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ABSTRACT

Semi-infinite programs as related to DEA with infinitely many DMUs will be solved by bisections within the framework of LPs. No gradient information is needed, contrary to the usual Newton-Raphson type methods for solving semi-infinite programs.

The rate of convergence is linear. The method has a stable convergency feature derived from the bisection rule.

1°. Problem .

Solve the following LP with an infinite set X of DMUs for a suitable subset of elements typically designated by x^o:

[Problem]

$$(P) \qquad V_p = \max_{w} w^T g(x^o) \qquad : \qquad x^o \in X$$

subject to

$$w^T g(x) = \mu^T(-f(x)) \leq 0 \qquad \text{for every } x \in X$$

$$\mu^T f(x^o) = 1$$

$$w \geq \epsilon \cdot \hat{e}$$

$$\mu \geq \epsilon \cdot e$$

where X: a compact convex set, dim (X) = L,

$W \in R^s, \quad \hat{e} = (1, 1, \ldots, 1)^T \in R^s$

$\mu \in R^m, \quad e = (1, 1, \ldots, 1)^T \in R^m$

$g(x) \in R^s$: continuous on X

$f(x) \in R^m$: continuous on X

ϵ : a positive infinitesimal non-Archimedean quantity.

[Dual Problem]

$$(D) \qquad V_D = \min (Z - \epsilon \cdot \hat{e}^T s^- - \epsilon \cdot e^T s^+)$$

subject to

$$\sum_{x \in X} g(x)\lambda(x) - s^- = g(x^o)$$

$$\sum_{x \in X} (-f(x))\lambda(x) - s^+ = z(-f(x^o))$$

$\lambda(x) \geq 0$: for every $x \in X$ and $\lambda(x) = 0$ except for a finite number of points.

$s^- \geq 0 \; , \; s^- \in R^s$

$s^+ \geq 0 \; , \; s^+ \in R^m$.

2°. Outline of the Method.

The method consists of three main parts: initial discretization, deletion and subdivision. The discretized problems are solved by the simplex method throughout the iterations.

Step 0. (Discretization)

The dual pair (P) – (D) is discretized, i.e., the infinite index set X is replaced by a finite set. Let the finite set be $\{x^1, \ldots, x^n\}$. We call such sets grid.

Solve the resulting dual pair of linear programs (P_0) – (D_0) by means of the simplex method.

$$(P_0) \qquad V_p = \max W^T g(x^o)$$

subject to

$$W^T g(x^i) + \mu^T(-f(x^i)) \leq 0 \qquad i = 1, \ldots, n$$

$$\mu^T f(x^o) = 1$$

$$w \geq \epsilon \cdot \hat{e}$$

$$\mu \geq \epsilon \cdot e.$$

$$(D_0) \quad V_D = \min (Z - \epsilon \cdot \hat{e}^T s^- - \epsilon \cdot e^T s^+)$$

subject to

$$\sum_{i=1}^{n} g(x^i)\lambda_i - s^- = g(x^o)$$

$$\sum_{i=1}^{n} (-f(x^i))\, \lambda_i - s^+ = z\,(-f(x^o))$$

$$\lambda_i \geq 0 \qquad (i = 1, \ldots, n)$$

$$s^- \geq 0$$

$$s^+ \geq 0.$$

Let optimal solutions to (P_0) and (D_0) be

$$w = (w_1, \ldots, w_s)^T, \qquad \mu = (\mu_1, \ldots, \mu_m)^T$$

and

$$z,\ \lambda = (\lambda_1, \ldots, \lambda_n)^T, \qquad s^- = (s_1^-, \ldots, s_s^-)^T, \qquad s^+ = (s_1^+, \ldots, s_m^+)^T.$$

Step 1. (Deletion)

Apply the 'Deletion rule' as explained later in Sections 3° and 4° to the grid $\{x_i\}$.

Step 2. (Subdivision or Bisection)

Apply the 'Subdivision (bisection) rule' as explained in Sections 3° and 4° to the grid.

Step 3. (New (P_0) and (D_0))

Formulate new dual LPs (P_0) - (D_0) by deleting / augmenting constraints / variables to (P_0)-(D_0). Solve them by the simplex method.

Step 4. (Convergence Check)

Stop the process if the subdivision parameter as explained in Sections 3° and 4° becomes less than the tolerance. Otherwise go back to Step 1.

3°. Details of the Method When X is One-Dimensional.

