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Abstract

As a natural extension of Roos and Vial's "Long steps with logarithmic penalty barrier function in linear programming" (1989) and Ye's "An $O(n^3L)$ potential reduction algorithm for linear programming" (1989), it will be shown that the classical logarithmic barrier function method can be adjusted so that it generates the optimal solution in $O(\sqrt{n}L)$ iterations, where n is the number of variables and L is the data length.

Keywords: Linear programming, interior point algorithm, barrier function, potential function, primal and dual, complexity

1. Introduction

Since the epoch-making breakthrough by Karmarkar[12], the interior point methods for linear programming have been extensively studied in many aspects. One of the focuses of the studies is on the central trajectory leading to the optimal point. (See, for example, Sonnevend [19], Renegar [17], Bayer and Lagarias [1], Megiddo [15], Kojima, Mizuno and Yosise [13], Monteiro and Adler

[16], Goldfarb and Liu [7], Ye and Todd [22], Todd and Ye [20].)

The algorithm dealing with the central trajectory can be classified into two groups: one that follows the central trajectory directly and the other that minimizes a substitute function of the problem so that the successive points of iteration remain in the proximity of the central trajectory consequently. Among the latter approach, some are called as large-step algorithms in the sense that the step size of the movement in an iteration has no a priori bound but is determined by minimizing the substitute function on a line segment.

There are several types of such functions. We will deal with two of them. One is the classical logarithmic barrier functions originated by Frisch [4] and studied by Fiacco and McCormick [2] as applied to linear programming by many authors.(See Gill,Murray, Saunders,Tomlin and Wright [6], Gonzaga [8], Kojima,Mizuno and Yoshise [14] and Roos and Vial [18].) The other is the modern potential function introduced by Karmarkar [12], studied and extended by many researchers.(See Gonzaga [10], Ye [21], Freund [3], Todd and Ye [20], among others.)

Recently, Roos and Vial [18] has proposed an O(nL) iteration large-step logarithmic barrier function algorithms and Ye [21] has developed an $O(\sqrt{n}L)$ iteration potential reduction algorithm based on the primal-dual potential function. (Freund [3] and Gonzaga [10] have presented similar results.) The $O(\sqrt{n}L)$ iteration seems to be the best theoretical bound as of November 1989.

The purpose of this paper is to show a new $O(\sqrt{n}L)$ iteration large-step logarithmic barrier function algorithm based on the results developed by Roos and Vial [18] and Ye [21]. Although Gonzaga [9] has presented an algorithm with the same polynomial bound in the same track, the formula for the control of the parameter is different from the present method. Gonzaga reduces it by a fixed rate when a centering condition comes to be satisfied, while our method reduces it adaptively.

2. Problem and Barrier Functions

We will deal with the primal form of the linear programming problem:

$$\langle P \rangle \ \min \{c^T x \ : \ Ax=b,\ x\geq 0\} \tag{2.1}$$

where A is an (m,n) matrix, b and c are m- and n-dimensional vectors respectively, x is the variable n-dimensional vector to be determined optimally and the symbol ^{T} denotes the transpose.

The dual form of <P> is expressed as

$$\langle D \rangle \ \max \{b^T y \ : \ s=c-A^T y\geq 0\} \tag{2.2}$$

where s and y are variable n- and m-dimensional vectors respectively.

For all x and y that are feasible for <P> and <D>, we have

$$b^T y \leq z^{op} \leq c^T x \tag{2.3}$$

where z^{op} denotes the minimal(maximal) objective value of <P>(<D>). As far as notations are concerned, e denotes the vector of all ones. The upper case letter (X) designates the diagonal matrix of the vector (x) in the lower case.

For <P> and <D>, we assume that

(1) the relative interior of the feasible regions of <P> and <D> is nonempty and we have an interior feasible solution x^0 and y^0 for <P> and <D> such that

$$Ax^0=b,\ x^0>0 \tag{2.4}$$

and

$$s^0=c-A^T y^0>0, \tag{2.5}$$

(2) A has full rank,

and

(3) the objective function value $c^T x$ is not a constant on the feasible region.

