Nonlinear Analysis and Estimation of Dynamic Stochastic General Equilibrium Models

A Dissertation

Submitted to the National Graduate Institute for Policy Studies (GRIPS)

in Partial Fulfillment of the Requirements for the Degree of

Ph.D. in Public Economics

by

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August 2023

Abstract

This dissertation investigates the estimation of Dynamic General Equilibrium models (DSGE) solved using nonlinear approximation. The DSGE model has been widely used as a theoretical framework for the study of the business cycle and monetary and fiscal policy. To solve DSGE models, it is common practice in macroeconomics to use linear methods to approximate solutions of nonlinear DSGE models.

Therefore, the linearization of DSGE models has become a standard tool for the approximation of solutions to the dynamic optimization problem (DOP) in the DSGE model. Linearization is typically obtained by using only the first term of a Taylor expansion around the steady state of the log of the equations representing the first-order conditions of the dynamic optimization problem. This method makes it possible to use formal statistical methods to estimate and test DSGE models. However, the linearization of this class of models also has a cost: some questions, such as welfare evaluation and risk premia in stochastic environments, cannot be fully addressed in a linearized model.

In the first study, we propose a new likelihood-based approach using perturbation methods to estimate nonlinear DSGE models. We implicitly use a nonlinear approximation to the policy function that is invertible with respect to the shocks, implying that in the approximation, the shocks can be recovered uniquely from some of the control variables. Based on this approximation, the likelihood can then be obtained by using a standard change of variables theorem and a Lagrange inversion formula. We implement this technique to estimate the DSGE model. In contrast with previous likelihood-based approaches, the method used here allows for unobserved non-stochastic state variables and requires neither additional shocks nor simulation to evaluate the likelihood. Using US data, we demonstrate the proposed approach to the well-known neoclassical growth model of Fernandez-Villaverde (2010). In addition to the baseline model, we also consider versions of the model in which the structural shocks have time-varying variances. We find that a nonlinear heteroscedastic model has much better empirical performance. It is a better fit for the observed data than the linearized model. In addition, we find that

the monetary policy shock primarily drives the time changes in the uncertainty in the economy.

In the second study, we develop a more general New Keynesian model with limited heterogeneity featuring two agent properties, referred to in the macroeconomic literature as the Two-Agent New Keynesian (TANK) DSGE model, and estimate the model using the method developed in the first study. Our model incorporates technology, monetary and fiscal shocks. The model features price and wage rigidity dynamic and capital adjustment cost. This study builds on recent empirical evidence of nonlinear relations between financial variables and aggregate fluctuations. Our work argues for the importance of explicitly considering nonlinearities when analyzing the behavior of the TANK DSGE model. Thus, we employ the likelihood-based approach using the perturbation method to estimate the model in linearized and nonlinearized forms and draw inferences from it. We used quarterly aggregate Korean data for 1999 Q4—2021 Q4. We analyze data for this time range since, as the literature shows, the Bank of Korea started to become more proactive in controlling inflation; and adopted an inflation-targeting rule in 1998, which allowed them to focus on keeping inflation under control and being less accommodating of the high demand for liquidity that preceded the Asian financial crisis.

As a result, we find that the estimation of the linear representation of the TANK model generates a better fit of the model to the data (as measured by log-likelihood at the posterior mode). However, the nonlinear model is preferable in terms of log marginal and predictive likelihoods to the linearized one. Similar to the literature, we find that the government expenditure shock has an expansionary effect on consumption and output. However, this effect differs for different types of households. We also find that distortionary taxation has a crowding-out effect, and contractionary monetary policy shock can effectively curb inflationary pressures in the economy.

In summary, the dissertation investigates the estimation of DSGE models using the nonlinear approximation of order two and proposes a new likelihood-based approach to estimate them. The studies reported here demonstrate the methodology's effectiveness and provide insights into the model under a different framework.

Dedication

To my beloved family.

Acknowledgment

First and foremost, I would like to express my most profound appreciation and gratitude to my principal advisor, Professor Roberto Leon-Gonzalez, for his unwavering support, guidance, and mentorship throughout my doctoral studies. His expertise and vast knowledge in the field have been an invaluable asset to me, and I am genuinely grateful for the time and effort he has dedicated to my research. His guidance, feedback, and encouragement have been instrumental in shaping my research, and I am grateful for the opportunity to have worked under his supervision.

I would also like to sincerely thank my sub-advisors, Professor Junichi Fujimoto and Professor Hidehiko Matsumoto, for their valuable insights and feedback on my work. Their guidance and support have been instrumental in shaping my research, and I am genuinely grateful for the knowledge and experience they have imparted to me. I am grateful for their time and effort in reviewing my work and for their valuable feedback, which has helped me to improve my research.

I also express my sincere gratitude to the Japan Ministry of Education, Culture, Sports, Science, and Technology (MEXT) for providing me with the scholarship that made my studies at National Graduate Institute for Policy Studies (GRIPS) possible. This scholarship has been a significant financial support for me and has allowed me to focus entirely on my studies and research. I am truly grateful for this opportunity and hope to make the most of it.

Furthermore, I would like to extend my sincere gratitude to the GRIPS and its dedicated administrative staff, especially the Policy Analysis Program coordinator, Aiko Hashimoto. I am grateful for the support I received from AST, IPT, GAT, and Student Office, who have provided me with invaluable support throughout my academic and daily life at GRIPS. Their assistance has been essential in ensuring a smooth and productive academic journey.