In this section, we will show details of the method in case X is one dimensional. Cases with dim (X) > 1 will be discussed in Section 4°.

3.1 Initial Discretization and Subdivision Parameter

Let the set X be $[a,b] \subset R$ and arrange the grid $x_0, \ldots, x_n$ as

$$a = x_0 < x_1 < \ldots < x_n = b \tag{3.1}$$

where

$$x_i = x_0 + i\,(b-a)/n \qquad (i = 0, \ldots, n). \tag{3.2}$$

We define the subdivision parameter (or mesh size) T to be

$$T = (b-a)/n \qquad \text{(the length of an interval)} \tag{3.3}$$

3.2 Solving (D_0)

We solve the dual program (D_0) by means of the simplex method. The reason for dealing with the dual program will be clarified later on. The optimal information related to the primal program is easily obtained from the optimal basis of (D_0).

Let the optimal solution to (P_0) and (D_0) be

$$w = (w_1, \ldots, w_s)^T, \qquad \mu = (\mu_1, \ldots, \mu_m)^T \tag{3.4}$$

and

$$z, \ \lambda = (\lambda_0, \ldots, \lambda_n)^T, \ s^- = (s_1^-, \ldots, s_s^-)^T, \ s^+ = (s_1^+, \ldots, s_m^+)^T. \tag{3.5}$$

3.3 Deletion / Subdivision Rules

Since the optimal solutions (3.4) - (3.5) solve the discretized problems, we have, at grid point x_i,

$$w^T g(x^i) + \mu^T (-f(x^i)) = 0 \qquad \text{if } \lambda_i > 0 \tag{3.6}$$

and

$$w^T g(x^i) + \mu^T (-f(x^i)) \leq 0 \qquad \text{if } \lambda_i = 0. \tag{3.7}$$

However, it is not certain if the relations

$$w^T g(x) + \mu^T (-f(x)) \geq 0 \tag{3.8}$$

hold for every $x \in X$.

Let $\quad \psi(x) \equiv w^T g(x) + \mu^T(-f(x)).$ (3.9)

The discrepancy $\delta(w, \mu)$ of (w, μ) is defined as

$$\delta(w, \mu) = \max_{x \in [a,b]} \psi(x) \tag{3.10}$$

An upper bound to $\delta(w, \mu)$ is given by

$$\Delta = (FM\,|T|^2)/8 \tag{3.11}$$

(Kortanek [1]),

where F is an upper bound to w_r ($r = 1, ..., s$) and λ_k ($k = 1, ..., m$)

$$M = \max_{x \in [a,b]} \left(\sum_{r=1}^{s} |g_r''(x)| + \sum_{k=1}^{m} |f_k''(x)| \right) < \infty \qquad (3.12)$$

and T is defined by (3.3).

It is easy to see that if at two successive grid points x_i and x_{i+1}, we have

$$\psi(x_i) < -\Delta \text{ and } \psi(x_{i+1}) < -\Delta,$$

then it follows that

$$\psi(x) < 0 \qquad \text{for every } x \in [x_i, x_{i+1}].$$

Thus, we have the deletion rule for grids.

[Deletion Rule]

If at three successive grid points x_i, x_{i+1}, and x_{i+2}, we have

$$\psi(x_i) < -\Delta, \ \psi(x_{i+1}) < -\Delta \text{ and } \psi(x_{i+2}) < -\Delta, \qquad (3.13)$$

then we delete x_{i+1} and hence the whole interval (x_i, x_{i+2}) from further consideration.

Notice that the rule needs to be changed a little at the boundary points.

[Subdivision Rule]

We subdivide the remaining intervals by introducing a new grid at the mid-point of each interval.

Thus, we have

$$\text{new } T = T/2 \quad \text{and} \qquad (3.14)$$

$$\text{new } \Delta = \Delta/4 \qquad (3.15)$$

[Remark 1] Usually it is not easy to determine Δ as defined by (3.11). In such a case, Δ (actually $-\Delta$) should be taken to be a threshold for deleting grid points. A smaller Δ (higher $-\Delta$) deletes more grid points. If $\psi(x)$ is well approximated by a quadratic curve at a local maximum, the relation (3.15) will generally hold after the subdivision.

3.4 Solving the New LP

We delete the columns corresponding to the deleted grid points from the dual tableau and introduce new columns corresponding to the new grid points to the tableau. The new columns will be priced out by using the optimal dual basis of the preceding iteration and the primal simplex method will determine the new optimal solution.

