Dynamic DEA with network structure
– A slacks-based measure approach –

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Abstract: We propose a dynamic DEA model involving network structure in each period within the framework of a slacks-based measure approach. We have previously published the network SBM (NSBM) and the dynamic SBM (DSBM) models separately. Hence, this article is a composite of these two models. Vertically, we deal with multiple divisions connected by links of network structure within each period and, horizontally, we combine the network structure by means of carry-over activities between two succeeding periods. This model can evaluate (1) the overall efficiency over the entire observed period, (2) dynamic change of period efficiency and (3) dynamic change of divisional efficiency. In addition, we also introduce dynamic Malmquist index by which we can compare divisional performances over time. We applied this model to a dataset of US electric utilities and compared the result with that of DSBM.
Keyword: Dynamic DEA, network DEA, SBM, Malmquist index

1. INTRODUCTION

Traditional DEA (data envelopment analysis) models deal with measurements of relative efficiency of decision making units (DMUs) regarding multiple inputs vs. multiple outputs. One of the drawbacks of these models is the omission of the internal structure of the DMUs. For example, many companies are comprised of several divisions that are linked to each other having division-specific inputs and outputs as well as links to other divisions. To reflect the actual world, the network DEA model was developed to take into account the internal structure of DMUs using link variables [7, 8, 22]. In addition, companies' activity generally continues across multiple periods. The dynamic DEA model was developed to evaluate DMUs performance from a long-term perspective using carry-over variables [1, 2, 5, 8, 12, 13, 15, 16 17, 19, 23].

We propose a model combining these two developed models, resulting in dynamic and network DEA. This combined model enables us not only to obtain the overall efficiency of DMUs over the entire observed period, but also to conduct further analysis, that is, observing dynamic change of the period efficiency and dynamic change of the divisional efficiency of DMUs. In addition, we propose a Malmquist index corresponding to the dynamic and network framework. Using our model, we can measure the efficiency score of DMUs in a more realistic manner that is not achieved by the traditional models so far.

The rest of this paper unfolds as follows. In Section 2, we describe mathematical formulations of dynamic and network SBM model. We discuss the uniqueness issue of period efficiencies in Section 3. Divisional dynamic Malmquist index is introduced in Section 4. An application to U.S. electric utilities is presented in Section 5, along with comparisons with results by the Dynamic SBM model. Section 6 concludes the paper.

2. Dynamic DEA with network structure

In this section, we define the dynamic DEA with network structure based on SBM framework [18, 20] (DNSBM) and formulate it as a programming problem.

2.1. Graphical explanation

The DNSBM model takes into account the internal structure of a DMU, in which Divisions are vertically connected by links (intermediate products). In addition, consecutive periods are horizontally connected by carry-overs. Finally, dynamic and network structure can be depicted as Figure 1.

2.2. Notations

We deal with n DMUs (j = 1,…, n) consisting of K divisions (k = 1,…, K) over T time periods (t = 1,…,T). Let m_k and r_k be the numbers of inputs and outputs to division k, respectively. We denote the link leading from division k to division h by $(k,h)_l$ and the set of links by L_{kh}. The observed data are as follows.

a) Inputs and outputs

$x^t_{ijk} \in R_+$ (i=1,...,m_k; j=1,...,n; k=1,...,K; t=1,...,T) is input resource i to DMU$_j$ for division k in period t, and $y^t_{ijk} \in R_+$ (i=1,...,r_k; j=1,...,n; k=1,...,K; t=1,...,T) is output product i from DMU$_j$, division k, in period t. If some outputs are undesirable, we treat them as inputs to division k.

b) Links

$z^t_{j(kh)_l} \in R_+$ (j=1,…,n; l=1,…,L_{kh}; t=1,…,T) is linking intermediate products of DMUj from division k to division h in period t, where L_{kh} is the number of items in links from k to h.

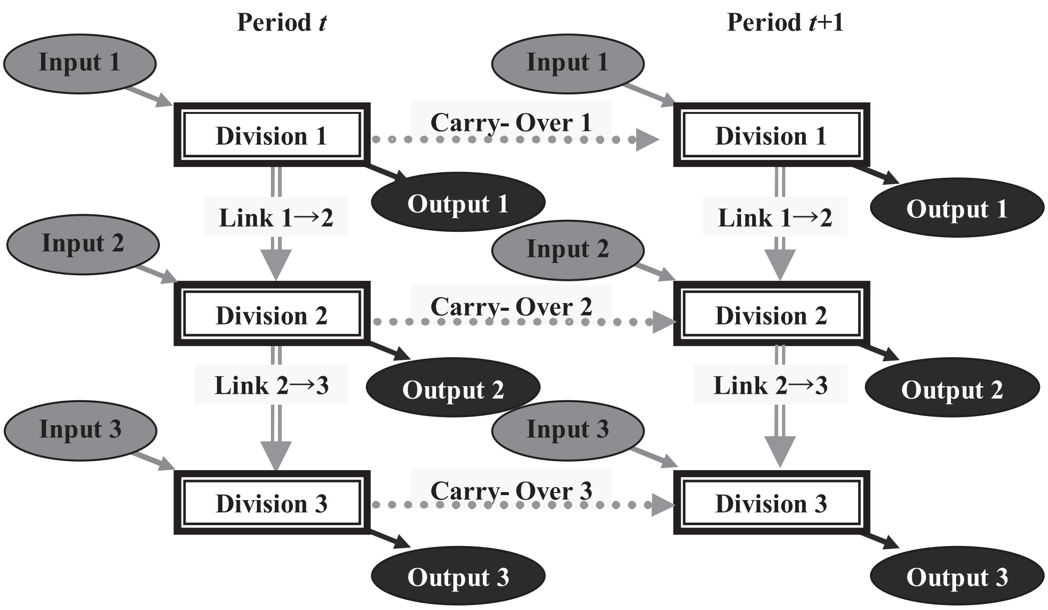

Figure 1: Dynamic model with network structure

<2>

c) Carry-overs

$z_{jk_l}^{(t,t+1)} \in R_+$ (j=1,...,n; l=1,...,L_k; k =1,...,K, t=1,...,T-1)

is carry-over of DMUj, at division k, from period t to period t+1, where L_k is the number of items in the carry-over from division k.