Finally, I would like to express my heartfelt thanks to my husband, Nikolai, and family for their constant love, support, and encouragement throughout my studies. Their unwavering belief in me and my abilities has been a continuous source of motivation and inspiration, and I am eternally grateful for their presence in my life. Thank you so much!

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Chapter 1 Introduction

Numerical techniques are often used to solve "true" nonlinear Dynamic Stochastic General Equilibrium Models (DSGE), which include the perturbation method, projection method, and dynamic programming tools such as value function and policy function iterations. Among them, the perturbation method is appealing because it is significantly faster than, for example, value function iteration, and has good convergence properties. These features are essential for estimation because the policy function needs to be solved repeatedly with many parameter values (Aruoba et al. (2006)). Within this dissertation we employ this method to solve the DSGE models.

The main idea of perturbation methods is to build a Taylor series expansion of policy functions around the deterministic steady state of an economy with a perturbation parameter. The first terms of the Taylor series result in linear policy approximation, known as the linearization method. Since the linearization method makes it possible to obtain of a linear approximation of policy functions, which can be solved by matrix algebra techniques, it has become a standard tool for solving and estimating the model (Blanchard and Kahn (1985), Sims (2000)). Moreover, many DSGE models exhibit behavior that is close to linear, especially in the neighborhood of the steady state, which makes the linearization method a reasonable approximation. As supported by a number of studies, this method is widely used in evaluating likelihood-based DSGE models because it allows for the transformation of these models into state-space representations compatible with Kalman filtering (An (2005)). Furthermore, with the availability of software platforms such as Dynare, the estimation of DSGE models using Bayesian methods becomes more straightforward, which has led to a surge in empirical studies in the last twenty years.

However, the linearization of these models can omit interesting aspects provided by nonlinearity, especially in the conduct of welfare analysis, examination of asset pricing, and analysis of the changes in uncertainty over time in an economy within DSGE models. In this regard, the second-order approximation of solutions of DSGE models has been explored since the seminal studies of Judd (1998), Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), Collard and Juillard (2001a), proved that a second-order approximation can be obtained from the second-order perturbation to equilibrium conditions and can provide algorithms for construction of an accurate second-order solution to the DSGE model.

To estimate the model resulting from the second and higher-order perturbation technique is challenging and requires a large computational effort. We discuss different methods of computing the likelihood function of DSGE models developed in Bayesian econometrics, including Sequential Monte Carlo methods (Fernández-Villaverde and Rubío-Ramirez (2007)), and Exact Likelihood computation techniques (Amisano and Tristani (2011)). While the former method involves the simulation of a large number of particles, which requires significant processing power, the latter applies to small models with quite restrictive assumptions. To fill the apparent gap, we propose a relatively simple and fast approach to estimation of nonlinear DSGE models, which is also a likelihood-based approach. The gist of our proposed technique is to compute the higher order approximation of the likelihood function of a nonlinear DSGE model using a Laplace approximation method and imposing invertibility of the policy function in the approximation of the likelihood.

Thus, the main contribution of the dissertation is the development of a likelihood-based approach to the computation of a likelihood function of nonlinear DSGE models using second-order perturbation methods. This enables us to form posterior density of parameters of a model, which is proportional to the product of the likelihood function and the prior density of the parameters. This addresses the central objective of this dissertation: to apply the proposed methodology to the estimation of DSGE models and make inferences about that estimation.

Accordingly, in the first study, presented in Chapter 2, we propose a new likelihood-based approach using perturbation methods to estimate nonlinear DSGE models. We note that in the literature it has been pointed out that in some DSGE models the shocks can be recovered uniquely from some of the control variables, which implies that the policy function is invertible with respect to the shocks (e.g. Stokey et al. (1989), Hopenhayn and Prescott (1992) and Gordon and Qiu (2018)). Accordingly we impose invertibility in the approximation by using an implicit nonlinear invertible approximation of the policy function. This allows us to construct the likelihood function using a standard change of variables theorem and a Lagrange inversion formula. We further develop this technique to implement a nonlinear estimation of the DSGE model.

In contrast with previous likelihood-based approaches, the method used here allows for unobserved non-stochastic state variables and requires neither additional shocks nor simulation to evaluate the likelihood. Using US data, we demonstrate the proposed approach in the case of the well-known neoclassical growth model of Fernández-Villaverde (2010). In addition to the baseline model, we modify the model by incorporating an uncertainty, defined as an increase in the standard deviation of the shocks that affect the economy (which implies that the shocks are heteroskedastic). We specify uncertainty by Generalized Autoregressive Conditional Heteroskedasticity (GARCH) processes. We estimated four structural models: the log-linearized model with homoscedastic shocks; the model resulting from second-order approximation with homoscedastic shocks; the model resulting from second-order approximation with unrestricted GARCH in the shocks; and the model resulting from the second-order approximation with a common factor in the GARCH processes.

As a result of the Bayesian estimation of four models, we find that the nonlinear heteroscedastic empirically performed better model than linearized models in terms of their log marginal likelihood, log-likelihood and predictive likelihood. The posterior analyses indicate that the monetary policy shock in a nonlinear heteroscedastic model is the main driving force of uncertainty in an economy.