Associated with <P>, we consider the logarithmic barrier function

$$f(x,\mu) = \frac{c^T x}{\mu} - \sum_{j=1}^{n} \ln(x_j) \tag{2.6}$$

where μ is a positive parameter.

The function f is strictly convex on the relative interior of the feasible region and achieves a minimum value at a unique point in it. In contrast to the classical barrier function $f(x,\mu)$, several authors have been studying extensively another types of functions motivated by Karmarkar [12]. ([8],[20],[21],[22]).

We will consider here two of them:

the primal potential function for an interior feasible x

$$f_P(x,\underline{z}) = \rho \ln(c^T x-\underline{z}) - \sum_{j=1}^{n} \ln(x_j) \tag{2.7}$$

and

the primal-dual potential function for an interior feasible pair (x,s)

$$f_{PD}(x,s) = \rho \ln(x^T s) - \sum_{j=1}^{n} \ln(x_j s_j) \tag{2.8}$$

where $\underline{z}$ is a lower bound to z^{op} and ρ is a positive parameter.

For a pair of interior feasible primal-dual solution (x,s), let $\underline{z}=b^T y$, then we have a relation between the primal and the primal-dual potential functions:

$$f_{PD}(x,s) = f_P(x,\underline{z}) - \sum_{j=1}^{n} \ln(s_j). \tag{2.9}$$

For an interior feasible x^0 and a positive parameter μ^0, the projected Newton (ascent) direction p associated with f is given by

$$p = \frac{X^0 s}{\mu^0} - e \tag{2.10}$$

where $s = c - A^T y$ (2.11)

and

$$y = (A(X^0)^2 A^T)^{-1} AX^0(X^0 c - \mu^0 e). \tag{2.12}$$

For an interior feasible x^0 and a lower bound $\underline{z}^0$ to z^{op}, the projected Newton direction p_P associated with f_P is given by

$$p_P = \frac{\rho}{c^T x^0 - \underline{z}^0} X^0 s - e \tag{2.13}$$

where

$$s = c - A^T y \tag{2.14}$$

and

$$y = (A(X^0)^2 A^T)^{-1} A X^0 (X^0 c - \frac{c^T x^0 - \underline{z}^0}{\rho} e). \tag{2.15}$$

For the derivation of the above formulae, see Hertog and Roos [11]. It is evident that if we choose the parameter μ^0 as

$$\mu^0 = \frac{c^T x^0 - \underline{z}^0}{\rho} \tag{2.16}$$

then we have

$$p = p_P. \tag{2.17}$$

This fact is the basis on which our algorithm stands.

3. Algorithm and Complexity

This algorithm generates successive pairs of interior feasible solutions $(x^0,s^0),(x^1,s^1),\ldots,$ from a given initial pair (x^0,s^0). Since (x^{k+1},s^{k+1}) is completely determined by (x^k,s^k), we describe the algorithm as the process to generate (x^1,s^1) from (x^0,s^0).

[Algorithm A]

Set $\rho = n + \nu\sqrt{n}$ with $\nu \geq 1$ and $O(\nu) = 1$ (3.1)

$\alpha = 0.4.$

Given x^0, s^0 and y^0 such that $Ax^0 = b$, $x^0>0$ and $s^0=c-A^T y^0>0$, (3.2)

Compute

$$\underline{z}^0 = b^T y^0 \tag{3.3}$$

$$\mu^0 = \frac{c^T x^0 - \underline{z}^0}{\rho} \tag{3.4}$$

$$y = (A(X^0)^2 A^T)^{-1} A X^0 (X^0 c - \mu^0 e). \tag{3.5}$$

$$s = c - A^T y \tag{3.6}$$

and

$$p = \frac{X^0 s}{\mu^0} - e. \tag{3.7}$$

If $\|p\| \geq \alpha$

then begin the primal-step as follows:

$$x^1 = x^0 - \beta^0 X^0 p \quad \text{with } \beta^0 = \text{argmin}_{\beta \geq 0} f(x^0 - \beta X^0 p, \mu^0) \tag{3.8}$$

$$y^1 = y^0 \tag{3.9}$$

$$\underline{z}^1 = \underline{z}^0 \tag{3.10}$$

$$s^1 = s^0$$

$$\mu^1 = \frac{c^T x^1 - \underline{z}^0}{\rho} \tag{3.11}$$

else begin the dual-step as follows:

$$x^1 = x^0 \tag{3.12}$$

$$y^1 = y \tag{3.13}$$

$$\underline{z}^1 = b^T y \tag{3.14}$$

$$s^1 = s \tag{3.15}$$

$$\mu^1 = \frac{c^T x^0 - b^T y}{\rho} \tag{3.16}$$

end.