2.3. Production possibility set

The production possibility set $P = \left\{\left(\mathbf{x}_k^t, \mathbf{y}_k^t, \mathbf{z}_{(kh)}^t, \mathbf{z}_{i_k}^{(t,t+1)}\right)\right\}$ is defined by

$$
\begin{aligned}
&\mathbf{x}_k^t \geq \sum_{j=1}^n \mathbf{x}_{jk}^t \lambda_{jk}^t \ (\forall k, \forall t) \\
&\mathbf{y}_k^t \leq \sum_{j=1}^n \mathbf{y}_{jk}^t \lambda_{jk}^t \ (\forall k, \forall t) \\
&\mathbf{z}_{(kh)_l}^t = \sum_{j=1}^n \mathbf{z}_{j(kh)_l}^t \lambda_{jk}^t \ (\forall l, \forall (kh)_l, \forall t) \\
&\qquad \text{(as outputs from } k \text{ in period } t) \\
&\mathbf{z}_{(kh)_l}^t = \sum_{j=1}^n \mathbf{z}_{j(kh)_l}^t \lambda_{jh}^t \ (\forall l, \forall (kh)_l, \forall t) \\
&\qquad \text{(as inputs to } h \text{ in period } t) \qquad (1) \\
&\mathbf{z}_{k_l}^{(t,t+1)} = \sum_{j=1}^n \mathbf{z}_{jk_l}^{(t,t+1)} \lambda_{jk}^t \ (\forall k_l, \forall k, t = 1, \ldots, T-1) \\
&\qquad \text{(as carry-over from } t) \\
&\mathbf{z}_{k_l}^{(t,t+1)} = \sum_{j=1}^n \mathbf{z}_{jk_l}^{(t,t+1)} \lambda_{jk}^{t+1} (\forall k_l, \forall k, t = 1, \ldots, T-1) \\
&\qquad \text{(as carry-over to } t+1) \\
&\sum_{j=1}^n \lambda_{jk}^t = 1 \ (\forall k, \forall t), \ \lambda_{jk}^t \geq 0 \ (\forall j, \forall k, \forall t),
\end{aligned}
$$

where $\boldsymbol{\lambda}_k^t = \left\{\lambda_{jk}^t\right\} \in R_+^n$ is the intensity vector corresponding to division k (k=1,...,K) at t (t=1,...,T).

We notice that the above model assumes the variable returns-to-scale (VRS) for production. That is, the production frontiers are spanned by the convex hull of the existing DMUs. However, if we omit the last constraint $\sum_{j=1}^n \lambda_{jk}^t = 1$, we can deal with the constant returns-to-scale (CRS) case as well.

2.4. Expression for DMU$_o$

DMU$_o$ (o=1,...,n)$\in P$ can be expressed as follows.

2.4.1. Inputs and outputs

Input and output constraints are listed below.

$$
\begin{aligned}
&\mathbf{x}_{ok}^t = \mathbf{X}_k^t \boldsymbol{\lambda}_k^t + \mathbf{s}_{ko}^{t-} \quad (k = 1, \ldots, K; t = 1, \ldots, T) \\
&\mathbf{y}_{ok}^t = \mathbf{Y}_k^t \boldsymbol{\lambda}_k^t - \mathbf{s}_{ko}^{t+} \quad (k = 1, \ldots, K; t = 1, \ldots, T) \\
&\mathbf{e}\boldsymbol{\lambda}_k^t = 1 \quad (k = 1, \ldots, K; t = 1, \ldots, T) \\
&\boldsymbol{\lambda}_k^t \geq \mathbf{0}, \mathbf{s}_{ko}^{t-} \geq \mathbf{0}, \mathbf{s}_{ko}^{t+} \geq \mathbf{0}, \ (\forall k, \forall t)
\end{aligned} \qquad (2)
$$

where $\mathbf{X}_k^t = (\mathbf{x}_{1k}^t, \ldots, \mathbf{x}_{nk}^t) \in R^{m_k \times n \times T}$ and $\mathbf{Y}_k^t = (\mathbf{y}_{1k}^t, \ldots, \mathbf{y}_{nk}^t) \in R^{r_k \times n \times T}$ are input and output matrices, and $\mathbf{s}_{ko}^{t-}$ and $\mathbf{s}_{ko}^{t+}$ are, respectively, input/output slacks.

2.4.2. Links

Link is an intermediate product, which is an output from Division k and also an input to Division h. Regarding to the linking constraints, we have several options of which we present four possible cases.

(a) "free" link value case (LF)

The linking activities are freely determined (discretionary) while keeping continuity between input and output:

$$\mathbf{Z}_{(kh)free}^t \boldsymbol{\lambda}_h^t = \mathbf{Z}_{(kh)free}^t \boldsymbol{\lambda}_k^t. \quad (\forall (k,h) free, \forall t) \qquad (3)$$

where $\mathbf{Z}_{(kh)free}^t = (\mathbf{z}_{1(kh)free}^t, \ldots, \mathbf{z}_{n(kh)free}^t) \in R^{L_{(kh)free} \times n}$.