In the second study, presented in Chapter 3, we develop a more general New Keynesian model with limited heterogeneity featuring two agent properties (referred to in the macroeconomic literature as the Two-Agent New Keynesian (TANK) DSGE model) and estimate the model using a likelihood-based approach via the perturbation method. Following existing studies on two-agent models, we introduce Ricardian (financially unconstrained) and non-Ricardian (financially constrained) households. The model features price and wage rigidity dynamic and capital adjustment cost, and innovation to technology, monetary and government spending rules, as well as labor income and consumption tax shocks. Also, we model real and nominal frictions nonlinearly, where the former includes investment costs, the latter price and wage frictions. This study builds on recent empirical evidence of nonlinear relations between financial variables and aggregate fluctuations, hence, our interest in employing the likelihood-based approach using the perturbation method to estimate the model in linearized and nonlinearized fashion and draw inferences from it. We used quarterly aggregate Korean data for 1999 Q4–2021 Q4. We chose this time range for analysis since, as the literature shows, the

Bank of Korea started to become more proactive in controlling inflation and adopted an inflation targeting rule in 1998, which allowed them to focus on (a) keeping inflation under control and (b) being less accommodating of the high demand for liquidity that preceded the Asian financial crisis.

Through likelihood-based estimation, we find that the linear representation of the TANK model generates a better fit of the model to the data than the nonlinear one, as measured by log-likelihood at the posterior mode. However, the nonlinear representation of the TANK model has better log marginal and predictive likelihood than the linearized model. These results still support the idea of including nonlinearities when analyzing the behavior of the TANK model. The standard deviations of the parameters are estimated to be much higher for the second-order approximation model than for the first-order approximation, indicating that the nonlinear likelihood is more dispersed than the linear.

The analyses of Bayesian impulse responses for both models indicate that, in general, they align with findings of existing studies, most of which have mostly involved linear estimation. However, the magnitude varies by model: we find that positive government expenditure shock has an expansionary effect on aggregate consumption and output in both linear and nonlinear models. Nevertheless, in both models this effect differs for constrained and unconstrained households. Thus the consumption level of constrained households increased due to increased wage levels in the economy, and that of unconstrained households decreased due to increased real interest rates. Distortionary taxation has a crowding-out effect on the aggregate economy and on the consumption of both agents. Regarding the innovation to monetary policy rules, a positive shock decreases output, aggregate consumption, investment, and hours worked.

The dissertation is structured as follows. The introduction is presented in Chapter 1. Chapter 2 proposes a new likelihood-based approach for estimating DSGE models using perturbation methods and Bayesian inference theory. Chapter 3 explores the two-agent model and applies the above mentioned methodology to estimation. Chapter 4 presents conclusions and implications for policy-makers; and proposes promising directions for future research.

Chapter 2 Likelihood-Based Estimation of Nonlinear Dynamic Stochastic General Equilibrium Models Through Perturbation Methods

2.1 Introduction

Linearization of DSGE models is a common tool for approximation of the solution to the dynamic optimization problem in DSGE (e.g., Blanchard and Kahn (1985), Sims (2002), Klein (2000)). This method has been used in numerous studies in the last twenty years. Linearization is typically achieved by using only the first term of a Taylor expansion around the steady state of the log of the equations representing the first-order conditions of the dynamic optimization problem.

Linearization has made it possible to use formal statistical methods to estimate and test DSGE models such as the General Method of Moments (e.g., Hansen and Singleton (1983), Christiano and Eichenbaum (1992)), Maximum Likelihood (e.g., Hansen and Sargent (1980), Altug (1989)) and Bayesian methods (e.g., DeJong et al. (2000), Schorfheide (2000), Otrok (2001)).

However, the linearization of this class of models also comes at a cost, because not all questions can be fully addressed in a linearized model. As argued by Schmitt-Grohé and Uribe (2004), two such questions are welfare evaluations and risk premia in stochastic environments. In a linearized model, the agents become risk-neutral, so it is impossible to analyze the impact of uncertainty on the economy. Furthermore, evaluating social welfare across alternative stochastic or policy environments using a linear approximation model leads to the omission of some critical second-order terms, resulting in spurious results.

To address the limitations of the linearization method, Schmitt-Grohé and Uribe (2004) proposed the use of perturbation methods (e.g., Judd (1998)) to obtain higher-order Taylor approximations of the policy functions. They also showed that when perturbation methods are used to obtain a first-order approximation, the result is the same as that obtained in previous studies on linearization (e.g., Blanchard and Kahn (1985)).

In terms of estimation, moving away from a linearized model towards a nonlinear one is challenging because the structural errors enter the model in a nonlinear fashion, and the likelihood function of a model is no longer normal or readily available. Despite this difficulty, Amisano and Tristani (2011) show how to obtain the likelihood in some restricted models when using a second-order approximation to solve the model. Their method is applicable only when there are no unobserved non-stochastic state variables, which implies that, for example, capital has to be observed in a model with capital. In addition, this method requires finding solutions of polynomial equations, which is computationally intensive and, therefore, in practice, implies that the number of structural shocks has to be relatively small.

Fernández-Villaverde and Rubío-Ramirez (2007) propose a particle filter approach that permits the numerical approximation of the likelihood. However, this approach requires that the number of shocks be greater than the number of observed variables and that some shocks (structural or measurement errors) have to enter linearly in the likelihood. Therefore, in practice, for econometric convenience, one has to add more shocks to the model, which might make identifying the shocks of interest more difficult. Furthermore, because evaluating the likelihood requires simulation with a potentially large number of particles, the method could be slow, especially in large models.

This chapter proposes a new likelihood-based approach to estimating nonlinear DSGE models. We exploit the condition where the shocks in a DSGE model can sometimes be recovered uniquely from some of the control variables, implying that the policy function is invertible with respect to shocks. Accordingly, we impose invertibility in the approximation by using an implicit nonlinear invertible approximation of the policy function. This allows us to construct the likelihood function by using a standard change of variables theorem and a Lagrange inversion formula. This likelihood can then be used for Bayesian analysis or Maximum Likelihood. In contrast with Amisano and Tristani (2011), this method allows for unobserved non-stochastic state variables, and unlike Fernández-Villaverde (2010) it requires neither the introduction of additional shocks nor simulation to evaluate the likelihood.

