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An Observation on the Cone-Ratio Model in Data Envelopment Analysis

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Abstract

In this paper, we will discuss subjects related to virtual multipliers in the cone-ratio model in DEA. Usually, there exists ambiguity in the virtual multipliers in the polyhedral cone-ratio method when some exemplary efficient DMUs' multipliers are employed as the admissible directions of the cone. Firstly, we will show a cell subdivision of the multiplier simplex. Then, three practical methods for resolving this ambiguity will be presented with an example.

1 Introduction

Data Envelopment Analysis (DEA) inaugurated by Charnes, Cooper and Rhodes (1978) has been widely applied for evaluating the relative efficiency of decision making units (DMUs) with multiple inputs and outputs. The relative efficiency is measured by a ratio scale of the virtual input *vs.* the virtual output, which are the weighted sums of inputs and outputs, respectively. The weights to inputs and outputs are usually non-negative and are

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decided to be most preferable to the DMU concerned, via a linear programming solution. Since the original Charnes, Cooper and Rhodes (CCR) model, many studies have been developed to cope with the actual situations of the problems. One of them is directed to research in the feasible region of the weights and has actually imposed some additional constraints to the weights. Representatively, such studies resulted in the Assurance Region (AR) model and the Cone-Ratio (CR) model. The assurance region method developed by Thompson, Singleton, Thrall and Smith (1986) confines the feasible region of the weights by imposing a lower and an upper bounds to the ratio of some selected pairs of weights. (See also Thompson, Langemeir, Lee, Lee and Thrall (1990) and Roll and Golany (1993)). On the other hand, the cone-ratio model by Sun (1987), Charnes, Cooper, Wei and Huang (1989) and Charnes, Cooper, Huang and Sun (1990) solves the CCR model first and chooses a few exemplary efficient DMUs from among all the efficient ones by consulting with experts on the problem. Then, the corresponding optimal weights to the selected efficient DMUs are used to construct a convex cone as the feasible region of the weights. However, usually the optimal weights are not uniquely determined and hence there is ambiguity in selecting the weights to form the cone.

In an effort to overcome this problem, this paper will propose three practical methods for deciding the convex cone in accordance with three principles which will be explained later. In Section 2, we will discuss the subdivision of the multiplier (weight) simplex. Based on the subdivision, the AR and CR models will be conceptually compared in Section 3, with emphasis on the ambiguity of weights in the CR model. Then, in Section 4, we will propose three practical methods to resolve the ambiguity under three principles, i.e. the most restricted, the most relaxed and the central. We will show an example of the proposed method in Section 5. Finally, in Section 6, the possible application of the vertex enumeration algorithm will be discussed.

2 Cell Subdivision of Multiplier Simplex

Suppose there are n DMUs with m inputs and s outputs. The i-th input and the r-th output of the j-th DMU are denoted by x_{ij} and y_{rj}, respectively. Let the input and output matrices X and Y be

$$X = (x_{ij}) \in R^{m\times n} \text{ and } Y = (y_{rj}) \in R^{s\times n}. \tag{1}$$

We assume $X > O$ and $Y > O$. The virtual input and output for DMU_j are defined by

$$V_j = \sum_{i=1}^{m} v_i x_{ij} \quad (j = 1, \ldots, n) \tag{2}$$

and

$$U_j = \sum_{r=1}^{s} u_r y_{rj}, \quad (j = 1, \ldots, n) \tag{3}$$

where (v_i) $((u_r))$ is the the *weight* (or *multiplier*) to the input (output) i (r). Again, we assume $v_i > 0$ $(\forall i)$ and $u_r > 0$ $(\forall r)$[1].

Now, we observe the ratio of the virtual input *vs.* output:

$$R_j = \frac{U_j}{V_j} = \frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}}. \quad (j = 1, \ldots, n) \tag{4}$$

Since the ratio R_j is invariant under any multiplication by a positive scalar t to (v, u), we impose hereafter the simplex constraint to (v, u) as follows:

$$\sum_{i=1}^{m} v_i + \sum_{r=1}^{s} u_r = 1. \tag{5}$$

[1]For computational purpose, we can relax these positive constraints to non-negativity. See Charnes, Cooper and Thrall (1991) and Tone (1993)

By this constraint, together with the positiveness of multipliers, the feasible (v, u) forms the interior of the $(m + s - 1)$ dimensional simplex denoted by S. Under the above assumptions, for each $(\bar{v}, \bar{u}) \in S$, there exists at least one DMU_{j_0} that maximizes the ratio R_j $(j = 1, \ldots, n)$ defined by (4). We call DMU_{j_0} *dominates* $(\bar{v}, \bar{u})$. It can be demonstrated that the $(m+s-1)$ dimensional simplex S is divided into a finite number of $(m+s-1)$ dimensional cells dominated by some DMUs. The cells are not necessarily convex. There may exist $(m + s - 2)$ or less dimensional dominant DMUs, with the extremal case 0 dimensional (point) dominant DMUs. Fig. 1 shows an example of the cell subdivision of the multiplier simplex S, where, for example, DMU_1, DMU_2 and DMU_3 are $(m+s-1)$ dimensional dominants, while DMU_{10} is a 0 dimensional dominant.