The process terminates if the relation

$$c^T x^k - b^T y^k < 2^{-L} \tag{3.17}$$

is satisfied for some k.

The following two lemmas are essentially proved by Roos and Vial [18].

[Lemma 1]

If $\|p\| < \alpha$ then (y,s), defined by (3.5) and (3.6), is an interior dual feasible solution and we have

$$c^T x^0 - z^{0p} < c^T x^0 - b^T y \leq \mu^0 (n + \alpha\sqrt{n}). \tag{3.18}$$

Hence, noting $x^1 = x^0$, $\mu^0 = (c^T x^0 - \underline{z}^0)/(n + \nu\sqrt{n})$, $s^1 = s$ and $\underline{z}^1 = b^T y$, we have

$$(x^1)^T s^1 = c^T x^1 - \underline{z}^1 \leq \frac{n + \alpha\sqrt{n}}{n + \nu\sqrt{n}} (c^T x^0 - \underline{z}^0) = \frac{n + \alpha\sqrt{n}}{n + \nu\sqrt{n}} (x^0)^T s^0. \tag{3.19}$$

Thus, the duality gap is reduced at least by a factor $(n+\alpha\sqrt{n})/(n+\nu\sqrt{n})$ (<1).

Proof. See Appendix 1.

[Lemma 2]

If a step length of $\bar{\beta}=(1+\|p\|_\infty)^{-1}$ is taken from x^0 along the direction $-X^0p$, then the change in the barrier function f, denoted by Δf satisfies

$$\Delta f \le -\|p\| + \ln(1+\|p\|). \tag{3.20}$$

If $\|p\| \ge \alpha = 0.4$, then

$$\Delta f \le -0.04. \tag{3.21}$$

Proof. See Roos and Vial [18].

On the line segment $x^0-\beta X^0p$ $(0<\beta<\beta_\infty$, $\beta_\infty=\min\{1/p_j : p_j>0, j=1,2,..,n\})$, we define a "gap function" $g(\beta)$ as

$$g(\beta) = f(x^0-\beta X^0p, \mu^0) - f_P(x^0-\beta X^0p, \underline{z}^0). \tag{3.22}$$

[Lemma 3]

$g(\beta)$ is increasing for β $(0<\beta<\beta_\infty)$.

Proof.

$$\frac{dg(\beta)}{d\beta} = \frac{\rho\beta(c^T X^0 p)^2}{(c^T x^0 - \underline{z}^0)(c^T x^0 - \underline{z}^0 - \beta c^T X^0 p)} > 0. \quad \text{Q.E.D.}$$

The following two lemmas are derived from those by Ye [21].

[Lemma 4]

Let $0<\alpha<0.7$ and $\rho=n+\nu\sqrt{n}$ with $\nu\ge1$.

If $\|p\| < \alpha$, then we have

$$f_{PD}(x^0,s) \le f_{PD}(x^0,s^0) - \nu + \frac{\nu}{1+\nu}(\alpha+1) + \frac{\alpha^2(1+\alpha/\sqrt{1-\alpha^2})}{2(1-2\alpha^2)}$$

$$\le f_{PD}(x^0,s^0) - \frac{1}{2} + \frac{\alpha}{2} + \frac{\alpha^2(1+\alpha/\sqrt{1-\alpha^2})}{2(1-2\alpha^2)} \tag{3.23}$$

where s is defined by (3.6).

If we set $\alpha=0.4$, then the reduction in the primal-dual potential function is as follows:

$$f_{PD}(x^0,s) \le f_{PD}(x^0,s^0) - 0.13. \qquad (3.24)$$

Proof. See Appendix 3.