This case can serve to see if the current link flow is appropriate or not in the light of other DMUs, i.e., the link flow may increase or decrease in the optimal solution of the linear programs which we will introduce in the next section. Between the current link value and the free link value we have the relationship

$$\mathbf{z}_{o(kh)free}^t = \mathbf{Z}_{(kh)free}^t \boldsymbol{\lambda}_k^t + \mathbf{s}_{o(kh)free}^t, \qquad (4)$$

where $\mathbf{s}_{o(kh)free}^t \in R^{L_{kh}}$ is slacks and free in sign.

(b) Non-discretionary "fixed" link value case (LN)

The linking activities are kept unchanged (non-discretionary):

$$
\begin{aligned}
&\mathbf{z}_{o(kh)fix}^t = \mathbf{Z}_{(kh)fix}^t \boldsymbol{\lambda}_h^t \quad (\forall (k,h) fix, \forall t) \\
&\mathbf{z}_{o(kh)fix}^t = \mathbf{Z}_{(kh)fix}^t \boldsymbol{\lambda}_k^t. \quad (\forall (k,h) fix, \forall t)
\end{aligned} \qquad (5)
$$

This case corresponds to the situation where the intermediate products are beyond the control of DMUs or discretion of management.

(c) "as-input" link value case (LB)

The linking activities are treated as input to the succeeding division and excesses are accounted for in the input inefficiency.

$$\mathbf{z}_{o(kh)in}^{t}=\mathbf{Z}_{(kh)in}^{t}\boldsymbol{\lambda}_{k}^{t}+\mathbf{s}_{o(kh)in}^{t}\quad ((kh)in=1,\ldots,linkin_{k})\quad (6)$$

where $\mathbf{s}_{o(kh)in}^{t}\in R^{L_{(kh)in}}$ is slacks and non-negative, and $linkin_k$ is the number of "as-input" link from division k.

(d) "as-output" link value case (LG)

The linking activities are treated as output from the preceding division and shortages are accounted for in the output inefficiency.

$$\mathbf{z}_{o(kh)out}^{t}=\mathbf{Z}_{(kh)out}^{t}\boldsymbol{\lambda}_{k}^{t}-\mathbf{s}_{o(kh)out}^{t}\quad ((kh)out=1,\ldots,linkout_{k})\quad (7)$$

where $\mathbf{s}_{o(kh)out}^{t}\in R^{L_{(kh)out}}$ is non-negative slack and $linkout_k$ is the number of "as-output" links from division k.

2.4.3. Carry-overs

Carry-over variable is an output at period t and becomes an input at period t+1. We classify carry-over activities into four categories as follows.

(a) Desirable (good) carry-over case (CG)

This indicates desirable carry-over, e.g. profit carried forward and net earned surplus carried to the next period. In our model, desirable carry-overs are treated as outputs and their values are restricted to be not less than the observed one. Comparative shortage of carry-overs in this category is accounted as inefficiency.

(b) Undesirable (bad) carry-over case (CB)

This belongs to undesirable carry-over, e.g. loss carried forward, bad debt and dead stock. In our model, undesirable carry-overs are treated as inputs and their values are restricted to be not greater than the observed ones. Comparative excess in carry-overs in this category is accounted as inefficiency.

(c) Discretionary (free) carry-over case (CF)

This corresponds to carry-over that DMU can handle freely. Its value can be increased or decreased from the observed one. The deviation from the current value is not directly reflected in the efficiency evaluation, but the continuity condition between two periods explained below exerts an indirect effect on the efficiency score.

(d) Non-discretionary (fixed) carry-over case (CN)

This indicates carry-over that is beyond the control of a DMU. Its value is fixed at the observed level. Similar to free carry-over, fixed carry-over affects the efficiency score indirectly through the continuity condition between two periods.

$$\sum_{j=1}^{n} z_{jk_l\alpha}^{(t,t+1)}\lambda_{jk}^{t}=\sum_{j=1}^{n} z_{jk_l\alpha}^{(t,t+1)}\lambda_{jk}^{t+1}\quad (\forall k;\forall k_l;\, t=1,\ldots,T-1)\quad (8)$$

where the symbol α stands for *good*, *bad*, *free* or *fix*.

Corresponding to each category of carry-overs we have the following constraints.

$$\begin{aligned}
&z_{ok_l good}^{(t,t+1)}=\sum_{j=1}^{n} z_{jk_l good}^{(t,t+1)}\lambda_{jk}^{t}-s_{ok_l good}^{(t,t+1)}\\
&\quad (k_l=1,\ldots,ngood_k;\, k=1,\ldots,K;\, t=1,\ldots,T)\\
&z_{ok_l bad}^{(t,t+1)}=\sum_{j=1}^{n} z_{jk_l bad}^{(t,t+1)}\lambda_{jk}^{t}+s_{ok_l bad}^{(t,t+1)}\\
&\quad (k_l=1,\ldots,nbad_k;\, k=1,\ldots,K;\; t=1,\ldots,T)\\
&z_{ok_l free}^{(t,t+1)}=\sum_{j=1}^{n} z_{jk_l free}^{(t,t+1)}\lambda_{jk}^{t}+s_{ok_l free}^{(t,t+1)}\\
&\quad (k_l=1,\ldots,nfree_k;\, k=1,\ldots,K;\, t=1,\ldots,T)\\
&z_{ok_l fix}^{(t,t+1)}=\sum_{j=1}^{n} z_{jk_l fix}^{(t,t+1)}\lambda_{jk}^{t}\\
&\quad (k_l=1,\ldots,nfix_k;\, k=1,\ldots,K;\, t=1,\ldots,T)\\
&s_{ok_l good}^{(t,t+1)}\ge 0,\; s_{ok_l bad}^{(t,t+1)}\ge 0 \text{ and } s_{ok_l free}^{(t,t+1)}:\, free\quad (\forall k_l;\forall t)
\end{aligned}\quad (9)$$

where $s_{ok_l good}^{(t,t+1)}$, $s_{ok_l bad}^{(t,t+1)}$ and $s_{ok_l free}^{(t,t+1)}$ are slacks denoting, respectively, carry-over shortfall, carry-over excess and carry-over deviation, and $ngood_k$, $nbad_k$, and $nfree_k$ indicate the number of desirable (good), undesirable (bad) and free carry-overs for each division k.