Fig. 1

3 Assurance Region and Cone-Ratio Models

In applying DEA to actual problems, we should be conscious of the economic/socioeconomic aspect of the problems, which is closely related with the virtual multiplier (weight) v (u) to the input (output) items. Although the original DEA models impose no restriction on v and u except positivity (or non-negativity), we can introduce the relative importance of weights by restricting the feasible region of weights. Along this line, two remarkable models have been proposed, i.e. the assurance region (AR) and the cone-ratio (CR) models.

The AR model imposes lower and upper bounds to the ratio of some selected pairs of weights. For example, we may add a constraint on the ratio of weights to Input 1 and Input 2 as follows:

$$l_{12} \leq \frac{v_2}{v_1} \leq u_{12}, \tag{6}$$

where l_{12} and u_{12} are the lower and upper bounds to the ratio, respectively. Likewise, similar constraints may be added to pairs of some output multipliers and even to multipliers between some input and output multipliers. See Thompson *et al.* (1986), (1990) and Roll and Golany (1993) for more details. The constraint such as (6) restricts the feasible region of multipliers to be in the polyhedral cone originated from a vertex of the simplex S. Thus, the assurance region, which satisfies all the ratio constraints will come to form a polyhedron $ABCD$ in Fig. 2, for example.

Fig. 2

On the other hand, in the cone-ratio model, especially in the polyhedral cone-ratio model, some exemplary DMUs will be chosen from among the CCR efficient DMUs as a result of expert knowledge. Then, the optimal weights corresponding to the selected DMUs will be used to form a polyhedral cone for an admissible region of multilpiers. (See Charnes *et al.* (1990) for details). However, usually the optimal weights are not uniquely determined. For example, let us observe the case when the experts chose DMUs 1, 2 and 3 in Fig. 1 as exemplary. Obviously, any point in the cell 1, 2 or 3 makes the DMU efficient. Therefore, we need some other criteria for selecting a reasonable point in the cell. There may be at least three principles for this purpose. The first one, the most restricted case, is to choose the cone as the minimum diameter convex set which makes the exemplary DMUs efficient. See the bold line EF in Fig. 3, for example.

Fig. 3

The next one, the most relaxed case, is to choose the cone as the convex hull of the exemplary cells, as designated by the region encircled by the bold lines in Fig. 4, for example.

Fig. 4

The last one chooses the cone generated by the 'central points' of each exemplary DMUs. See Fig. 5, for example.

Fig. 5

However, it is not easy to implement the above three principles. In fact, the first two might belong to NP-hard problems and the last one depends on the method of choosing the central point for each cell.

4 Practical Methods for Three Cases

Corresponding to the above mentioned general principles, we will propose three practical methods which approximately implement them.

4.1 The Most Restricted Case

This approach aims to obtain the cone depicted in Fig. 3.

Let the chosen exemplary DMUs be $DMU_{\alpha_1}, \ldots, DMU_{\alpha_p}$. We solve the following fractional program (FP$_k$) for each DMU_{α_k} $(k = 1, \ldots, p)$.

$$\text{(7)} \qquad (\mathrm{FP}_k) \quad \max \quad \frac{\sum_{r=1}^{s} u_r \sum_{j \neq k} y_{r\alpha_j}}{\sum_{i=1}^{m} v_i \sum_{j \neq k} x_{i\alpha_j}}$$

$$\text{(8)} \qquad \text{subject to} \quad \sum_{i=1}^{m} v_i x_{i\alpha_k} = \sum_{r=1}^{s} u_r y_{r\alpha_k} = 1$$

$$\text{(9)} \qquad \sum_{i=1}^{m} v_i x_{i\alpha_j} \geq \sum_{r=1}^{s} u_r y_{r\alpha_j} \quad (j = 1, \ldots, p)$$

$$\text{(10)} \qquad v_i \geq 0 \ (\forall i) \qquad u_r \geq 0 \ (\forall r).$$

It is easy to see that the objective value of (FP$_k$) is not greater than 1 and (FP$_k$) tries to find a point in the cell DMU_{α_k} that maximizes the composite ratio scale of other exemplary DMUs, although a scaling is needed to transfer the the optimal (v^*, u^*) of (FP$_k$) onto the cell DMU_{α_k}. The objective value attains 1 if, and only if, the intersection of the cells corresponding to $DMU_{\alpha_1}, \ldots, DMU_{\alpha_p}$ is not empty. In this case, it is sufficient to solve only (FP$_1$) and the vertex found will be used as the cone, actually the ray, for the cone-ratio model. Otherwise, we will solve (FP$_k$) for $k = 1, \ldots, p$ and find optimal vertices $(v_1^*, u_1^*), \ldots, (v_p^*, u_p^*)$ which will be used to form the cone. The fractional program (FP$_k$) can be solved as a linear programming problem via the Charnes and Cooper transformation (1962), which will be briefly described in Appendix A.