[Lemma 5]

Let $\rho=n+\nu\sqrt{n}$ with $\nu\ge1$ and $(x^k,s^k)(k=0,1,2,\dots)$ be a series of interior primal-dual feasible solutions with $f_{PD}(x^0,s^0)=O(\sqrt{n}L)$. If, for a positive δ independent of n, the relation

$$f_{PD}(x^{k+1},s^{k+1}) \le f_{PD}(x^k,s^k) - \delta \qquad (3.25)$$

holds for each k, then in $O(\nu\sqrt{n}L)$ iterations, we have

$$c^T x^k - b^T y^k = (x^k)^T s^k < 2^{-L}.$$

If $O(\nu)=1$ moreover, then the polynomial bound of iteration is $O(\sqrt{n}L)$.

Proof. See Appendix 4.

Now we are ready to show the theorem.

[Theorem 1]

If Algorithm A starts from an interior primal-dual feasible solution (x^0,s^0) with $f_{PD}(x^0,s^0)=O(\sqrt{n}L)$, then it terminates in $O(\sqrt{n}L)$ iterations.

Proof.

Let the series of the interior feasible primal-dual solutions generated by Algorithm A be $(x^k,s^k)(k=0,1,2,\dots)$. For each (x^k,s^k), we have three potential functions f, f_P and f_{PD} defined by (2.6), (2.7) and (2.8) respectively. We will show that, for each iteration, the primal-dual potential function reduces at least by a positive value $\delta=0.04$ and then we have the conclusion by Lemma 5.

The case $\|p\| \ge \alpha$:

In this case we move in the primal space from x^0 to x^1 as defined by (3.8). From Lemma 2, we have

$$f(x^1,\mu^0) - f(x^0,\mu^0) \le 0.04.$$

Using the gap function g , the change in the primal potential function is expressed as

$$f_P(x^1,\underline{z}^0) - f_P(x^0,\underline{z}^0) = f(x^1,\mu^0) - g(\beta^0) - f(x^0,\mu^0) + g(0)$$

$$\leq -0.04 - g(\beta) + g(0).$$

By Lemma 3, we have $g(\beta^0) \geq g(0)$.

Hence,

$$f_P(x^1,\underline{z}^0) - f_P(x^0,\underline{z}^0) \leq -0.04.$$

Noting $s^1=s^0$ in this case, we have

$$\begin{aligned} f_{PD}(x^1,s^1) &= f_P(x^1,\underline{z}^0) - \sum_{j=1}^{n} \ln(s_j{}^0) \\ &\leq f_P(x^0,\underline{z}^0) - \sum_{j=1}^{n} \ln(s_j{}^0) - 0.04 \\ &\leq f_{PD}(x^0,s^0) - 0.04. \end{aligned}$$

The case $\|p\| < \alpha$:

From Lemma 5, we have

$f_{PD}(x^1,s^1) \leq f_{PD}(x^0,s^0)-0.13$ Q.E.D.

4. Concluding Remarks

We will point out several features of our algorithm.

4.1 On primal- and dual-step

In Algorithm A, we choose either the primal or the dual step depending on $\|p\|$. Specifically, if $\|p\|\geq\alpha(=0.4)$, then we employ the primal, otherwise the dual. The value 0.4 is not mandatory, but is used to assure a constant reduction in the primal-dual potential function even in the worst case. So, in the implementational phase of the algorithm, the following procedure may be recommendable;

If $\|p\|<1$ and (x^0,s)(s defined by (3.6)) reduces f_{PD} a certain amount, then we go into the dual step, otherwise into the primal. Also, the minimization of f_{PD} with respect to $\underline{z}$ may be considerable. Anyway the dual step is cheap compared with the primal.

4.2 On updating $\underline{z}$

If $\|p\|<1$, then we have, from (3.19),

$$z^{op} \geq \underline{z}^1 > \underline{z}^0. \qquad (4.1)$$

Thus, we can update the lower bound strictly. This fact means that if we start from $\underline{z}^0 = z^{op}$, then the centering condition $\|p\|<1$ never holds and so we never visit the dual step.

It should benoted that to be in a proximity of the center, as characterized by $\|p\|<1$, is not the object or goal of the path-following algorithm, but just a stimulus. By choosing $\mu=(cx-\underline{z})/\rho$, we change the stimulus, in a sense, adaptively and continuously. This shows a sharp contrast to Roos and Vial [18] and Gonzaga [9] where the centering condition is a necessity to promote their "outer step".