2.5. The objective function

This section deals with the overall-, period- and divisional efficiencies in the case of the non-oriented (i.e., both input- and output-oriented) model. The overall-efficiency is evaluated by the following program.

$$\theta_o^* = \min \frac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k + linkin_k + nbad_k} \times \left(\sum_{i=1}^{m_k} \frac{s_{iok}^{t-}}{x_{iok}^t} + \sum_{(k,h)_l=1}^{linkin_k} \frac{s_{o(k,h)_l in}^t}{z_{o(k,h)_l in}^t} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l bad}^{(t,t+1)}}{z_{ok_l bad}^{(t,t+1)}} \right) \right]}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} w^k \left[1 + \frac{1}{r_k + linkout_k + ngood_k} \times \left(\sum_{i=1}^{r_k} \frac{s_{iok}^{t+}}{y_{iok}^t} + \sum_{(k,h)_l=1}^{linkout_k} \frac{s_{o(k,h)_l out}^t}{z_{o(k,h)_l out}^t} + \sum_{k_l=1}^{ngood_k} \frac{s_{ok_l good}^{(t,t+1)}}{z_{ok_l good}^{(t,t+1)}} \right) \right]} \tag{10}$$

subject to (2), (3), and (5) to (9), where W^t $(t = 1,\dots,T)$ is the weight to period *t* and w^k $(k = 1,\dots,K)$ is the weight to division *k*. These weights satisfy the condition: $\sum_{t=1}^{T} W^t = 1$, $\sum_{k=1}^{K} w^k = 1$, $W^t \geq 0 (\forall t)$, $w^k \geq 0\ (\forall k)$. They are supplied exogenously. The input-(output-) oriented model can be defined by dealing with the numerator (denominator) of the above objective function.

2.6. Period and divisional efficiencies

Period efficiency τ_o^{t*} and divisional efficiency δ_{ok}^* are defined by

$$\tau_o^{t*} = \frac{\sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k + linkin_k + nbad_k} \times \left(\sum_{i=1}^{m_k} \frac{s_{iok}^{t-}}{x_{iok}^t} + \sum_{(k,h)_l=1}^{linkin_k} \frac{s_{o(k,h)_l in}^t}{z_{o(k,h)_l in}^t} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l bad}^{(t,t+1)}}{z_{ok_l bad}^{(t,t+1)}} \right) \right]}{\sum_{k=1}^{K} w^k \left[1 + \frac{1}{r_k + linkout_k + ngood_k} \times \left(\sum_{i=1}^{r_k} \frac{s_{iok}^{t+}}{y_{iok}^t} + \sum_{(k,h)_l=1}^{linkout_k} \frac{s_{o(k,h)_l out}^t}{z_{o(k,h)_l out}^t} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l good}^{(t,t+1)}}{z_{ok_l good}^{(t,t+1)}} \right) \right]} \quad (t = 1,\dots,T) \tag{11}$$

$$\delta_{ok}^* = \frac{\sum_{t=1}^{T} W^t \left[1 - \frac{1}{m_k + linkin_k + nbad_k} \times \left(\sum_{i=1}^{m_k} \frac{s_{iok}^{t-}}{x_{iok}^t} + \sum_{(k,h)_l=1}^{linkin_k} \frac{s_{o(k,h)_l in}^t}{z_{o(k,h)_l in}^t} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l bad}^{(t,t+1)}}{z_{ok_l bad}^{(t,t+1)}} \right) \right]}{\sum_{t=1}^{T} W^t \left[1 + \frac{1}{r_k + linkout_k + ngood_k} \times \left(\sum_{i=1}^{r_k} \frac{s_{iok}^{t+}}{y_{iok}^t} + \sum_{(k,h)_l=1}^{linkout_k} \frac{s_{o(k,h)_l out}^t}{z_{o(k,h)_l out}^t} + \sum_{k_l=1}^{ngood_k} \frac{s_{ok_l good}^{(t,t+1)}}{z_{ok_l good}^{(t,t+1)}} \right) \right]} \quad (k = 1,\dots,K) \tag{12}$$

where variables on the right hand side indicates optimal values for the overall efficiency θ_o^*.

Finally, period-divisional efficiency is defined by

$$\rho_{ok}^{t*} = \frac{1 - \frac{1}{m_k + linkin_k + nbad_k} \times \left(\sum_{i=1}^{m_k} \frac{s_{iok}^{t-}}{x_{iok}^t} + \sum_{(k,h)_l=1}^{linkin_k} \frac{s_{o(k,h)_l in}^t}{z_{o(k,h)_l in}^t} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l bad}^{(t,t+1)}}{z_{ok_l bad}^{(t,t+1)}} \right)}{1 + \frac{1}{r_k + linkout_k + ngood_k} \times \left(\sum_{i=1}^{r_k} \frac{s_{iok}^{t+}}{y_{iok}^t} + \sum_{(k,h)_l=1}^{linkout_k} \frac{s_{o(k,h)_l out}^t}{z_{o(k,h)_l out}^t} + \sum_{k_l=1}^{ngood_k} \frac{s_{ok_l good}^{(t,t+1)}}{z_{ok_l good}^{(t,t+1)}} \right)} \quad (k = 1,\dots,K; t = 1,\dots,T) \tag{13}$$

In the input- (output-) oriented model, the numerator (denominator) of the above formulas is applied. We notice that, although the overall-efficiency is uniquely determined, the period, divisional and period-divisional efficiencies are not necessarily unique. Furthermore, in the input-oriented model, the overall efficiency is the weighted arithmetic mean of the period-efficiencies and, in the output-oriented model, the overall efficiency is the weighted harmonic mean of the period-efficiencies, whereas in the non-oriented model the overall efficiency is neither the arithmetic nor the harmonic mean of the period-efficiencies.