4.2 The Most Relaxed Case

This case aims to obtain a cone approximately realizing the convex hull in Fig. 4. Instead of maximizing the objective function in (FP$_k$), we try to minimize it, subject to the same constraints. Thus the objective is:

$$\text{minimize} \quad \frac{\sum_{r=1}^{s} u_r \sum_{j \neq k} y_{r\alpha_j}}{\sum_{i=1}^{m} v_i \sum_{j \neq k} x_{i\alpha_j}}. \tag{11}$$

This program will find a vertex in the cell DMU_k which is, in a sense, farthest from other exemplary DMUs. Let the optimal solution be $(\bar{v}_k^*, \bar{u}_k^*)$ $(k = 1, \ldots, p)$, which will be utilized to form the cone for the cone-ratio model.

4.3 The Central Case

This case corresponds to finding a central (relatively interior) point for each CCR-efficient DMU as depicted in Fig. 5.

The CCR model is formulated for each DMU_o $(o = 1, \ldots, n)$ as the following LP:

$$
\begin{array}{llrcl}
(12) & (CCR_o) & \max\ u^T y_o & & \\
(13) & \text{subject to} & v^T x_o & = & 1 \\
(14) & & -v^T X + u^T Y & \leq & 0 \\
(15) & & v \geq 0,\ u & \geq & 0.
\end{array}
$$

The computation is usually done on the dual side of (CCR_o) which is :

$$
\begin{array}{llrcl}
(16) & (LP_o) & \min\ \theta & & \\
(17) & \text{subject to} & \theta x_o - X\lambda - s_x & = & 0 \\
(18) & & y_o - Y\lambda + s_y & = & 0 \\
(19) & & \lambda & \geq & 0,\ s_x \geq 0,\ s_y \geq 0.
\end{array}
$$

Let optimal solutions for (CCR_o) and (LP_o) be (v^*, u^*) and $(\theta^*, \lambda^*, s_x^*, s_y^*)$, respectively, for which we have the complementarity:

$$(v^*)^T s_x^* = 0 \text{ and } (u^*)^T s_y^* = 0. \tag{20}$$

Furthermore, for a CCR-efficient DMU, we have

$$\theta^* = 1, \quad s_x^* = 0 \text{ and } s_y^* = 0.$$

However, by the strong theorem of complementarity, it can be seen that for a CCR-efficient DMU, there exists, in addition, an optimal solution of (CCR_o) with

$$v^* > 0 \text{ and } u^* > 0. \tag{21}$$

There may be two methods for finding a strictly complementary solution for a CCR-efficient DMU.

4.3.1 Primal-Dual Interior Point Method

The primal-dual interior point methods for linear programming (see Kojima, Mizuno and Yoshise (1989), McShane, Monma and Shanno (1989) and Choi, Monma and Shanno (1990), among others) will theoretically converge to the center of the optimal facet of the problem and the solution is strictly complementary. However, practical implementations of the interior point methods usually employ a long step size to the boundary of the feasible region for attaining to the next interior iterate and hence the optimal solution is strictly complementary but not central.

4.3.2 Parametric Linear Programming Approach

If a strictly complementary solution, i.e. $v^* > 0$ and $u^* > 0$, is required instead of the central one, we can obtain one, by a simplex based method, as follows:

If a CCR-efficient solution is not strictly complementary, we will solve the following parametric linear program in a scalar t:

$$\max w = t(e^T s_x + e^T s_y) \tag{22}$$

$$\text{subject to } x_o = X\lambda + s_x \tag{23}$$

$$y_o = Y\lambda - s_y \tag{24}$$

$$\lambda \geq 0, \quad s_x \geq 0, \quad s_y \geq 0. \tag{25}$$

Since we have the optimal solution $w^* = 0$ for $t = 0$ at the end of the (CCR_o) solution, we try to tend t positive, while keeping $s_x = 0$ and $s_y = 0$. A positive t^* is guaranteed to exist by the strong theorem of complementarity, which turns out, in the (CCR_o) side, positive weights such that

$$v^* \geq t^* e > 0 \quad \text{and} \quad u^* \geq t^* e > 0. \tag{26}$$

From the (v^*, u^*), we can get a strictly complementary solution for (CCR_o).

5 An Example

Table 1 exhibits data for 14 general hospitals operated under similar environments. As input items, we employ the working hours per month of doctors and nurses while, as outputs, the amounts of medical expense insurance for outpatients and inpatients are used.