Using US data, we demonstrate the approach by applying it to the well-known neoclassical growth model of Fernández-Villaverde (2010). In addition to the baseline model, we modify the model by incorporating uncertainty, defined as an increase in the standard deviation of the shocks that affect the economy. This implies that the shocks are heteroskedastic. We specify uncertainty by GARCH processes with both an unrestricted variance component and a common factor that affects the volatility of aggregate shocks. We estimated four structural models: the log-linearized model with homoscedastic shocks; the model resulting from second-order approximation with homoscedastic shocks; the model resulting from second-order approximation with unrestricted GARCH in the shocks; and the model resulting from the second-order approximation with a common factor in the GARCH processes. We find that among the four estimated models a nonlinear heteroscedastic model performs best empirically as measured by the log marginal likelihood, log likelihood and predictive likelihood. The posterior analyses indicate that the monetary policy shock in a nonlinear heteroscedastic model is the main driving force of the uncertainty in the economy.

The remainder of this chapter is structured as follows: section 2.2 reviews related literature on the current topic. Section 2.3 presents the perturbation method for obtaining the policy function and estimating the model. Section 2.4 shows how to a) approximate the inverse of the policy function using the perturbation methods and b) estimate the model. Section 2.5 illustrates that the Laplace based solution to the model that we propose gives similar results to the previous approach in the literature. Then, section 2.6 illustrates an application of the proposed method to the neoclassical growth DSGE model and discusses the results. Finally, section 2.7 concludes.

2.2 Literature review

As Fernández-Villaverde (2010) emphasizes, DSGE models do not have simple pencil and paper solutions. In order to obtain this solution, it is necessary to resort to numerical approximation to characterize the equilibrium dynamics of a model.

One of the first approaches for solving DSGE models comes from control theory. For example, the seminal study of Kydland and Prescott (1982) substituted a quadratic approximation for the original problem. In particular, they integrated the growth model and business cycle theory, where they approximated the policy function using a quadratic approximation in the neighborhood of steady state. They computed the equilibrium decision rules for obtaining the approximate economy and showed that because the rules they obtained are linear, the resulting equilibrium is generated by a system of stochastic difference equation.

In response to findings of Kydland and Prescott (1982), Christiano (1990) compared two quadratic approximations to value function iterations as a method of solving DSGE models. The method involves the iteration of the value function, which represents the optimal decision of agents in the economy given an expectation about future variables. Christiano (1990) shows that the quadratic approximation can be easier to implement and faster than value function iteration. However, he points out that value function iteration gives more accurate solutions of dynamic models and can solve more complex methods than the method provided by Kydland and Prescott (1982).

Another method for solving functional equations of DSGE models is the projection method. Essentially, projection method finds a function that 1) approximates a given function, 2) and satisfies additional constraints by expressing it as a linear combination of a set of basis functions. Judd (1992) introduced this technique in the economics literature.

An alternative approach, linearization procedure, has become prevalent as a result of its tractability and effectiveness in approximating nonlinear systems. The concept of linearization involves replacing a complex model (which has a state-space representation) with a simpler one that we can solve, and then using the solution to estimate the solution of the original problem. Researchers favor this approach since most linear models can approximate arbitrary "true" nonlinear models. As the literature has shown (Jin and Judd (2002), Blanchard and Kahn (1985), Sims (2002)), linearization has been successfully applied in many fields, notably within the discipline of working with DSGE models. Essentially, this technique finds the Taylor expansion of the policy function that describes the model's variables around the deterministic steady state. Thus, linearization is the first term of Taylor expansion. Judd and Guu (1993) confirm that this is indeed a first-order perturbation method. The authors explain that perturbation methods involve using a Taylor series expansion around the a known solution to approximate the solution to a complex model using a simpler one. They demonstrate this method by applying it to a simple growth model, proving that the perturbation method gives an accurate approximate solution. They also compared the accuracy and computational efficiency of the perturbation method with that of value function iteration. As a result, the authors conclude that perturbation method sufficiently approximates the solution of complex economic dynamic models. Since the publication of Judd and Guu (1993), the approximation method has become a widespread macroeconomics practice for approximating solutions nonlinear models.

However, first-order approximation techniques cannot fully address many problems related to macroeconomics, including zero-lower bound on nominal interest rate; welfare evaluation; structural shocks with time-varying variances; and stochastic volatility. For example, as demonstrated by Kim and Kim (2003), welfare evaluation using first-order approximation can produce a spurious outcome. They argue that evaluating individual welfare (utility) using a linear approximation of the policy function may result in a misleading conclusion. Specifically, the evaluation indicates that welfare is higher under autarky (incomplete-markets economy) than under full risk sharing (complete-markets economy), which contradicts the first welfare theorem. The point is that when using a linear approximation of the policy function, specific higher-order terms of the equilibrium welfare function are excluded, while others are included. The exclusion and inclusion of terms can result in a distorted comparison of actual welfare under autarky and full risk sharing. An accurate analysis would require all relevant terms of the welfare function be considered.