4.3 On the choice of ρ

Although we employ $\rho=n+\nu\sqrt{n}$ with $\nu\geq 1$, $O(\nu)=1$, it is interesting to observe the case $\rho=\theta(n+\nu\sqrt{n})$, with $\theta>1$, $O(\theta)=1$. From Lemma 5, the polynomial bound of iteration is $O(nL)$, worse than $O(\sqrt{n}L)$ of the present algorithm. Then, if $\|p\|<\alpha$ and we go into the dual step, it holds

$$(x^1)^T s^1 = c^T x^1 - \underline{z}^1 = c^T x^0 - \underline{z}^1 \leq \frac{n+\alpha\sqrt{n}}{\theta(n+\nu\sqrt{n})}(c^T x^0 - \underline{z}^0) < \frac{1}{\theta}(c^T x^0 - \underline{z}^0)$$

$$= \frac{1}{\theta}(x^0)^T s^0. \qquad (4.2)$$

Thus, duality gap reduces at least by a factor $1/\theta(<1)$. If we set $\theta=2$, then Algorithm A will behave similarly to Roos and Vial [18] although the correspondence is not exact.

4.4 On the step size of the algorithm

Algorithm A uses the logarithmic barrier function f to determine the step size in the primal step. If, instead, we employ the primal potential function f_P for this purpose, then Algorithm A coincides with Ye's primal potential reduction algorithm [21]. In this context, it may be possible that other types of the substitute functions with the same polynomial bound exist.

As for the step size, let

$$\beta^0 = \operatorname{argmin}_{\beta\geq 0} f(x^0 - \beta X^0 p, \mu^0) \qquad (4.3)$$

and

$$\beta^1 = \operatorname{argmin}_{\beta\geq 0} f_P(x^0 - \beta X^0 p, \underline{z}^0). \qquad (4.4)$$

Then, we have

$$\beta^0 \le \beta^1, \tag{4.5}$$

as otherwise Lemma 3 does not hold.

Appendix

Appendix 1. (Proof of Lemma 1)(Roos and Vial [18])

Since

$$\|p\| = \| \frac{X^0 s}{\mu^0} - e \| < \alpha < 1,$$

(s,y) is an interior dual feasible solution and so $\underline{z}^1 = b^T y < z^{op}$.

On the other hand,

$$|e^T(\frac{X^0 s}{\mu^0} - e)| \le \|e\| \, \| \frac{X^0 s}{\mu^0} - e \| \le \alpha\sqrt{n}$$

and

$$e^T(\frac{X^0 s}{\mu^0} - e) = \frac{(x^0)^T s}{\mu^0} - n = \frac{c^T x^0 - b^T y}{\mu^0} - n.$$

Hence, noting $x^1 = x^0$, $\mu^0 = (c^T x^0 - \underline{z}^0)/(n + \nu\sqrt{n})$, $s^1 = s$ and $\underline{z}^1 = b^T y$, we have

$$(x^1)^T s^1 = c^T x^1 - \underline{z}^1 \le \frac{n+\alpha\sqrt{n}}{n+\nu\sqrt{n}}(c^T x^0 - \underline{z}^0) = \frac{n+\alpha\sqrt{n}}{n+\nu\sqrt{n}}(x^0)^T s^0.$$

Q.E.D.

Appendix 2.

[Lemma 6]

Given two numbers α and ν with $0<\alpha<\nu$, the function

$$h(x) = x \ln((x+\alpha)/(x+\nu))$$

is decreasing for $x>0$.

Proof.

$$h'(x) = \ln(1+\frac{\alpha-\nu}{x+\nu}) + \frac{(\nu-\alpha)x}{(x+\alpha)(x+\nu)}.$$

Since $(\alpha-\nu)/(x+\nu) > -1$ for $x>0$, we have

$$h'(x) \le \frac{\alpha-\nu}{x+\nu} + \frac{(\nu-\alpha)x}{(x+\alpha)(x+\nu)} = \frac{\alpha(\alpha-\nu)}{(x+\alpha)(x+\nu)} < 0.$$ Q.E.D.

