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Abstract

The purposes of this paper are (i) to present DEA (Data Envelopment Analysis) without using the non-Archimedian infinitesimal ϵ and (ii) to introduce a new measure of efficiency, which takes account of slacks in inputs and shortages in outputs and expresses the relative efficiency of decision making units more properly than the traditional one.

1 Introduction and Historical Background

In their ingenious paper [10], Charnes, Cooper and Rhodes introduced a fractional programming method to measure the relative efficiency of a decision making unit (DMU), which was solved by transforming the fractional programming into a linear programming problem via the Charnes-Cooper scheme [6]. The method is called DEA (Data Envelopment Analysis). In the paper, it was assumed that the weights to inputs and outputs be *nonnegative.* In the subsequent "short communication " [11], they changed their problems and required that the weights be strictly *positive.* Thus, the non-Archimedian infinitesimal ϵ was introduced to distinguish between nonnegative and positive values. Although, through the subsequent discussions

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(notably, Boyd and Färe [5], Charnes and Cooper [7], Charnes, Cooper and Thrall [12], among others), the role of ϵ has become clear and weakened, it is still frequently used in the literature (for example, [4],[8]) and in particular, in some cases of computational situations, values such as $\epsilon = 10^{-5}, 10^{-6}$ (single precision) or $\epsilon = 10^{-12}$ (double precision) are employed instead of the non-Archimedian infinitesimal ϵ. Theoretically, this is a contradiction and we need a completely ϵ-free development of DEA from both theoretical and computational points of view. For this purpose, in Section 2, we will define an input oriented DEA model based on the production possibility set. Its dual is the Charnes-Cooper-Rhodes (CCR) model with the weights to inputs and outputs as variables. Then, we define a DMU as *slackless* if, for every optimal solution to the DEA model, it has no slack in inputs and no shortages in outputs. By a theorem of the alternative, it will be proved that for a slackless DMU there is a strictly positive weight solution in the corresponding CCR model. Subsequently, for a DMU with non-zero slacks in an optimal solution to the DEA model, there exist no positive weight solutions in the CCR model. This fact is the source of the non-Archimedian infinitesimal ϵ and, in the author's view, has not been explicitly pointed out in the literature. In Section 3, we will define the *max-slack* solution and show a procedure to find it. The max-slack solution can be used for deciding if the DMU is slackless or not. Thus, all jobs of the CCR model can be successfully achieved with no recourse to ϵ. In Section 4, we will introduce a new measure of relative efficiency, based on the max-slack solution, which takes account of slacks in the inputs and shortages in the outputs of the objective DMU and expresses the relative efficiency of DMUs more properly than the traditional one.

2 Input Oriented DEA and Slackless DMU

We consider n decision making units (DMUs) with an input matrix $X = [x_{ik}] \in R^{m \times n}$ and an output matrix $Y = [y_{jk}] \in R^{s \times n}$, where x_{ik} is the amount of input i consumed by DMU k and y_{jk} is the amount of output j produced by DMU k. The unit k has the input vector $x_k = (x_{ik}) \in R^m$ composed of m kinds of resources and the output vector $y_k = (y_{jk}) \in R^s$ composed of s kinds of productions. We assume $X > O$ and $Y > O$. The production possibility set P ([2], [3], [9]) is defined by

$$P = \{(x, y) | x \geq X\lambda,\ y \leq Y\lambda,\ \lambda \geq 0\}, \tag{1}$$

where x, y and λ are m-, s- and n- vectors, respectively.