3. Uniqueness issue of period efficiencies

Although the overall efficiency is uniquely determined by the program (10), slacks are not necessarily unique. Hence, the period efficiency in (11) may suffer from plurality. Comparing the importance of periods, it would be reasonable that the last period *T* has the top priority and those of *T*-1, *T*-2,…, 1 decrease in this order. Under this priority principle, we propose the following scheme for overcoming this plurality problem.

3.1. Period efficiency in T

First, we solve the program (10) and obtain the overall efficiency θ_o^*. Then we minimize period efficiency in *T* while keeping the overall efficiency at θ_o^*.

Let us denote the period efficiency in *T* thus obtained by τ_o^{T*}.

$$\tau_o^{T*} = \min \frac{\sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k + linkin_k + nbad_k} \times \left(\sum_{i=1}^{m_k} \frac{s_{iok}^{T-}}{x_{iok}^{T}} + \sum_{(k,h)_l=1}^{linkin_k} \frac{s_{o(k,h)_l in}^{T}}{z_{o(k,h)_l in}^{T}} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l bad}^{(T,T+1)}}{z_{ok_l bad}^{(T,T+1)}} \right) \right]}{\sum_{k=1}^{K} w^k \left[1 + \frac{1}{r_k + linkout_k + ngood_k} \times \left(\sum_{i=1}^{r_k} \frac{s_{iok}^{T+}}{y_{iok}^{T}} + \sum_{(k,h)_l=1}^{linkout_k} \frac{s_{o(k,h)_l out}^{T}}{z_{o(k,h)_l out}^{T}} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l good}^{(T,T+1)}}{z_{ok_l good}^{(T,T+1)}} \right) \right]} \qquad (15)$$

subject to

$$\frac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k + linkin_k + nbad_k} \times \left(\sum_{i=1}^{m_k} \frac{s_{iok}^{t-}}{x_{iok}^{t}} + \sum_{(k,h)_l=1}^{linkin_k} \frac{s_{o(k,h)_l in}^{t}}{z_{o(k,h)_l in}^{t}} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l bad}^{(t,t+1)}}{z_{ok_l bad}^{(t,t+1)}} \right) \right]}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} w^k \left[1 + \frac{1}{r_k + linkout_k + ngood_k} \times \left(\sum_{i=1}^{r_k} \frac{s_{iok}^{t+}}{y_{iok}^{t}} + \sum_{(k,h)_l=1}^{linkout_k} \frac{s_{o(k,h)_l out}^{t}}{z_{o(k,h)_l out}^{t}} + \sum_{k_l=1}^{ngood_k} \frac{s_{ok_l good}^{(t,t+1)}}{z_{ok_l good}^{(t,t+1)}} \right) \right]} = \theta_o^* \qquad (16)$$

and (2), (3), and (5) to (9).

3.2. Period efficiency in t

We repeat this process until *t*=2. Thus, period efficiency in *t* (τ_o^{t*}) is measured by the following program.

$$\tau_o^{t*} = \min \frac{\sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k + linkin_k + nbad_k} \times \left(\sum_{i=1}^{m_k} \frac{s_{iok}^{t-}}{x_{iok}^{t}} + \sum_{(k,h)_l=1}^{linkin_k} \frac{s_{o(k,h)_l in}^{t}}{z_{o(k,h)_l in}^{t}} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l bad}^{(t,t+1)}}{z_{ok_l bad}^{(t,t+1)}} \right) \right]}{\sum_{k=1}^{K} w^k \left[1 + \frac{1}{r_k + linkout_k + ngood_k} \times \left(\sum_{i=1}^{r_k} \frac{s_{iok}^{t+}}{y_{iok}^{t}} + \sum_{(k,h)_l=1}^{linkout_k} \frac{s_{o(k,h)_l out}^{t}}{z_{o(k,h)_l out}^{t}} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l good}^{(t,t+1)}}{z_{ok_l good}^{(t,t+1)}} \right) \right]} \qquad (17)$$

subject to

$$\frac{\sum_{k=1}^{K} w^k \left[1 - \frac{1}{m_k + linkin_k + nbad_k} \times \left(\sum_{i=1}^{m_k} \frac{s_{iok}^{T-}}{x_{iok}^{T}} + \sum_{(k,h)_l=1}^{linkin_k} \frac{s_{o(k,h)_l in}^{T}}{z_{o(k,h)_l in}^{T}} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l bad}^{(T,T+1)}}{z_{ok_l bad}^{(T,T+1)}} \right) \right]}{\sum_{k=1}^{K} w^k \left[1 + \frac{1}{r_k + linkout_k + ngood_k} \times \left(\sum_{i=1}^{r_k} \frac{s_{iok}^{T+}}{y_{iok}^{T}} + \sum_{(k,h)_l=1}^{linkout_k} \frac{s_{o(k,h)_l out}^{T}}{z_{o(k,h)_l out}^{T}} + \sum_{k_l=1}^{nbad_k} \frac{s_{ok_l good}^{(T,T+1)}}{z_{ok_l good}^{(T,T+1)}} \right) \right]} = \tau_o^{T*} \qquad (18)$$

and (2), (3), (5) to (9) and (16).