Another example of an issue that cannot be solved with linearization is time-varying variances (changes in uncertainty over time). There is a significant strand of the literature devoted to investigating it, defined as an increase in the standard deviation of the shocks that affect the economy (Justiniano and Primiceri (2008), Bloom (2009), Fernández-Villaverde et al. (2015), Bloom et al. (2018), Fernández-Villaverde and A.Guerrón-Quintana (2020)). These studies provide evidence that uncertainty shock significantly impacts aggregate fluctuations. Thus, in recent survey study, Fernández-Villaverde and A.Guerrón-Quintana (2020) examine uncertainty shocks using a representative DSGE model with financial frictions. The authors argue the importance of a higher-order approximation (perturbation methods) for analysis of models with uncertainty shocks, given that those models are high-dimensional, meaning that it is necessary to keep track of additional state variables, such as the volatility of shocks. In that regard, they argue that in contrast to other methods, particularly projection and dynamic programming, perturbation techniques can handle DSGE models with large state spaces and produce accurate results even at a significant distance from the perturbation point. Aruoba et al. (2006) and Caldara et al. (2012) have demonstrated the effectiveness of perturbation techniques in the solution of DSGE models, particularly their ability to accurately approximate the policy function.

In this regard, obtaining higher-order expansions is an essential task: Collard and Juillard (2001a) and Collard and Juillard (2001b) propose a fixed-point algorithm for solving dynamic models. Their method involves transforming the DSGE model into a system of nonlinear equations, starting with an initial guess at the values of endogenous variables and iteratively updating the guesses until they converge to the true fixed point of the model. However, Schmitt-Grohé and Uribe (2004) argue that the solution proposed by Collard and Juillard (2001b) differs from a second-order Taylor approximation. To address this issue, they derived the second-order approximation to the solution of the DSGE model, and employed the perturbation method incorporating scale parameters for the standard deviation of exogenous shocks as an argument of the policy function. In approximating the policy function, they employed a second-order Taylor expansion with respect to the state variables and a scale parameter, and proved that in the second-order approximation, the presence of uncertainty affects the constant terms of decision rules, which is evidently missing in the first-order approximation.

When it comes to estimation, moving away from a linearized model towards a nonlinear one is challenging because structural errors enter nonlinearly in the model, and the likelihood is no longer normal or readily available.

Despite this difficulty, in macroeconomics, Fernández-Villaverde and Rubío-Ramirez (2007) suggest a Sequential Monte Carlo (SMC) method, also known as particle filtering, that permits numerical approximation of the likelihood of macroeconomic models. However, this approach requires that the number of shocks is greater than the number of observed variables and that some shocks (structural or measurement errors) must enter linearly in the likelihood. Therefore, for econometric convenience, it is necessary to add more shocks to the model, which could make identifying the shocks of interest more difficult. In addition, as Amisano and Tristani (2011) indicate, when using particle filters, as the number of particles used in each evaluation increases to infinity, the approximation of the likelihood function converges to the actual likelihood.

In that regard, because evaluating the likelihood requires simulation with a potentially immense number of particles, the method could be slow, especially when used with extensive models. Moreover, it is not guaranteed that approximation errors are negligible, even when a large number of particles is employed.

Amisano and Tristani (2011) propose a procedure for the exact calculation of the likelihood function of a nonlinear DSGE model. In particular, they show a method for obtaining the likelihood in some restricted models, where variances in structural shocks to the state vector are subject to stochastic regime switches. First, the authors employed a second-order perturbation method to solve the model, since (unlike studies including

Coleman (1991), Andolfatto and Gomme (2003), Sims and Zha (2006)) they assume that regime switching only affects the variance of structural shocks. Then, to obtain the model's unobserved state variables, they invert observation equations. Finally, they determine the likelihood by finding the solutions of polynomial equations, similar to linearised models with i.i.d. shocks. Nevertheless, the method of Amisano and Tristani (2011) only applies when a model does not feature unobserved non-stochastic state variables. This assumption implies, for example, that in a model featuring capital, the capital has to be observed. As mentioned above, their method requires the solving of polynomial equations, which is computationally intensive and, therefore, in practice, implies that the number of structural shocks has to be relatively small.

The findings in the literature motivate a search for a better estimation approach to evaluation of the parameters of DSGE models, an approach that does not resort to numerous assumptions considered by the above-mentioned studies. Moreover, different from the particle filter approach, we seek to establish a computationally efficient way to compute the likelihood function.

2.3 Using Perturbation Methods to Obtain the Policy Function and Estimate the Model

Using the first-order conditions of the dynamic optimization problem, we can see that the general nonlinear form of a DSGE model can be cast as (e.g., Amisano and Tristani (2011)):

$$E_t[f(y_{t+1}, y_t, x_{t+1}, x_t)] = 0 \tag{2.1}$$

where E_t is the expectation operator conditional on information available at time t; y_t represents a vector of non-predetermined variables; and x_t denotes a vector of predetermined variables. The vector x_t can be partitioned as $x_t = (K_t, A_t)$, where K_t is a vector of endogenous predetermined state variables and A_t is a vector of exogenous predetermined state variables. Schmitt-Grohé and Uribe (2004) assume that A_{t+1} follows the stochastic process: $A_{t+1} = \Lambda A_t + \sigma\varepsilon_{t+1}$, where the scalar σ is a perturbation parameter, and ε_{t+1} is a vector of zero mean innovations, independently and identically distributed with Σ as the variance-covariance matrix. The matrix Λ has all eigenvalues within the unit circle. Given $x_t = (K_t, A_t)$, the solution of the model (DSGE) consists of the policy functions which give the optimal value of y_t:

$$\begin{aligned} y_t &= g_y(K_t, A_t, \sigma) \\ K_{t+1} &= h_K(K_t, A_t, \sigma) \\ A_{t+1} &= h_A(K_t, A_t, \sigma) + \sigma\varepsilon_{t+1} \end{aligned} \tag{2.2}$$

Schmitt-Grohé and Uribe (2004) obtain a Taylor approximation of the above policy functions, g_y, h_K and h_A around the deterministic steady state, $x_t = x_{ss}$ and $\sigma = 0$, using perturbation methods. The deterministic steady state is defined as vectors (y_{ss}, x_{ss}) such that $f(y_{ss}, y_{ss}, x_{ss}, x_{ss}) = 0$.