For a given DMU denoted by o with the input x_o and the output y_o, we consider the following input oriented DEA model as expressed by the linear programming (LP_o).

$$\begin{array}{lrcl} (LP_o) & \min\ \theta & & \\ \text{subject to} & \theta x_o & \geq & X\lambda, \\ & y_o & \leq & Y\lambda, \\ & \lambda & \geq & 0, \end{array} \tag{2}$$

where $\theta \in R$ and $\lambda \in R^n$ are variables.
This model contracts inputs as far as possible while controlling for outputs. The dual of (LP_o) is:

$$\begin{array}{lrcl} (DP_o) & \max y_o^T u & & \\ \text{subject to} & x_o^T v & = & 1, \\ & v^T X & \geq & u^T Y, \\ & v & \geq & 0, \quad u \geq 0, \end{array} \tag{3}$$

where $v \in R^m$ and $u \in R^s$ are variables.

As is well known, (DP_o) is the linear programming version of the original CCR fractional programming problem. The dual variables u and v are the weights for outputs and inputs, respectively.

For (LP_o), we define the *slacks* s_x and s_y by

$$s_x = \theta x_o - X\lambda \quad \text{and} \quad s_y = Y\lambda - y_o. \tag{4}$$

Let optimal solutions for (LP_o) and (DP_o) be $(\theta^*, \lambda^*, s_x^*, s_y^*)$ and (u^*, v^*), respectively. Note that these problems are often degenerate and the optimal solutions are not always unique.

Definition 1 (slackless DMU) *If, for every optimal solution for* (LP_o), *we have*

$$s_x^* = 0 \quad \textit{and} \quad s_y^* = 0, \tag{5}$$

then we call DMU_o *as* slackless.

Theorem 1 DMU_o *is slackless if and only if there exists an optimal dual solution* (u^*, v^*) *for* (DP_o) *with* $u^* > 0$ *and* $v^* > 0$.

Proof. If DMU_o is slackless, there is no solution (λ, s_x, s_y) for the system:

$$\begin{aligned} \theta^* x_o &= X\lambda + s_x, & (6)\\ y_o &= Y\lambda - s_y, & (7)\\ \lambda &\geq 0, & (8)\\ (s_x, s_y) &\geq 0 \text{ and } (s_x, s_y) \neq 0. & (9) \end{aligned}$$

By applying Slater's theorem of the alternative (see Appendix 1), modified for the nonhomogenous system, to the system (6)-(9), it is concluded that the system

$$v^T X \geq u^T Y, \tag{10}$$

$$\theta^*(v^T x_o) - u^T y_o + z = 0, \tag{11}$$

has a solution $\quad v \in R^m, \quad u \in R^s \quad$ and $\quad z \in R \quad$ with

$$z > 0, \quad u \geq 0, \quad v \geq 0 \tag{12}$$

or

$$z \geq 0, \quad u > 0, \quad v > 0. \tag{13}$$

Since $v^T x_o \geq 0$, we have two cases:

Case (i) $v^T x_o > 0$.

Let $v^* = v/v^T x_o$, $u^* = u/v^T x_o$ and $z^* = z/v^T x_0$. Then, we have:

$$(v^*)^T x_o = 1, \tag{14}$$

$$(v^*)^T X \geq (u^*)^T Y, \tag{15}$$

$$\theta^* - (u^*)^T y_o + z^* = 0, \tag{16}$$

$$z^* \geq 0, \quad u^* \geq 0, \quad v^* \geq 0. \tag{17}$$

Hence, (u^*, v^*) is feasible for (DP_o). By the duality relation, we have, from (16),

$$\theta^* = (u^*)^T y_o \quad \text{and} \quad z^* = 0, \tag{18}$$

which demonstrates the optimality of (u^*, v^*) for (DP_o). Using (13), we have $u^* > 0$ and $v^* > 0$.

Case (ii) $v^T x_o = 0$.

From (10), we have $0 = v^T x_o \geq u^T y_o \geq 0$ and hence $u^T y_o = 0$. The equation (11) results in $z = 0$ and, from (13), we have $u > 0$ and $v > 0$. This contradicts with $v^T x_o = 0$, since $x_o > 0$ by assumption. Thus, Case (ii) never occurs.