Appendix 3. (Proof of Lemma 4)

Ye [21] proved that if $\|p\|<\gamma\sqrt{n/(n+\gamma^2)}$ with $\gamma<1$ then the following inequality holds

$$n \ln(x^0)^T s^1 - \sum_{j=1}^{n} \ln(x_j^0 s_j^1) \le n \ln(x^0)^T s^0 - \sum_{j=1}^{n} \ln(x_j^0 s_j^0) + \frac{\gamma^2}{2(1-\gamma)}.$$

Thus, for $\|p\|<\alpha$ with $\alpha<0.7$, we have

$$\begin{aligned} n \ln(x^0)^T s^1 - \sum_{j=1}^{n} \ln(x_j^0 s_j^1) &\le n \ln(x^0)^T s^0 - \sum_{j=1}^{n} \ln(x_j^0 s_j^0) + \frac{\alpha^2(1+\alpha\sqrt{n/(n-\alpha^2)})}{2(1-\alpha^2-\alpha^2/n)} \\ &\le n \ln(x^0)^T s^0 - \sum_{j=1}^{n} \ln(x_j^0 s_j^0) + \frac{\alpha^2(1+\alpha/\sqrt{1-\alpha^2})}{2(1-2\alpha^2)}. \end{aligned} \quad \text{(A1)}$$

On the other hand, we have from (3.19),

$$\sqrt{n}(\ln(x^0)^T s^1 - \ln(x^0)^T s^0) \le \sqrt{n}(\ln \frac{n+\alpha\sqrt{n}}{n+\nu\sqrt{n}}). \quad \text{(A2)}$$

By Lemma 6 above, the right hand side of (A2) attains a minimum at n=1 for $n\ge 1$. Thus,

$$\begin{aligned} \sqrt{n}(\ln(x^0)^T s - \ln(x^0)^T s^0) &\le \ln \frac{1+\alpha}{1+\nu} = \ln(1+ \frac{\alpha-\nu}{1+\nu}) \\ &< \frac{\alpha-\nu}{1+\nu} = -1+ \frac{1+\alpha}{1+\nu}. \end{aligned} \quad \text{(A3)}$$

From (A1) and (A3), we have, for $\alpha<0.7$,

$$\begin{aligned} f_{PD}(x^0,s) &\le f_{PD}(x^0,s^0) - \nu + \frac{\nu}{1+\nu}(\alpha+1) + \frac{\alpha^2(1+\alpha/\sqrt{1-\alpha^2})}{2(1-2\alpha^2)} \\ &\le f_{PD}(x^0,s^0) - \frac{1}{2} + \frac{\alpha}{2} + \frac{\alpha^2(1+\alpha/\sqrt{1-\alpha^2})}{2(1-2\alpha^2)}, \end{aligned}$$

because $-\nu+\nu(\alpha+1)/(1+\nu)$ attains its maximum at $\nu=1$ for $\nu\ge 1$.

Q.E.D.

Appendix 4. (Proof of Lemma 5)(Ye [21])

$$f_{PD}(x,s) = \rho \ln(x^T s) - \sum_{j=1}^{n} \ln(x_j s_j)$$

$$= (\rho-n) \ln(x^T s) - \sum_{j=1}^{n} \ln((x_j s_j)/(x^T s)).$$

From the inequality of the geometric mean and the arithmetic mean, we have

$$- \sum_{j=1}^{n} \ln((x_j s_j)/(x^T s)) \geq n \ln n.$$

Hence,

$$(\rho-n) \ln(c^T x-b^T y) = (\rho-n) \ln(x^T s) \leq f_{PD}(x,s) - n \ln n$$

$$\leq f_{PD}(x,s).$$

Thus, if we can reduce f_{PD} at least by δ (a constant independent of n), at each iteration, we have, after $-(\rho-n)L/\delta$ iterations,

$$(\rho-n) \ln(c^T x-b^T y) \leq -(\rho-n)L + f_{PD}(x^0,s^0).$$

Assume that $f_{PD}(x^0,s^0)=O(\sqrt{n}L)$ and $\rho=n+\nu\sqrt{n}$, then after $O(\nu\sqrt{n}L)$ iterations, we have

$$c^T x - b^T y \leq 2^{-L}.$$ Q.E.D.

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