The intuition is make sense because for absentee landlord case the marginal increase in aggregate differential land rent goes out of the city.

The optimality condition found by Kanemoto (1984) is that the optimum length of the railway should be such that the cost of doing so is equal to the aggregate differential land rent. One may consider our findings in proposition 1 as replication of that found by Kanemoto (1984) specially when one considers the length of railway track as the network geometry η to be optimized. However, this findings distinctly differs from our findings in several capacity; firstly, our optimality condition (proposition 1) equates the marginal change in construction cost with aggregate differential land rent and thus we do not require to consider the total change in land rent. Secondly, our proposition demonstrates that we do not need to consider the change in land rent after the network improvement; it is sufficient to calculate the change in aggregate differential land rent with respect to current level of land rent. Thus partial differentiation of aggregate differential land rent with respect to network design is enough to establish the optimal criteria. Thirdly, Kanemoto (1984) assumes that the shape and center of the city as given. It also assumes only radial system of railway network. Whereas our paper derives the general optimality condition irrespective of the city shape, number of city center, type of transport network, and the number of transport mode available to the commuters.

Our findings is also different from capitalization hypothesis of public investment, measurement of benefit based on real income and evaluation of public investment project based on general equilibrium demand function. The above are the measure of benefit of a public project; that is how much benefit can be achieved from a public invested project. For example the capitalization hypothesis implies that the investment in a public project is fully reflected into a the increase in land rent. According to Kanemoto (2011) the benefits under general equilibrium settings can be determined by the increase in real national income at least for a small change in transport capacity. Another findings by Kanemoto (2011) says that under perfect competition the benefit of a public investment project can be determined by the reduction in transport cost only. The change in social surplus is the change in the increase in consumer surplus over the project cost. Our optimality condition (proposition 1) is clearly different from these findings. It particularly says that at that point where marginal capital loss from investing a project must be equal to the marginal gain of increased total differential land rent evaluated at current level of land rent.It directs us the extend to which we have to invest in order to ensure highest benefit by improving/constructing a transport network. To our best knowledge, our optimality condition is unique one specially under general setting.

However, on can extend our model by several dimension. The congestion externality can be introduced in the model. It will be interesting to see how the optimality condition behaves under congestion pricing. The analysis can be extended for open city model. Also one can formulate the model under general equilibrium framework which may be tedious but practically useful for the optimal allocation of fund in transport sector.

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A Appendix A

A.1 Total differentiation of the equation (12)

Taking total differentiation of the equation (12) we get

$$dN = \int_0^{\overline{G}} d\left(\frac{\xi(G)}{Q^*(G)}\right) dG + \frac{\xi(\overline{G})}{Q^*(\overline{G})} d\overline{G}.$$

Since number of population N is given so we have

$$0 = \int_0^{\overline{G}} \left[\frac{1}{Q^*(G)} \frac{\partial \xi(G)}{\partial \eta} d\eta - \frac{\xi(G)\, dQ^*}{\{Q^*(G)\}^2}\right] dG + \frac{\xi(\overline{G})}{Q^*(\overline{G})} d\overline{G}$$

which becomes

$$0 = \int_0^{\overline{G}} \frac{1}{Q^*(G)} \left[-\frac{\xi(G)}{Q^*(G)} dQ^* + \frac{\partial \xi(G)}{\partial \eta} d\eta\right] dG + \frac{\xi(\overline{G})}{Q^*(\overline{G})} d\overline{G}.$$

A.2 Derivation of dQ*, dZ*, dL*

Total differentiation of the equations (7) through (10) yields

$$\begin{aligned} dQ^* &= Q_R^c dR + Q_U^c d\overline{U} \\ dZ^* &= Z_R^c dR + Z_U^c d\overline{U} \\ dL^* &= L_R^c dR + L_U^c d\overline{U} \end{aligned}$$

as well as

$$Q^* dR + R dQ^* = dy^d - dZ^* - w dL^*.$$

By noting that $Z_R^c + RQ_R^c + wL_R^c = 0$ we have

$$\begin{aligned} dR &= \frac{1}{Q^*(G)} \left[dy^d(G) - \left(Z_R^c dR + Z_U^c d\overline{U}\right) - R(G)\left(Q_R^c dR + Q_U^c d\overline{U}\right) - w\left(L_R^c dR + L_U^c d\overline{U}\right)\right] \\ &= \frac{dy^d(G) - \lambda(G)\, d\overline{U}}{Q^*(G)} \end{aligned}$$