Table 1: Data of General Hospital and Results

	Input		Output	
Hospital	Doctor	Nurse	Outpatient	Inpatient
H1	3008	20980	97775	101225
H2	3985	25643	135871	130580
H3	4324	26978	133655	168473
H4	3534	25361	46243	100407
H5	8836	40796	176661	215616
H6	5376	37562	182576	217615
H7	4982	33088	98880	167278
H8	4775	39122	136701	193393
H9	8046	42958	225138	256575
H10	8554	48955	257370	312877
H11	6147	45514	165274	227099
H12	8366	55140	203989	321623
H13	13479	68037	174270	341743
H14	21808	78302	322990	487539

The CCR-efficiency along with weights to inputs and outputs is shown in Table 2. Now, suppose that we chose H6 and H10 as exemplary among the five CCR-efficient DMUs. The three approaches mentioned in the preceding section gave the following results.

Table 2: Efficiency and Weight by CCR Model

Hospital	CCR Eff.	Doctor	Nurse	Outpatient	Inpatient
		v_1	v_2	u_1	u_2
H1	0.955	.332E-03	0	.959E-05	.167E-06
H2	1	.242E-03	.140E-05	.714E-05	.225E-06
H3	1	.104E-03	.204E-04	.339E-05	.325E-05
H4	0.702	.282E-03	.128E-06	0	.699E-05
H5	0.827	0	.245E-04	0	.384E-05
H6	1	.784E-04	.154E-04	.256E-05	.245E-05
H7	0.844	.133E-03	.102E-04	0	.505E-05
H8	1	.209E-03	.458E-07	.111E-07	.516E-05
H9	0.995	0	.233E-04	.426E-05	.136E-06
H10	1	.319E-04	.149E-04	.215E-05	.143E-05
H11	0.913	.162E-03	.739E-07	0	.402E-05
H12	0.969	.793E-04	.611E-05	0	.301E-05
H13	0.786	0	.147E-04	0	.230E-05
H14	0.974	0	.128E-04	0	.200E-05

5.1 The Most Restricted Case

We solved the fractional program (FP) in Subsection 4.1 for H6 and obtained the optimal solution:

$$\begin{array}{ll} v_1^* = 7.8408 \times 10^{-5} & v_2^* = 1.5401 \times 10^{-5} \\ u_1^* = 2.5567 \times 10^{-6} & u_2^* = 2.4503 \times 10^{-6}, \end{array}$$

with the optimal objective value = 1.

Thus, the cells of H6 and H10 have the above common vertex which was used as the admissible direction for solving the cone-ratio model. The results are exhibited in the 'Restricted' column of Table 3.

5.2 The Most Relaxed Case

We solved the fractional program (FP) with the minimizing objective function (11) for H6 and H10 and obtained the following results:

For H6

$$\begin{array}{ll} v_1^* = 1.8601 \times 10^{-4} & v_2^* = 0 \\ u_1^* = 5.3658 \times 10^{-6} & u_2^* = 9.3452 \times 10^{-8}, \end{array}$$

with the optimal objective value = 0.8863.

For H10

$$\begin{array}{ll} v_1^* = 0 & v_2^* = 2.0427 \times 10^{-5} \\ u_1^* = 0 & u_2^* = 3.1961 \times 10^{-6}, \end{array}$$

with the optimal objective value = 0.9065.

The two directions thus obtained were used to form the cone and the results are exhibited in the 'Relaxed' column of Table 3. Eventually, the efficiency coincides with that of the CCR model.

5.3 The Central Case

For the central case, we used the strong complementary directions (v, u) for H6 and H10 in Table 2, which were obtained by the method mentioned in Subsection 4.3.2. The results are shown in the 'Central' column of Table 3.

Fig. 6 depicts the three efficiencies. As a matter of course, the restricted case is severest in efficiency and two out of five CCR-efficient DMUs dropped from the peer set of efficiency. The relaxed case happens to coincide with the CCR score and the central case is in-between. Numbers in the last column 'Fukuda' will be explained in the next section.

Table 3: Efficiency by Cone-Ratio Models

Hospital	Restricted	Relaxed	Central	Fukuda
H1	0.8910	0.9546	0.9024	0.8870
H2	0.9434	1	0.9626	0.9359
H3	1	1	1	1
H4	0.5456	0.7018	0.5454	0.5826
H5	0.7418	0.8270	0.7749	0.7812
H6	1	1	1	1
H7	0.7361	0.8441	0.7361	0.7601
H8	0.8428	1	0.8428	0.9033
H9	0.9318	0.9946	0.9504	0.9452
H10	1	1	1	1
H11	0.8276	0.9125	0.8276	0.8556
H12	0.8701	0.9690	0.8700	0.8921
H13	0.6096	0.7859	0.6251	0.6576
H14	0.6929	0.9742	0.7632	0.7937

Fig. 6

6 Enumeration of Optimal Vertices

It is interesting to know all the vertices (v, u) of the convex polyhedron, which makes a DMU efficient, not only for the purposes mentioned in the preceding sections but also for understanding the overall positioning of the efficient DMU in the (v, u) space.