4. A dynamic Malmquist index

The concept of Malmquist productivity index was first introduced by S. Malmquist [14] and has further been developed by several authors in the non-parametric framework. For example see Färe and Grosskopf [9]. It is an index representing Total Factor Productivity (TFP) growth of a DMU, in that it reflects (a) progress or regress in efficiency along with (b) progress or regress of the frontier technology.

The traditional dynamic DEA and the proposed dynamic and network DEA models in the current study generate relative period efficiency scores based on efficiency frontiers of each period, while they do not capture the absolute position of each frontier. In this case, the absolute progress or regress of efficiency performance of each DMU cannot be measured. The Malmquist index will be an effective measure to incorporate frontier-shift effect into evaluation, and thus result in capturing the absolute productivity change of each DMU in the dynamic DEA model.

In this section, we define dynamic overall and divisional Malmquist indices as follows.

4.1. Divisional dynamic catch-up index

As the ratio of the period-divisional efficiencies between t and t+1, we define the divisional dynamic catch-up index (d-DCU) as

$$\text{d-DCU} = \gamma_{ok}^{t \to t+1} = \frac{\rho_{ok}^{t+1^*}}{\rho_{ok}^{t^*}} \quad (o = 1,\ldots,n; k = 1,\ldots,K; t = 1,\ldots,T-1). \tag{19}$$

d-DCU >1, = 1, and <1 indicate respectively progress, status quo and regress in catch-up effect, respectively.

4.2. Divisional dynamic frontier-shift effect

We define divisional dynamic frontier-shift effect (d-DFS) from t to t+1 as

$$\text{d-DFS} = \sigma_{ok}^{t \to t+1} = \left[\frac{\rho_{ok}^{t*}}{\pi_{ok}^{t(t+1)}} \times \frac{\pi_{ok}^{t+1(t)}}{\rho_{ok}^{t+1*}} \right]^{1/2} \quad (o = 1,\ldots,n; k = 1,\ldots K; t = 1,\ldots,T-1), \tag{20}$$

where $\pi_{ok}^{t(t+1)}$(or $\pi_{ok}^{t+1(t)}$) is the SBM (Tone [20]) or Super-SBM (Tone [21]) score of DMU$_{ok}$ at period t (or t+1) evaluated by the division k frontier at t+1 (or t). If the division k has no inputs (m_k=0, $linkin_k$=0, $nbad_k$=0) or no outputs (r_k=0, $linkout_k$=0, $ngood_k$=0), we define DFS=1.

4.3. Divisional dynamic Malmquist index

Using the above catch-up index (d-DCU) and frontier-shift effect (d-DFS), we define the dynamic divisional Malmquist index (d-DMI) as $\mu_{ok}^{t \to t+1}$ at $t \to t+1$ in division k.

$$\text{d-DMI} = \text{d-DCU} \times \text{d-DFS} = \mu_{ok}^{t \to t+1} = \gamma_{ok}^{t \to t+1} \sigma_{ok}^{t \to t+1} \quad (o = 1,\ldots,n; k = 1,\ldots,K; t = 1,\ldots,T-1). \tag{21}$$

4.4. Overall dynamic Malmquist index

Overall dynamic Malmquist index (O-DMI) can be obtained as the weighted geometric mean of the dynamic divisional Malmquist indices (d-DMIs) as

$$\text{O-DMI} = \mu_{o}^{t \to t+1} = \Pi_{k=1}^{K} (\mu_{ok}^{t \to t+1})^{w_k} \quad (o = 1,\ldots,n, t = 1,\ldots,T-1), \tag{22}$$

where $w_k \geq 0$ is the weight to division k with $\sum_{k=1}^{K} w_k = 1$.

4.5. Cumulative dynamic Malmquist index

Although the above Malmquist index is defined between two-period (t→t+1) base, we can find the divisional cumulative dynamic Malmquist indices (d-CDMI) based on the Period 1 to t, which can be divided into divisional cumulative dynamic catch-up index (d-CDCU) and divisional cumulative dynamic frontier-shift effect (d-CDFS) as follows:

$$\begin{aligned} \text{d-CDMI} &= \mu_{ok}^{1 \to t} = \Pi_{t'=1}^{t} \mu_{ok}^{t' \to t'+1} \\ &= \text{d-CDCU} \times \text{d-CDFS} \\ &= \Pi_{t'=1}^{t} (\gamma_{ok}^{t' \to t'+1} \cdot \sigma_{ok}^{t' \to t'+1}) \\ &(o = 1,\ldots,n; k = 1,\ldots,K; t = 1,\ldots,T-1). \end{aligned} \tag{23}$$

Overall cumulative dynamic Malmquist index (O-CDMI) is defined as follows:

$$\text{O-CDMI} = \mu_{o}^{1 \to t} = \Pi_{k=1}^{K} (\mu_{ok}^{1 \to t})^{w_k} \quad (o = 1,\ldots,n; t = 1,\ldots,T-1). \tag{24}$$

This index enables us to capture continuous productivity change of each DMU from the first period.