and

$$\begin{aligned} dQ^* &= Q_R^c \frac{dy^d(G) - \lambda(G)\, d\overline{U}}{Q^*(G)} + Q_U^c d\overline{U} \\ dZ^* &= Z_R^c \frac{dy^d(G) - \lambda(G)\, d\overline{U}}{Q^*(G)} + Z_U^c d\overline{U} \\ dL^* &= L_R^c \frac{dy^d(G) - \lambda(G)\, d\overline{U}}{Q^*(G)} + L_U^c d\overline{U} \end{aligned}$$

or

$$\begin{aligned} dQ^* &= \frac{Q_R^c}{Q^*(G)} \left(-dD + \frac{d\Phi}{N}\right) + \left[Q_U^c - \frac{\lambda(G)\, Q_R^c}{Q^*(G)}\right] d\overline{U} \\ dZ^* &= \frac{Z_R^c}{Q^*(G)} \left(-dD + \frac{d\Phi}{N}\right) + \left[Z_U^c - \frac{\lambda(G)\, Z_R^c}{Q^*(G)}\right] d\overline{U} \\ dL^* &= \frac{L_R^c}{Q^*(G)} \left(-dD + \frac{d\Phi}{N}\right) + \left[L_U^c - \frac{\lambda(G)\, L_R^c}{Q^*(G)}\right] d\overline{U} \end{aligned}$$

where $\lambda(G) \equiv \partial E / \partial U = Z_U^c + R(G) Q_U^c + wL_U^c$.

A.3 Taking total differentiation of the Aggregated Land Rent Equation

Taking total differentiation of equation (14) we get,

$$d\Phi = \int_0^{\overline{G}} [\xi(G) dR(G)] dG + \int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} d\eta dG.$$

Replacing the value of $R(G)$ from equation (10) and rearranging we get,

$$\begin{aligned} d\Phi &= \int_0^{\overline{G}} \xi(G) \left[\left\{ \frac{1}{Q^*(G)} \left[-dD + \frac{d\Phi}{N} - R(G) dQ^*(G) - dZ^*(G) - wdL^*(G) \right] \right\} \right] dG \\ &\quad + \int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} d\eta dG \end{aligned}$$

$$\begin{aligned} d\Phi &= \int_0^{\overline{G}} \xi(G) \left[\left\{ \frac{1}{Q^*(G)} \left[-dD + \frac{d\Phi}{N} \right] \right\} \right] dG - \int_0^{\overline{G}} \xi(G) \left[\left\{ \frac{1}{Q^*(G)} [R(G) dQ^*(G)] \right\} \right] dG \\ &\quad - \int_0^{\overline{G}} \xi(G) \left[\left\{ \frac{1}{Q^*(G)} [dZ^*(G)] \right\} \right] dG - \int_0^{\overline{G}} \xi(G) \left[\left\{ \frac{1}{Q^*(G)} [wdL^*(G)] \right\} \right] dG \\ &\quad + \int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} d\eta dG \end{aligned}$$

Now putting the value of dQ^*, dZ^* and dL^* from equation (20) and using the fact that $Z_R^c + RQ_R^c + wL_R^c = 0$ and $\lambda(G) = Z_U^c + R(G) Q_U^c + wL_U^c$ this reduces to

$$\begin{aligned} d\Phi &= \int_0^{\overline{G}} \xi(G) \left[\left\{ \frac{1}{Q^*(G)} \left[-dD + \frac{d\Phi}{N} \right] \right\} \right] dG - \int_0^{\overline{G}} \xi(G) \left[\left\{ \frac{1}{Q^*(G)} \lambda(G) d\overline{U} \right\} \right] dG \\ &\quad + \int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} d\eta dG \end{aligned}$$

and finally

$$d\Phi = \int_0^{\overline{G}} \left[\left(\frac{-dD + \frac{d\Phi}{N} - \lambda(G) d\overline{U}}{Q^*(G)} \right) \xi(G) + (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} d\eta \right] dG. \qquad (33)$$