Recently, Fukuda (1993) has developed an algorithm and software[2] for enumerating all vertices of a convex polyhedron defined by a system of linear inequalities, base on the Double Description Method (1958). This software

[2]The free software "cdd.c" is available via anonymous ftp from ftp.epfl.ch (directory incoming/dma).

works efficiently for medium size problems. In fact, for an actual problem with 42 DMUs, 4 inputs and 2 outputs, all vertices of an efficient DMU could be obtained in 0 second on SUN SparkServer 1000 and in 15 seconds on Mac Powerbook Duo 210 (68030, 25mhz). The problem has 6 efficient DMUs among the all 42 DMUs and the polyhedron corresponding to the said DMU has 22 vertices. It has been observed that the number of vertices increases exponentially in the number of tight (active) constraints in the optimal solution of (CCR_o). It increases also as the number of inputs and outputs does.

We applied this software for the sample problem above, especially for DMUs H6 and H10 and obtained 6 and 5 vertices for H6 and H10, respectively, as follows (CPU-time was negligible):

Table 4: All Vertices of H6

No.	v_1	v_2	u_1	u_2
1	1.8542E-4	8.4472E-8	0	4.5953E-6
2	1.8601E-4	0	1.9038E-8	4.5793E-6
3	1.8601E-4	0	5.3658E-6	9.3452E-8
4	1.2090E-4	9.3195E-6	0	4.5953E-6
5	7.3557E-5	1.6095E-5	4.0202E-6	1.2224E-6
6	7.8408E-5	1.5401E-5	2.5567E-6	2.4503E-6

From these vertices, we can obtain the 'center', as the average of each coordinate value, for each polyhedron as shown in Table 6. By using these two directions as admissible for the cone, we have the CCR efficiency as denoted in the column 'Fukuda' in Table 3.

Table 5: All Vertices of H10

No.	v_1	v_2	u_1	u_2
1	5.5037E-5	1.0810E-5	1.7946E-6	1.7199E-6
2	5.1905E-5	1.1357E-5	2.8369E-6	8.6257E-7
3	0	2.0427E-5	3.7409E-6	1.1890E-7
4	0	2.0427E-5	0	3.1961E-6
5	3.2349E-5	1.4774E-5	0	3.1961E-6

Table 6: Center of Gravity of Polyhedron

No.	v_1	v_2	u_1	u_2
H6	1.3838E-4	6.8166E-6	1.9936E-6	2.9227E-6
H10	2.7858E-5	1.5559E-5	1.6745E-6	1.8187E-6

7 Concluding Remarks

In this paper, we have proposed three approaches to the cone-ratio model with respect to the virtual input/output multipliers. The choice of the admissible directions for the cone is crucial for the successful application of the cone-ratio model. If experts' knowledge on the virtual multipliers (weights) could be combined with the three cases proposed, progress could be made in the evaluation of the relative efficiency of DMUs. Also, we have mentioned the possible applications of a vertex enumeration algorithm for efficient DMUs, which would be an interesting future research subject.

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studies. Also, I am grateful to Professor R.M. Thrall for helpful comments and corrections.

Appendix A. Solution of Fractional Programming

The fractional program (FP_k) can be transformed into equivalent linear program as follows:

$$\text{(LP}_k\text{)} \quad \max \quad \sum_{r=1}^{s} U_r \sum_{j \neq k} y_{r\alpha_j} \tag{27}$$

$$\text{subject to} \quad \sum_{i=1}^{m} V_i \sum_{j \neq k} x_{i\alpha_j} = 1 \tag{28}$$

$$\sum_{i=1}^{m} V_i x_{i\alpha_k} = \sum_{r=1}^{s} U_r y_{r\alpha_k} = t \tag{29}$$

$$\sum_{i=1}^{m} V_i x_{i\alpha_j} \geq \sum_{r=1}^{s} U_r y_{r\alpha_j} \quad (j = 1, \ldots, p) \tag{30}$$

$$V_i \geq 0 \ (\forall i), \qquad U_r \geq 0 \ (\forall r), \qquad t \geq 0. \tag{31}$$

Let an optimal solution of (LP_k) be ($\boldsymbol{V}^*$, $\boldsymbol{U}^*$, t^*). Then we have the optimal solution of (FP_k) by

$$\boldsymbol{v}^* = \boldsymbol{V}^*/t^* \ \text{ and } \ \boldsymbol{u}^* = \boldsymbol{U}^*/t^*. \tag{32}$$