3.5 Convergence Check

We stop the iterations if T comes to satisfy, for some tolerance T_{tol},

$$T < T_{tol} \quad . \tag{3.16}$$

[Remark 2] A typical process of subdivision (or bisection) is sketched in Fig. 1, where the curves represent $\psi(x)$ with x as abscissa and the tolerance $(-\Delta)$ for each iteration is given by the dashed line.

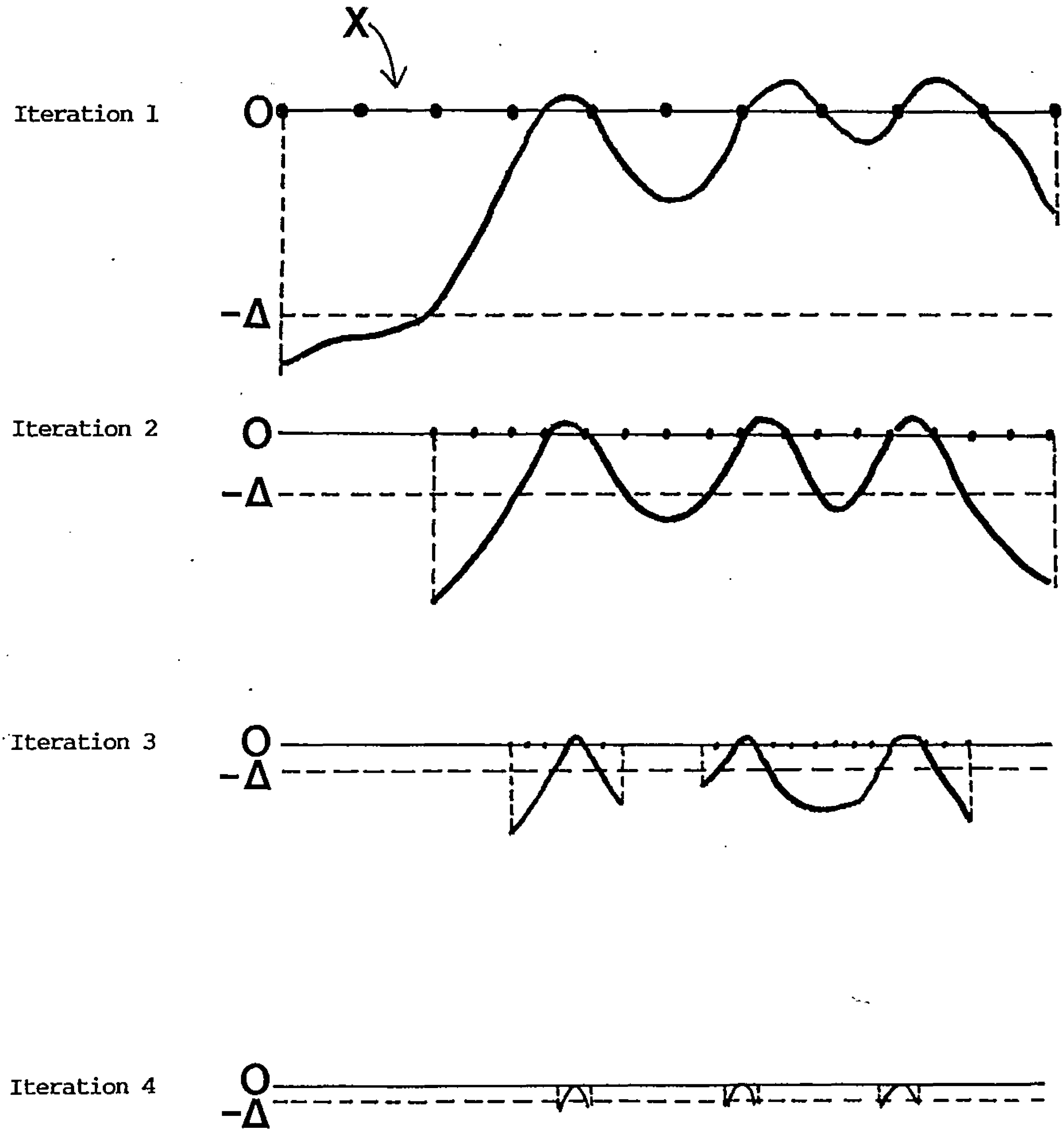

Fig. 1

4° General Case.