A.4 Derivation of Optimality Conditions for Resident Landlord Case

We can write equation (21) as follows:

$$
\begin{aligned}
0 &= -\left[\int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^3} Q_R^c dG\right]\left(-dD + \frac{d\Phi}{N}\right) - \left[\int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^2}\left(Q_U^c - \frac{\lambda(G) Q_R^c}{Q^*(G)}\right) dG\right] d\overline{U} \\
&\quad + \left[\int_0^{\overline{G}} \frac{1}{Q^*(G)} \frac{\partial \xi(G)}{\partial \eta} dG\right] d\eta + \frac{\xi(\overline{G})}{Q^*(\overline{G})} d\overline{G} \\
d\overline{G} &= -dD + \frac{d\Phi}{N} - \lambda(\overline{G})\, d\overline{U} \\
d\Phi &= \left[\int_0^{\overline{G}} \frac{\xi(G)}{Q^*(G)} dG\right]\left(-dD + \frac{d\Phi}{N}\right) - \left[\int_0^{\overline{G}} \lambda(G) \frac{\xi(G)}{Q^*(G)} dG\right] d\overline{U} \\
&\quad + \left[\int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} dG\right] d\eta \\
dD &= \frac{1}{N} \frac{\partial K}{\partial \eta} d\eta
\end{aligned}
$$

where the second equation uses the fact that $Z_R^c + RQ_R^c + wL_R^c = 0$. Noting that $\int_0^{\overline{G}} \xi(G)/Q^*(G)\, dG = N$ in the third equation yields

$$
\begin{aligned}
0 &= -\left[\int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^3} Q_R^c dG\right]\left(-dD + \frac{d\Phi}{N}\right) - \left[\int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^2}\left(Q_U^c - \frac{\lambda(G) Q_R^c}{Q^*(G)}\right) dG\right] d\overline{U} \\
&\quad + \left[\int_0^{\overline{G}} \frac{1}{Q^*(G)} \frac{\partial \xi(G)}{\partial \eta} dG\right] d\eta + \frac{\xi(\overline{G})}{Q^*(\overline{G})} d\overline{G} \\
d\overline{G} &= -dD + \frac{d\Phi}{N} - \lambda(\overline{G})\, d\overline{U} \\
d\Phi &= N\left(-dD + \frac{d\Phi}{N}\right) - \left[\int_0^{\overline{G}} \lambda(G) \frac{\xi(G)}{Q^*(G)} dG\right] d\overline{U} \\
&\quad + \left[\int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} dG\right] d\eta \\
dD &= \frac{1}{N} \frac{\partial K}{\partial \eta} d\eta.
\end{aligned}
$$

Eliminating dD and $d\Phi$ using the third and the fourth equations gives

$$
\begin{aligned}
0 &= -\left[\int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^3} Q_R^c dG\right]\left(d\overline{G} + \lambda(\overline{G})\, d\overline{U}\right) - \left[\int_0^{\overline{G}} \frac{\xi(G)}{(Q^*(G))^2}\left(Q_U^c - \frac{\lambda(G) Q_R^c}{Q^*(G)}\right) dG\right] d\overline{U} \\
&\quad + \left[\int_0^{\overline{G}} \frac{1}{Q^*(G)} \frac{\partial \xi(G)}{\partial \eta} dG\right] d\eta + \frac{\xi(\overline{G})}{Q^*(\overline{G})} d\overline{G}
\end{aligned}
$$

and

$$
\left[\int_0^{\overline{G}} \lambda(G) \frac{\xi(G)}{Q^*(G)} dG\right] d\overline{U} = -\left[\frac{\partial K}{\partial \eta}\right] d\eta + \left[\int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} dG\right] d\eta
$$

which can be expressed as (20) and (23).

A.5 Optimum Aggregate Differential Land Rent for Absentee Landlord Case

For absentee landlord case the equation (33) of Appendix A.3 becomes

$$d\Phi = \int_0^{\overline{G}} \left[\left(\frac{-dD - \lambda(G)\, d\overline{U}}{Q^*(G)} \right) \xi(G) + (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} d\eta \right] dG.$$

We can be written this as

$$\frac{d\Phi}{d\eta} = \int_0^{\overline{G}} \left[\frac{\xi(G)}{Q^*(G)} \left(-\frac{dD}{d\eta} - \lambda(G) \frac{d\overline{U}}{d\eta} \right) + (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} \right] dG.$$

At optimum

$$\begin{aligned} \frac{d\Phi}{d\eta} &= -\frac{dD}{d\eta} \int_0^{\overline{G}} \frac{\xi(G)}{Q^*(G)} dG + \int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} dG \\ &= -N \frac{dD}{d\eta} + \int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} dG \\ &= -\frac{\partial K}{\partial \eta} + \int_0^{\overline{G}} (R(G) - R_A) \frac{\partial \xi(G)}{\partial \eta} dG. \end{aligned}$$