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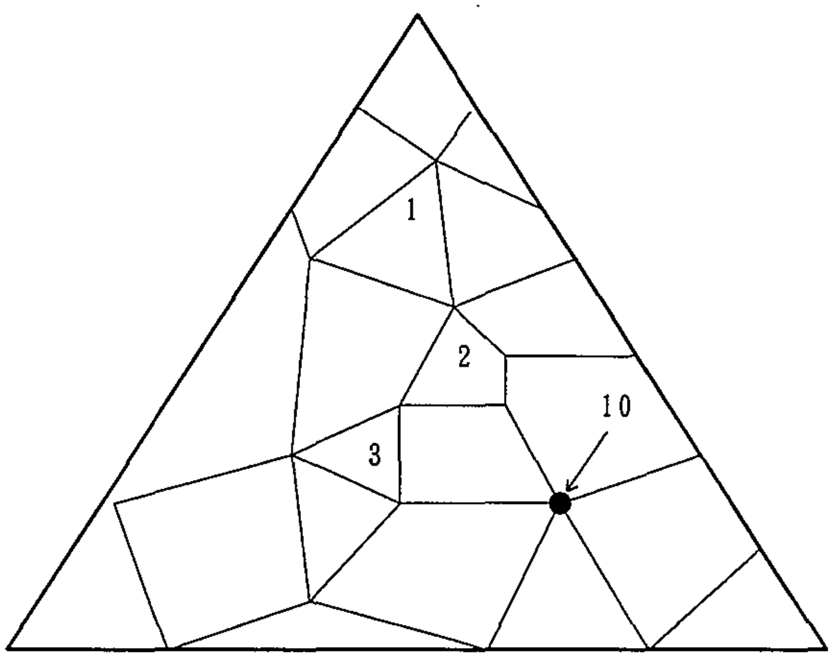