In order to obtain a higher order approximation, Schmitt-Grohé and Uribe (2004) use a perturbation method that incorporates the scale parameter σ as an argument of the policy function. They apply perturbation methods (e.g., Fleming (1971) and Judd (1998)) by taking a higher order Taylor expansion with respect to the state variables x_t as well as the scale parameter σ. Consequently, their method can be applied to finding higher-order approximations to the policy function. In addition, Schmitt-Grohé and Uribe (2004) presented a set of MATLAB programs designed to compute coefficients of the higher-order approximations. A similar approach was proposed by Sims (2000) and Collard and Juillard (2001b).

In order to write the higher-order approximation around the deterministic steady state, let us define Y_t and X_t as:

$$Y_t = \begin{pmatrix} y_t - y_{ss}, \\ K_{t+1} - K_{ss} \\ A_t - A_{ss} \end{pmatrix}, \qquad X_t = \begin{pmatrix} K_t - K_{ss} \\ A_{t-1} - A_{ss} \end{pmatrix}.$$

where y_{ss}, K_{ss}, A_{ss} are the deterministic steady state values of y_t, K_t, A_t. Using this notation the solution in (2.2) can be written as:

$$Y_t = g(X_t, \varepsilon_t, \sigma) \tag{2.3}$$

Following Dynare (2021) a second-order approximation to (2.3) can be written using Kronecker products as follows:

$$Y_t = G^{0,0} + G^{1,0}\varepsilon_t + G^{0,1}X_t + G^{2,0}(\varepsilon_t \otimes \varepsilon_t) + G^{0,2}(X_t \otimes X_t) + G^{1,1}(\varepsilon_t \otimes X_t) \tag{2.4}$$

where $G^{0,0}, \ldots, G^{1,1}$ are matrices of coefficients that depend on σ and other parameters of the model.

Suppose that some of the variables in Y_t are observed. Let Y_t^o be the variables of Y_t for which there are observed data, for $t = 1, ..., T$. Taking as given the model solution in (2.4), the method of Amisano and Tristani (2011) can be used to obtain the likelihood provided that Y_t^o and ε_t have the same dimension n_o and K_t is empty. However, this method requires finding all the 2^{n_o} solutions of polynomial equations for each $t = 1, ..., T$, so in practice it can only be used when n_o is small. For example, Amisano and Tristani (2011) provide an empirical application in which the dimension of the structural errors ε_t is two. Due to its computational difficulty, the method cannot be readily extended to the case in which the dimension of ε_t is greater than that of Y_t^o or to orders of approximation higher than 2.

Fernández-Villaverde and Rubío-Ramirez (2007) propose a particle-filtering approach that permits the numerical approximation of the likelihood in nonlinear DSGE models. This method can be used with various models, including those solved using perturbation methods and those solved with global solution methods (e.g., value function iteration). The main requirement for particle filtering is that the model must have a state-space representation.

However, this method does not apply to (2.4) unless it is assumed that either: 1) Y_t^o is observed with measurement error; or 2) some elements of ε_t enter linearly in (2.4). Either of these linearity assumptions is necessary for calculations of the importance weights in the particle filter. However, adding measurement errors to the model could weaken the identification of structural errors and parameters.

Regarding the second option, although it is possible in some models, in general it is not evident how to add structural errors that enter linearly in the solution (2.4). Furthermore, adding more structural errors for econometric convenience might make identifying the structural errors of interest more difficult. For these reasons, estimation of the theoretical model without adding additional linear errors is of interest. Despite the acknowledged discrepancy between the available macro data and the variables in the model, it is still of academic significance to compare the empirical fit obtained with and without taking into account measurement errors; that way, the necessity of introducing empirical errors could be determined empirically rather than accepted simply because of the econometric imperative.

2.4 Using Perturbation Methods to Approximate the Inverse of the Policy Function and Estimate the Model

Writing Y_t as $Y_t = (Y_t^o, Y_t^n)$, the solution in (2.3) can be written as:

$$\begin{aligned} Y_t^o &= g_o(X_t, \varepsilon_t, \sigma) \\ Y_t^n &= g_n(X_t, \varepsilon_t, \sigma) \end{aligned} \tag{2.5}$$

We assume that Y_t^o and ε_t have the same dimension n_o and that both are defined in $\mathbb{R}^{n_o}$. If the function $g_o(X_t, \varepsilon_t, \sigma)$ was globally invertible with respect to ε_t in an area of probability 1, then the following inverses would be well-defined:

$$\begin{aligned} \varepsilon_t &= m_o(X_t, Y_t^o, \sigma) \\ Y_t^n &= m_n(X_t, Y_t^o, \sigma) \end{aligned} \tag{2.6}$$

Following Galewski (2016), the following conditions are sufficient for global invertibility:

1. $g_o(X_t, \varepsilon_t, \sigma)$ is differentiable in ε_t with continuous derivatives.
2. The determinant of the Jacobian of $g_o(X_t, \varepsilon_t, \sigma)$ with respect to ε_t is never 0.
3. $\|g_o(X_t, \varepsilon_t, \sigma)\| \to \infty$ as $\|\varepsilon_t\| \to \infty$, where $\|.\|$ is the norm operator.