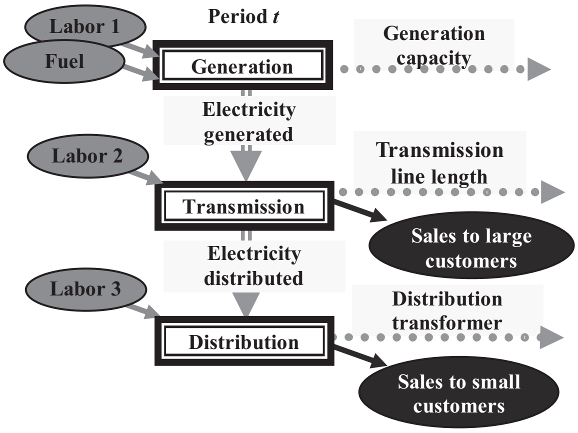

Figure 2: Network structure of Vertically integrated electric power company

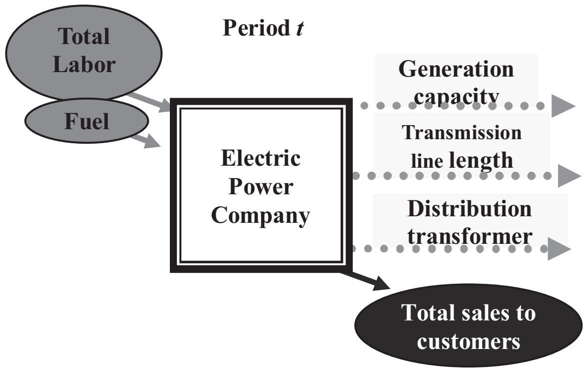

Figure 3: Aggregated structure

Tabel 1: Dataset of vertically integrated electric power companies

Division		Items		Units
1	Generation	Input	Labor 1	number of employees
			Consumed Fuel	10^9 British Thermal Unit (BTU)
		Link	Electricity Generated	Giga Watt hour (GWh)
		Carry-Over	Generation Capacity	Mega Watt (MW)
2	Transmission	Input	Labor 2	number of employees
		Output	Sales to large customers	GWh
		Link	Electricity Disributed	GWh
		Carry-Over	Transmission line length	km
3	Distribution	Input	Labor 3	number of employees
		Output	Sales to small customers	GWh
		Carry-Over	Distribution transformer	number of transformers

5. An application study

In this section we apply the DNSBM model to a dataset comprised of 21 U.S. electric utilities over five years and compare the results with those given by the dynamic SBM (DSBM) model.

5.1. Dataset of U.S. electric utilities

Figure 2 exhibits typical vertically integrated electric companies consisting of three divisions: Generation (Div 1), Transmission (Div 2) and Distribution (Div 3). We chose 21 DMUs over 5 years (1991–1995). Each division has inputs, outputs, links and carry-overs items as shown in Table 1.

In order to clarify the advantage of the DNSBM model over the previous model, we compare the results with those of dynamic model (DSBM), for which we aggregate the three divisions into a single “black box” as exhibited in Figure 3. In this model, labor input is the sum of those in divisions 1, 2 and 3. Output is total sales to customer which is measured as the sum of sales to large and small customers. Fuel and carry-overs are the same with the DNSBM. Thus, we neglect the internal structure of the company.

5.2. Overall efficiency of DNSBM

We applied the DNSBM model to this dataset under the following assumptions.

Weights to period are: 0.122 (1991), 0.122 (1992), 0.2195 (1993), 0.2439 (1994), 0.2927 (1995). Weights to

<8>

division are: 0.666 (Generation), 0.166 (Transmission), 0.166 (Distribution). All links and carry-overs are assumed free, i.e., (LF) and (CF). We chose the input-oriented constant returns-to-scale model.

Figure 4 compares the overall efficiencies between DNSBM and DSBM.

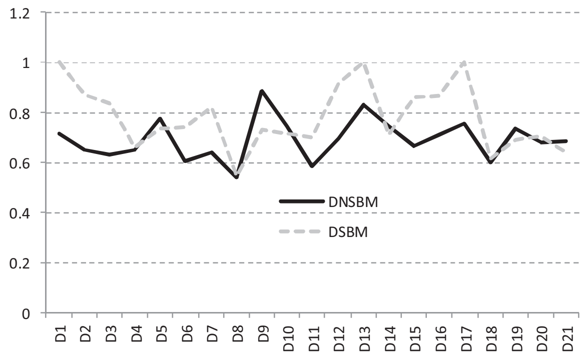

Figure 4: Comparison of DNSBM and DSBM

We cannot compare both scores directly, because problem schemes are different in DNSBM and DSBM models. However, there are some DMUs which are judged as efficient in DSBM but inefficient in DNSBM. This gap comes from the characteristics of the applied models: network structure in DNSBM and the aggregated one in DSBM. Let us observe the factor productivity index, which is measured as output divided by input. As for the labor, in DNSBM, we deal with labor separately in generation, transmission and distribution, whereas, in DSBM, they are merged into a single labor input. This neglect of the inner structure results in difference in the overall scores.

Actually, D1's overall score is 1 (efficient) by DSBM, but 0.7176 (inefficient) by DNSBM. In the latter model, labor productivity indices are evaluated in three divisions separately. The divisional labor productivity of D1 is superior to other DMUs in Divs 1 and 3, while labor productivity in Div 2 is worse than average. Hence, in a comprehensive manner, its overall efficiency score comes down eventually affected by the low labor productivity of Div 2. However, in the former model, D1 ranks at the top in the labor productivity index as defined by Total Sales/Total labor, because the grand total of sales and labor offset worse divisional labor productivity in Div2 by those in Divs 1 and 3. This contributes to giving it an overall score of 1 in the DSBM model. This is a suitable example of efficiency bias caused by neglecting the network structure.