Figure 1. Subdivision of Simplex S

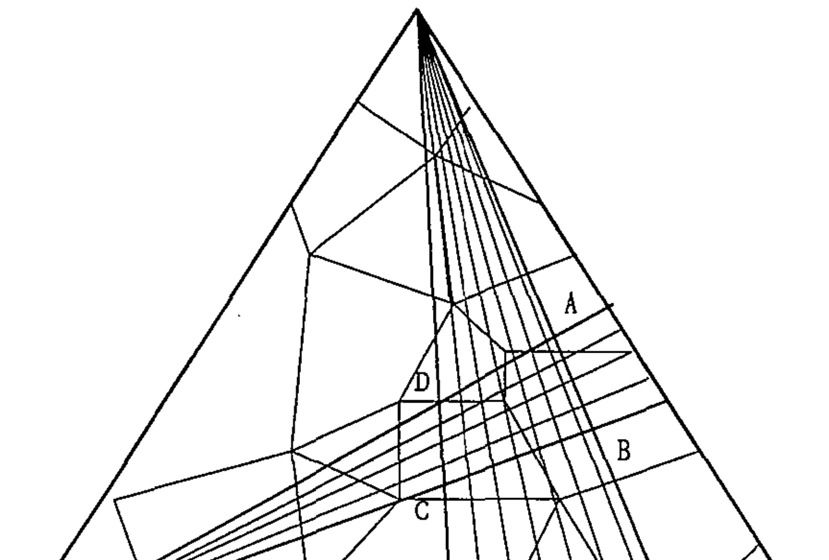


Figure 2. Assurance Region

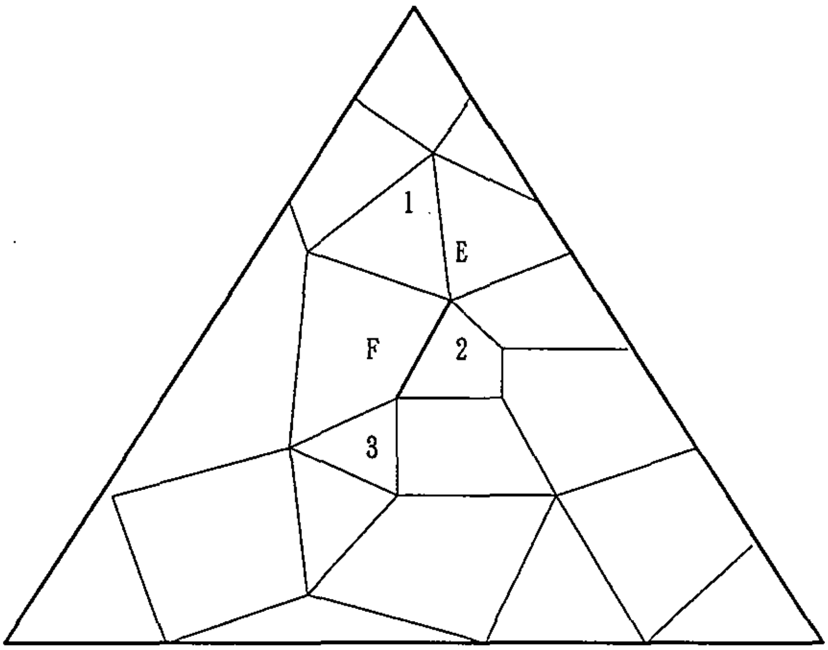


Figure 3. The Most Restricted Case

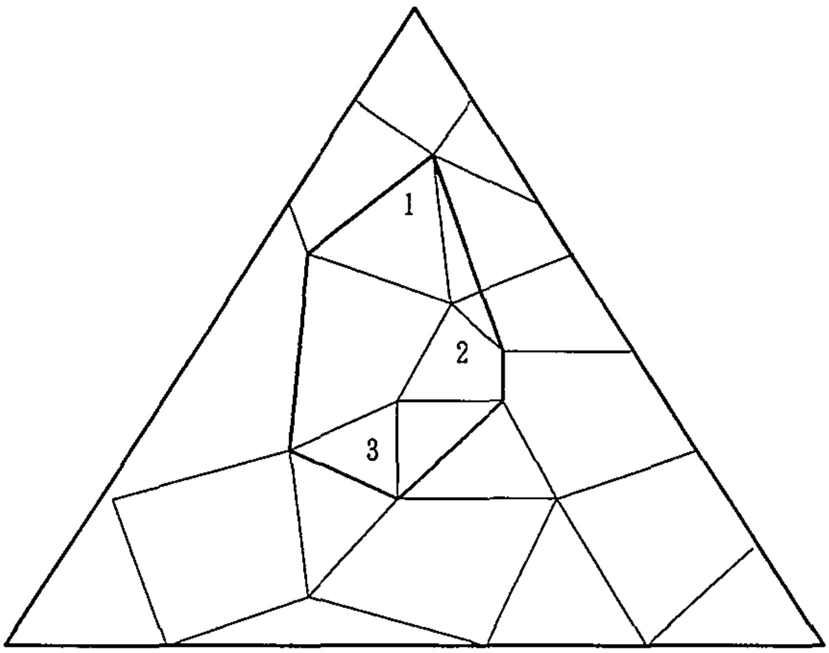


Figure 4. The Most Relaxed Case