The second condition is guaranteed if the policy function is strictly monotonic in ε_t. There is a large literature which gives conditions for policy functions to be strictly monotonic and proves the condition holds in many important models, for example: Topkis (1978), Hopenhayn and Prescott (1992), Stokey et al. (1989), Gordon and Qiu (2018). In addition, the assumption of monotonicity has often been exploited in the literature on DSGE models to obtain the policy function more efficiently (e.g. Christiano (1990), Judd (1998), Gordon and Qiu (2018)).

We assume that the policy function $Y_t^o = g_o(X_t, \varepsilon_t, \sigma)$ is locally invertible with respect to ε_t at the steady state. This condition holds if the Jacobian of the transformation from Y_t^o to ε_t is not zero at the steady state, which is a condition that can be verified numerically. This will allow us to invert the Taylor polynomial using a Lagrange inversion formula. In addition we assume that there exists a globally invertible approximation of the policy function $Y_t^o = g_o(X_t, \varepsilon_t, \sigma)$ and we denote it with $Y_t^o = \hat{g}_o(X_t, \varepsilon_t, \sigma)$, and its inverse as $\varepsilon_t = \hat{m}_o(X_t, Y_t^o, \sigma)$. If the policy function was indeed globally invertible, then we would use $\hat{g}_o(X_t, \varepsilon_t, \sigma) = g_o(X_t, \varepsilon_t, \sigma)$. Otherwise $\hat{g}_o(X_t, \varepsilon_t, \sigma)$ could be an invertible function that has the same Taylor polynomial as $g_o(X_t, \varepsilon_t, \sigma)$ up to a high order at the steady state. Two functions will have the same Taylor polynomial up to some order r, if they have the same value for the derivatives of order up to r at the point of approximation. Although there is a literature that constructs invertible approximations of functions (for example approximations of the cumulative density function of a normal distribution (e.g. Lipoth et al. (2022)), or approximations of multivariate functions with invertible neural networks (Teshima et al. (2020), Ishikawa et al. (2022)), for our purposes we do not need to obtain the invertible approximation explicitly: it is enough to assume that it exists.

We can use $\hat{m}_o(X_t, Y_t^o, \sigma)$ to obtain an approximation of the density of Y_t^o conditional on X_t, if we apply a change of variables theorem (e.g. Billingsley (1999))

$$\pi_y(Y_t^o|X_t) = \pi_\varepsilon(\hat{m}_o(X_t, Y_t^o, \sigma)) \left| \frac{\partial \hat{m}_o(X_t, Y_t^o, \sigma)}{\partial Y_t^o} \right| \tag{2.7}$$

where $\partial \hat{m}_o(X_t, Y_t^o, \sigma)/\partial Y_t^o$ is the Jacobian of the transformation.

The density in (2.7) can be approximated using a Laplace approximation, which we denote as $\hat{\pi}_{(y,2)}(Y_t^o|X_t)$. A Laplace approximation is a second-order Taylor approximation of the log-density around the mode of the distribution. In order to obtain the Laplace approximation, we only need the derivatives of $\varepsilon_t = \hat{m}_o(X_t, Y_t^o, \sigma)$ at the mode. Therefore we do not need to obtain the function $\varepsilon_t = \hat{m}_o(X_t, Y_t^o, \sigma)$ explicitly, only its derivatives. We approximate the mode of (2.7) and obtain the derivatives at the mode using a Taylor polynomial obtained through standard perturbation methods (as provided by Dynare (2021)) plus a Lagrange inversion formula to invert the Taylor polynomial.

Because the Laplace approximation is only a second-order Taylor approximation, it might be desirable to approximate the density in (2.7) with higher accuracy, by matching derivatives of order higher than two. However, we leave this problem for future research and limit ourselves to the use of the Laplace approximation here.

Provided that $Y_t^o = g_o(X_t, \varepsilon_t, \sigma)$ is locally invertible at the steady state, its Taylor polynomial can be inverted using a Lagrange inversion formula. As a matter of notation, let the inverse of the Taylor polynomial of $(Y_t^o = g_o(X_t, \varepsilon_t, \sigma), Y_t^n = g_n(X_t, \varepsilon_t, \sigma))$ be denoted as $(Y_t^o = \tilde{m}_{o,s}(X_t, Y_t^o, \sigma), Y_t^n = \tilde{m}_{n,s}(X_t, Y_t^o, \sigma))$. Specific formulas for these inversions are provided in Proposition 3. Then the derivatives of the likelihood in (2.7) can be approximated by those of the following function:

$$\tilde{\pi}_{(y,s)}(Y_t^o|X_t) = \pi_\varepsilon(\tilde{m}_{o,s}(X_t, Y_t^o, \sigma))\left|\frac{\partial \tilde{m}_{o,s}(X_t, Y_t^o, \sigma)}{\partial Y_t^o}\right| \tag{2.8}$$

Because the Taylor polynomial $\tilde{m}_{o,s}(X_t, Y_t^o, \sigma)$ need not be invertible, the function $\tilde{\pi}_{(y,s)}(Y_t^o|X_t)$ in (2.8) is not guaranteed to be a proper density function, in the sense that the area under the curve does not need to add up to one. For this reason it cannot be used as an approximation for the likelihood. However, because the derivatives of the Taylor polynomial $\tilde{m}_{o,s}(X_t, Y_t^o, \sigma)$ approximate those of $\hat{m}_o(X_t, Y_t^o, \sigma)$ near the point of approximation, we can use the derivatives of $\tilde{\pi}_{(y,s)}(Y_t^o|X_t)$ in (2.8) to approximate the derivatives of the likelihood $\pi_y(Y_t^o|X_t)$ in (2.7). We can then use these derivatives to construct the Laplace approximation, which is a proper density in the sense that it integrates up to one.