The reverse is also true by the nature of the theorem of the alternative. □

Evidently, we have:

Corollary 1 *(DP_o) has no strictly positive optimal solution (u^*, v^*) if and only if (LP_o) has an optimal solution $(\theta^*, \lambda^*, s_x^*, s_y^*)$ with $(s_x^*, s_y^*) \geq 0$ and $(s_x^*, s_y^*) \neq 0$.*

3 Max Slack Solution

The preceding Theorem and Corollary reveal the equivalence between the slackless solution in (LP_o) and the positive solution in (DP_o). So, hereafter, we will mainly deal with (LP_o), since it can be more easily handled theoretically and computationally than the latter and needs no ϵ.

Definition 2 (max slack solution) *An optimal solution of (LP_o) is called as* max slack, *if it maximizes $w = e^T s_x + e^T s_y$, where the e^T are vectors of ones.*

The max slack solution can be obtained by a 2-phase process as follows:
In the first phase, we minimize θ of (LP_o). Then, in the second phase, we maximize $w = e^T s_x + e^T s_y$ while keeping $\theta = \theta^*$ (the optimal θ value). It hardly needs pointing out that DMU_o is slackless if and only if its max slack solution satisfies $w = e^T s_x + e^T s_y = 0$. As for the definition of efficiency, it

is natural to state that a DMU is *efficient* if it has $\theta^* = 1$ and is slackless. Otherwise, it is *inefficient.* (The above procedure and the definition of efficiency are given in [12], too.)

Note: In the original CCR model [10], the weights u and v were required to be *nonnegative* and then, in the subsequent paper [11], the problem was changed and required u and v to be *positive*, considering the slackness in (LP_o). Specifically, they introduced the non-Archimedian infinitesimal ϵ and replaced the condition $u > 0,\ v > 0$ by $u \geq \epsilon e,\ v \geq \epsilon e$. If the optimal solution (u^*, v^*) for (DP_o) under the latter condition happened to be *Archimedian positive*, then DMU_o is slackless by Theorem 1. However, if some elements of (u^*, v^*) are non-Archimedian infinitesimal, we cannot decide from (u^*, v^*) whether DMU_o is slackless or not. We will be free from this kind of information gap so long as we deal with the max slack solution of (LP_o).

4 New Measure of Efficiency

The traditional DEA regards θ^* as the measure of efficiency. However, θ^* is indifferent to the level of slacks in inputs and outputs and hence is misleading as a practical means for comparing DMUs. Now, we can define another type of efficiency by the following principles: (1) it should be the same as θ^* when the DMU is slackless, and (2) it should be decreasing in the relative value of slacks in inputs and outputs.

As a candidate for this purpose, we propose a new measure of efficiency, defined by

$$\eta^* = \left(\theta^* - \frac{e^T s_x^*}{e^T x_o}\right)\left(\frac{e^T y_o}{e^T y_o + e^T s_y^*}\right), \tag{19}$$

where s_x^* and s_y^* are slacks of the max slack solution. It is easy to see that η^* thus defined satisfies the above criteria. Furthermore, η^* can be rewritten as

$$\eta^* = \frac{e^T X\lambda^*}{e^T x_o}\frac{e^T y_o}{e^T Y\lambda^*}. \tag{20}$$

Let us define a DMU (x_e, y_e) by

$$x_e = X\lambda^* \quad \text{and} \quad y_e = Y\lambda^*. \tag{21}$$

Theorem 2 *The DMU* (x_e, y_e) *is efficient.*

Proof. We estimate the efficiency of (x_e, y_e) by solving

$$\begin{array}{lrcl} (LP_e) & \min\ \theta_e & & \\ \text{subject to} & \theta_e x_e & = & X\lambda + s_x, \\ & y_e & = & Y\lambda - s_y, \\ & \lambda & \geq & 0, \quad s_x \geq 0, \quad s_y \geq 0. \end{array}$$