5.3. Dynamic Malmquist index

Figures 5 to 7 depict the averages of divisional cumulative dynamic Malmquist index (d-CDMI), Catch-up index (d-CDCU) and Frontier-shift effect (d-CDFS). With regard to Generation division, the average d-CDCU went slowly down (Figure 6) and the average d-CDFS remained status quo (Figure 7). As the result, the average d-CDMI went slowly down toward the last year (Figure 5). In both Transmission and Distribution divisions, relatively large productivity improvements were observed (Figure 5). In Transmission division, both d-CDCU and d-CDFS were improved, which resulted in the progress of d-CDMI. On the other hand, in Distribution division, large d-CDFS was the main cause of development in the d-CDMI.

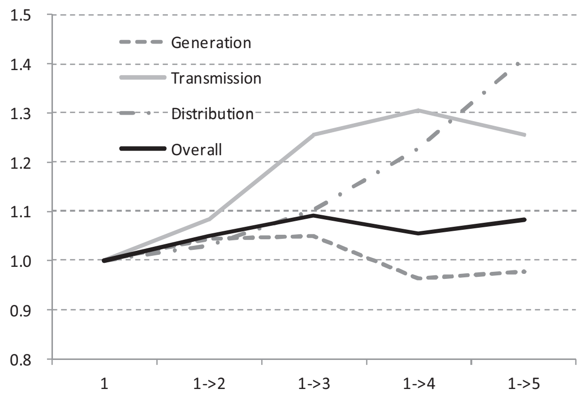

Figure 5: Average divisional cumulative dynamic Malmquist index (d-CDMI)

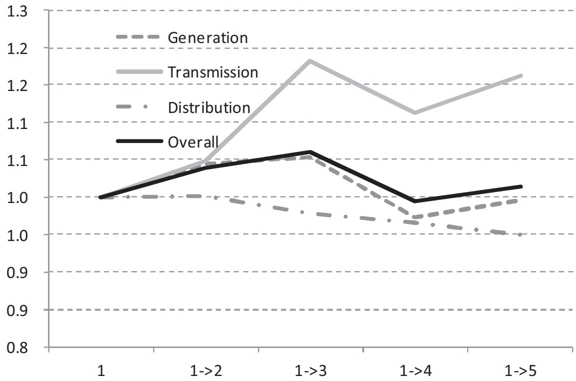

Figure 6: Average divisional cumulative dynamic catch-up index (d-CDCU)

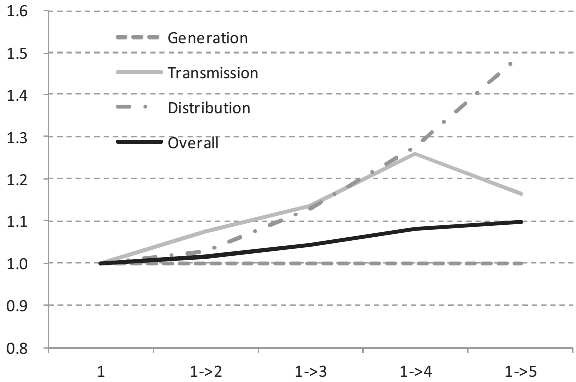

Figure 7: Average divisional cumulative dynamic frontier-shift effect (d-CDFS)

6. Conclusion

In this paper we have developed a dynamic DEA model with network structure (DNSBM) as a composition of the dynamic SBM (DSBM) and the network SBM (NSBM). Furthermore, we have proposed the divisional and overall dynamic Malmquist indices by which we can identify divisional differences in productivity growth along with overall productivity change. As a numerical example, we applied DNSBM to a dataset of electric power companies. We compared DNSBM with DSBM and demonstrated that the DNSBM model can reveal the efficiency status more accurately than the DSBM, because the DNSBM model includes the internal network structure of DMUs.

Future research subjects include the uniqueness issue of the divisional efficiency scores and extensions to the radial DEA model.

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7. References

[1] Bogetoft P, Färe R, Grosskopf S, Hayes K, Taylor L. Proceedings of DEA Symposium 2008 (Feb. 2008).

[2] Chang H, Choy HL, Cooper WW, Ruefli TW. Omega 37(5) 2009, 951-60.

[3] Charnes A, Clark T, Cooper WW, Golany B. Annals of Operations Research 2(1) (1985).

[4] Chen CM. European Journal of Operational Research 194(3) 2009, 687-99.

[5] Färe R, Grosskopf S. Socio-Economic Planning Sciences, 34 (2000) 35-49.

[6] Färe R, Grosskopf S. Intertemporal Production Frontiers: with Dynamic DEA, Kluwer, 1996.

[7] Färe R, Grosskopf S, Norris S, Zhang Z. The American Economic Review 84(1) (1994).

[8] Kao C. Network data envelopment analysis: current development and future research, Asia-Pacific Productivity Conference (APPC) 2008 (Jul. 2008).

[9] Malmquist S. Trabajos de Estadistia 4(1) (1953) 209-242.

[10] Nemoto J, Goto M. Journal of Productivity Analysis, 19(2-3) (2003).

[11] Nemoto J, Goto M. Economic Letters, 64(1) (1999).

[12] Park KS, Park K. European Journal of Operational Research 193(2) 2009,567-80.

[13] Pastor JT, Ruiz JL, Sirvent I. European Journal of Operational Research 115(3) (1999) 596-607.

[14] Sueyoshi T, Sekitani K. European Journal of Operational Research 161(2) (2005).

[15] Tone K. European Journal of Operational Research 130(3) (2001) 498-509.

[16] Tone K. European Journal of Operational Research 143 (2002) 32-41.

[17] Tone K, Tsutsui M. European Journal of Operational Research 197(1) (2009) 243-252.

[18] Tone K, Tsutsui M. Omega 38 (2010) 145-156.