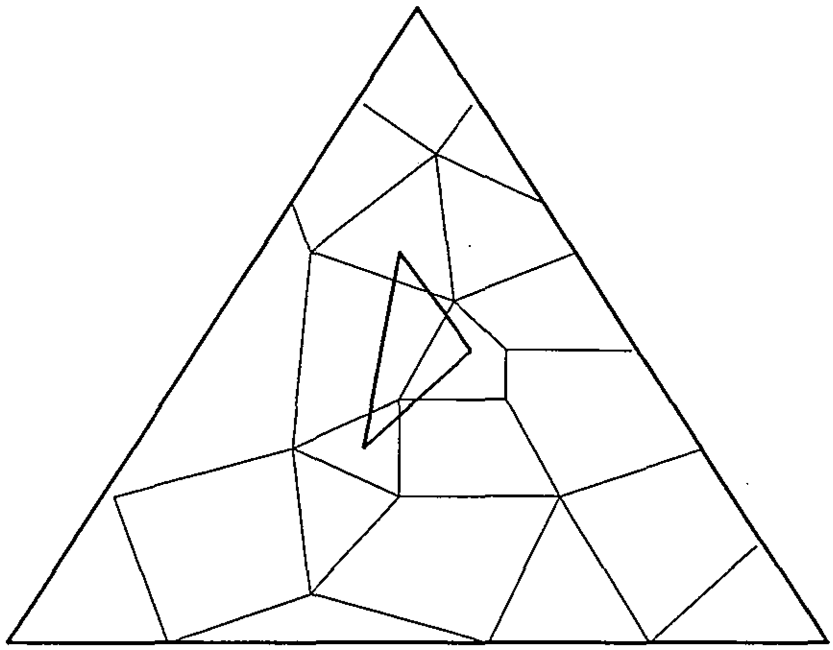
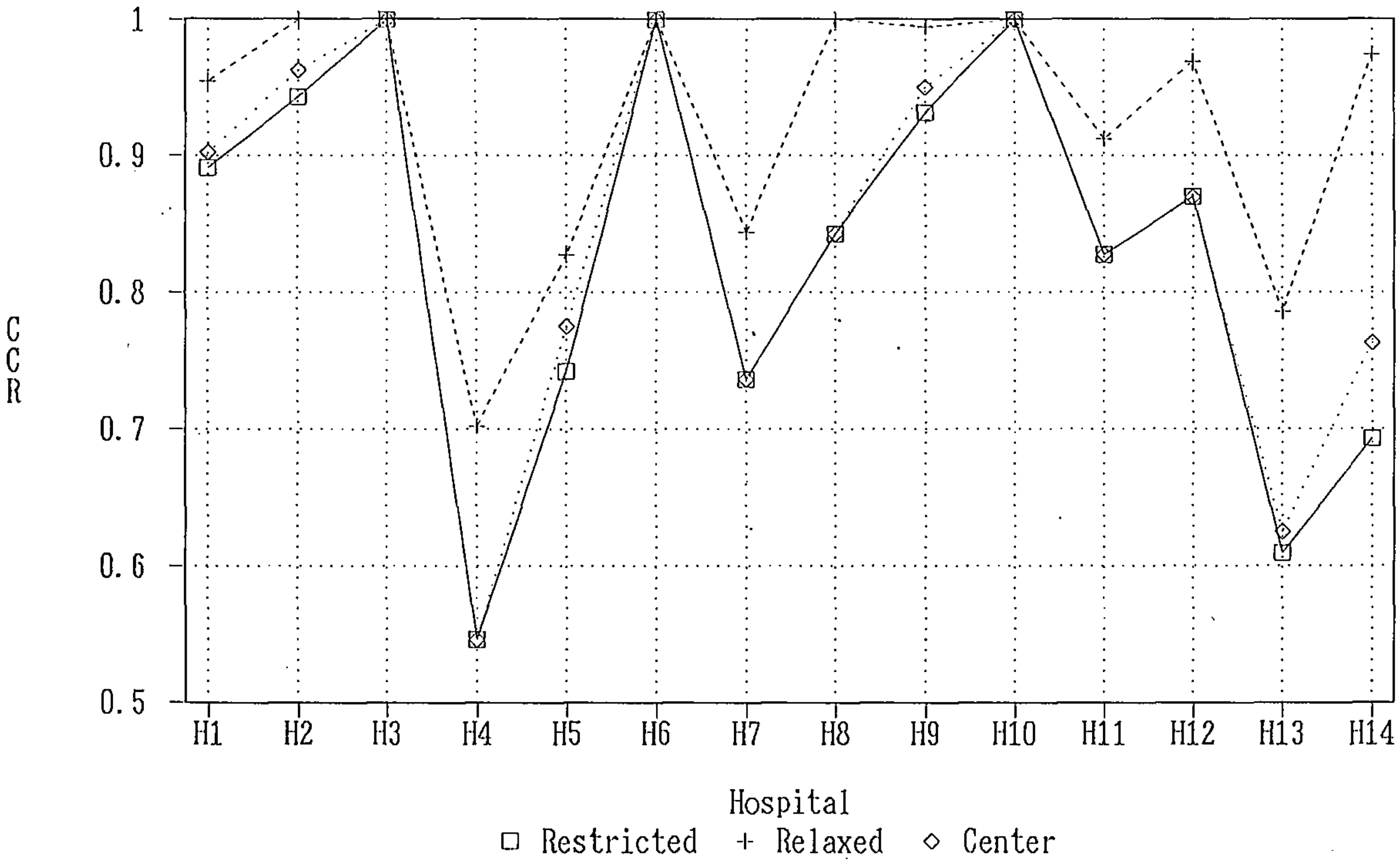


Figure 5. The Central Case

Figure 6. Three Cone-Ratio Models



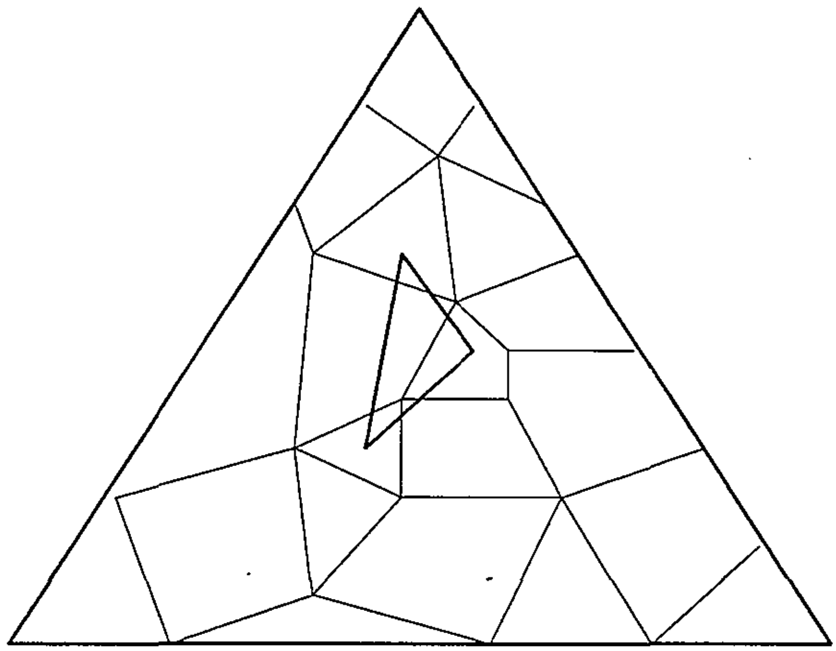


Figure 5. The Central Case

Figure 6. Three Cone-Ratio Models