Let a max slack optimal solution of (LP_e) be $(\theta_e^*, \lambda_e^*, s_{x_e}^*, s_{y_e}^*)$. From $\theta^* x_o = x_e + s_x^*$ and $y_0 = y_e - s_y^*$, we have

$$\begin{aligned} \theta_e^* \theta^* x_o &= X\lambda_e^* + s_{x_e}^* + \theta_e^* s_x^*, \\ y_o &= Y\lambda_e^* - s_{y_e}^* - s_y^*. \end{aligned}$$

Since $(\theta^*, \lambda^*, s_x^*, s_y^*)$ is a max slack optimal solution for (x_o, y_o) and $\theta_e^* \leq 1$, we have $\theta_e^* = 1$, $s_{x_e}^* = 0$ and $s_{y_e}^* = 0$. Thus, (x_e, y_e) is efficient. □

Thus, (x_e, y_e) is a projection of (x_o, y_o) onto its efficiency facet. η^* can be interpreted as the product of the average input efficiency $e^T x_e / e^T x_o (\leq 1)$ with the average output efficiency $e^T y_o / e^T y_e (\leq 1)$.

Example

Table 1 shows the input X and the output Y for six DMUs along with the max slack solutions. As to DMU_6, it has $\theta_6^* = 1$ which looks like better than DMU_1 and DMU_2. Taking account of $s_{x_1}^* = 2$ in DMU_6, its new efficiency is $\eta_6^* = 9/11 = 0.82$. Thus, DMU_6 drops to the lowest level.

Table 1: Efficiency: Old and New

DMU	1	2	3	4	5	6
X	4	6	8	4	2	10
	3	2	1	2	4	1
Y	1	1	1	1	1	1
$s^*_{x_1}$	0	0	0	0	0	2
$s^*_{x_2}$	0	0	0	0	0	0
s^*_y	0	0	0	0	0	0
θ^*	.86	.86	1	1	1	1
η^*	.86	.86	1	1	1	.82

5 Concluding Remarks

In this paper, we observed the primal and the dual sides of the DEA model and pointed out the equivalence of the slackless solution in the primal and the existence of a positive weight in the dual. Then, we proposed a new measure of efficiency. Although we have been mainly concerned with the input oriented DEA, we can easily extend the results to the output oriented DEA which is usually represented by:

$$\begin{aligned} & \max \ \xi \\ \text{subject to} \quad x_o & \geq X\lambda, \\ \xi y_o & \leq Y\lambda, \\ \lambda & \geq 0, \end{aligned}$$

where $\xi(\geq 1)$ is the expansion factor of the outputs for DMU_o.
By an analogous reasoning, we can define a new measure of efficiency τ^* for the output oriented DEA, using the max slack solution $(\xi^*, \lambda^*, s_x^*, s_y^*)$ by the formula:

$$\tau^* = \left(\xi^* + \frac{e^T s_y^*}{e^T y_o}\right)\left(\frac{e^T x_o}{e^T x_o - e^T s_x^*}\right). \tag{22}$$

Several DEA models (Banker-Charnes-Cooper [3], increasing returns to scale, decreasing returns to scale, among others) are presented and extensively studied. (See for example [1],[4].) The new measure of efficiency proposed here can be easily incorporated within these models as long as the models are derived from some production possibility set.

Appendix

Appendix 1(Slater [13])

Let A, B, C and D be given matrices with A and B being nonvacuous. Then the system (I) *or* (II) has a solution but never *both.*

(I)

$$Ax > o, \quad Bx \geq 0, \quad Bx \neq 0, \quad Cx \geq 0, \quad Dx = 0$$

has a solution x.

(II)

$$y_1^T A + y_2^T B + y_3^T C + y_4^T D = 0$$

with $y_1 \geq 0,\ y_1 \neq 0,\ y_2 \geq 0,\ y_3 \geq 0$
or
$y_1 \geq 0,\ y_2 > 0,\ y_3 \geq 0$

has a solution (y_1, y_2, y_3, y_4).

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