In this section, we will deal with the dual pair of problems (P) – (D) when X is a compact convex set with L = dim (x) > 1.

4.1 Initial Discretization

We discretize X by using L – dimensional cubes with edge length T. The mesh points are the initial grid points $\{x^1, \ldots, x^n\}$. The grid points are used to formulate (P_0) and (D_0), which are solved by the simplex method. Let the optimal solutions be w, μ, z, λ, s^-, and s^+.

4.2 Deletion and Subdivision (Bisection) Rules

Every grid has at most 2L neighbors.

[Deleting Rule]

If the relation

$$\psi(x) \equiv w^T g(x) + \mu^T(-f(x)) < -\Delta \tag{4.1}$$

holds at a grid and its neighbors, then we delete the center grid point and edges connecting the center with its neighbors from further consideration. $-\Delta$ is a threshold similar to (3.11) (see also [Remark 1]). For higher dimensional Ls, it would be difficult to estimate Δ by a formula such as (3.11). A practical way to estimate Δ is as follows: After the initial LPs are solved, we estimate the discrepancy $\delta(w, \mu)$ by sampling x from X. The value will be used as the initial Δ, which will be updated by dividing by 4 at each iteration.

[Subdivision Rule]

We divide the remaining edges by introducing a new grid at the mid-point of each edge. Thus we have

$$\text{new } T = T / 2 \tag{4.2}$$

and

$$\text{new } \Delta = \Delta / 4 \tag{4.3}$$

4.3 Solving New LPs and Checking Convergence

These steps are quite similar to those in the one-dimensional case as explained in subsections 3.4 and 3.5.

[Remark 3] A typical subdivision process of the two-dimensional X is depicted in Fig. 2.

Iteration 1

Iteration .2

Iteration 3

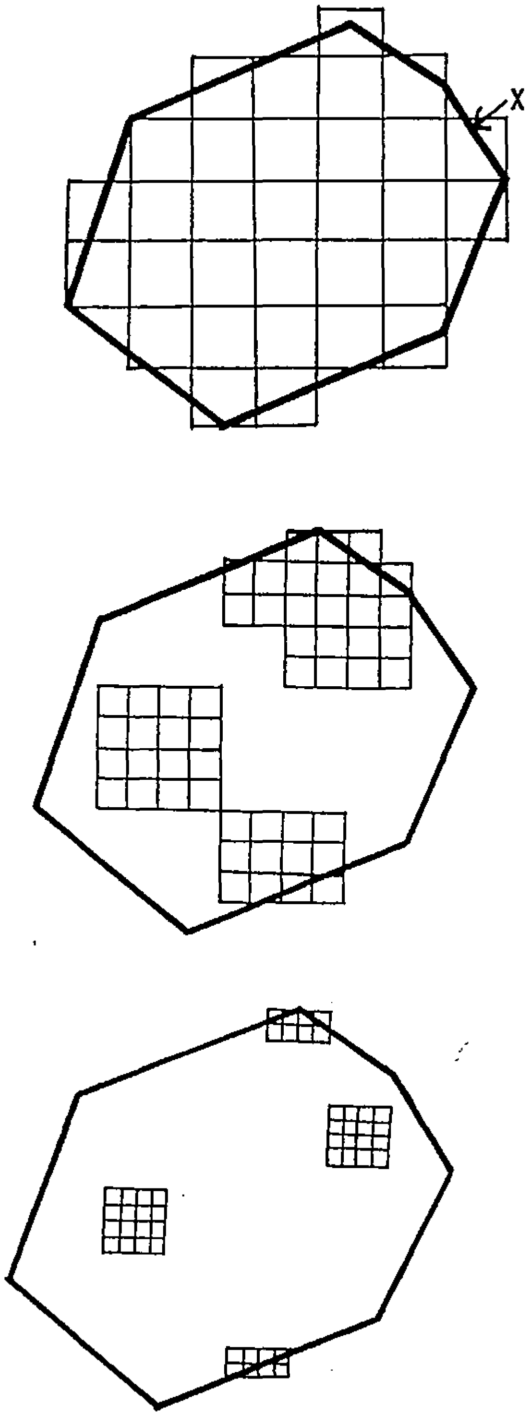

Fig. 2

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[1] KORTANEK, K.O. "Interpolation and Error Bounds for Semi-Infinite Programs and Solution of Nonlinear Systems of Equations" (1979) Manuscript.