We therefore propose to obtain the approximation $\hat{\pi}_{(y,2)}(Y_t^o|X_t)$ to the likelihood $\pi_y(Y_t^o|X_t)$ using the following procedure.

1. Obtain the Taylor polynomials of the policy function $Y_t^o = g_o(X_t, \varepsilon_t, \sigma)$ and $Y_t^n = g_n(X_t, \varepsilon_t, \sigma)$ of order $s = 2$ through perturbation methods.

2. Invert the Taylor polynomials using a Lagrange inversion formula to obtain $\varepsilon_t = \tilde{m}_{o,s}(X_t, Y_t^o, \sigma)$ and $Y_t^n = \tilde{m}_{n,s}(X_t, Y_t^o, \sigma)$.

3. Calculate the mode of $\log(\tilde{\pi}_{y,s}(Y_t^o|X_t))$, and the second-order derivatives at the mode $Y_t^o = Y^L$.

4. Use the mode and second-order derivatives to construct the Laplace approximation.

The procedure is started at $t = 1$ with $X_1 = 0$, which assumes that the initial value is the deterministic steady state. Because Y_t contains X_{t+1}, for each t we can obtain X_{t+1} by using the observed values Y_t^o and the relationship $Y_t^n = \tilde{m}_{n,s}(X_t, Y_t^o, \sigma)$. Because the dimensions of ε_t and Y_t^o are the same, we do not need any Kalman Filter to calculate the likelihood. Note that when ε_t is normally distributed, using $s = 1$ gives the same likelihood as in the literature for linear DSGE models (e.g. Fernández-Villaverde (2010)).

Using the Newton-Raphson algorithm, we calculate Y_L as the point that maximizes $\tilde{\pi}_{(y,s)}(Y_t^o|X_t)$ in (2.8). The Laplace approximation is a normal density with mean equal to Y_L and variance-covariance matrix equal to the inverse of minus the Hessian of

$\log\tilde{\pi}_{(y,s)}(Y_t^o|X_t)$. The following propositions provide the formulas for implementing the procedure. Proposition 1 gives the Hessian and gradient of $\log\pi_y(Y_t^o|X_t)$ at any point Y_t^o as a function of the derivatives of $\hat{m}_o(X_t, Y_t^o, \sigma)$ when ε_t is normally distributed.

Proposition 2 shows how to obtain the gradient and Hessian of $\log\tilde{\pi}_{(y,2)}(Y_t^o|X_t)$.

Proposition 3 explains how to invert the Taylor approximations of the policy functions $g_o(X_t, \varepsilon_t, \sigma)$ and $g_n(X_t, \varepsilon_t, \sigma)$.

Proposition 1. *Define J as the $n_o \times n_o$ Jacobian of $\hat{m}_o(X_t, Y_t^o, \sigma)$*

$$J = \frac{\partial \hat{m}_o(X_t, Y_t^o, \sigma)}{\partial (Y_t^o)'} \tag{2.9}$$

Let $Y_t^o = (Y_{t,1}^o, ..., Y_{t,n_o}^o)'$ and define F_i as the $n_o \times n_o$ matrix:

$$F_i = \frac{\partial J}{\partial Y_{t,i}^o}, \ \ i = 1, ..., n_o \tag{2.10}$$

and C as a $1 \times n_o$ vector

$$C = \begin{pmatrix} tr(J^{-1}F_1) & ... & tr(J^{-1}F_{n_o}) \end{pmatrix} \tag{2.11}$$

where $tr(.)$ is the trace operator. The gradient of $\log(\pi_y(Y_t^o|X_t))$ with respect to Y_t^o is

$$\frac{\partial \log(\pi_y(Y_t^o|X_t))}{\partial (Y_t^o)'} = -(\hat{m}_o(X_t, Y_t^o, \sigma))'\Sigma^{-1}J + C. \tag{2.12}$$

Let A be a $n_o \times n_o$ matrix defined as:

$$A = (a_{ij}), \ \ \textit{where } a_{ij} = tr(J^{-1}F_iJ^{-1}F_j) \tag{2.13}$$

and let V be a $n_o \times n_o$ matrix defined as:

$$V = \begin{pmatrix} V_1 & ... & V_{no} \end{pmatrix}, \ \ \textit{where } V_i = -F_i'\Sigma^{-1}(\hat{m}_o(X_t, Y_t^o, \sigma)). \tag{2.14}$$

Then the Hessian of $\log(\pi_y(Y_t^o|X_t))$ with respect to Y_t^o is

$$H = -J'\Sigma^{-1}J + V - A + F \tag{2.15}$$

where F is the $R \times R$ matrix, defined as

$F = (f_{ij})$, where $f_{i,j} = tr(J^{-1}F_{ij})$ and F_{ij} is the $n_o \times n_o$ matrix defined as:

$$F_{i,j} = \frac{\partial F_i}{\partial Y^o_{t,j}}$$

Proof. Let $\pi_o = \log(\pi_y(Y_t^o|X_t))$. From (2.7), assuming normality for ε_t we have that:

$$\pi_o = \log(\pi_o(Y_t^o|X_t)) = -\frac{1}{2}\varepsilon_t^{'}\Sigma^{-1}\varepsilon_t + \log|J| - \frac{1}{2}\log|\Sigma| - \frac{n_o}{2}\log(2\pi)$$

Using matrix differential calculus, the derivatives of a determinant can be calculated (e.g Magnus and Neudecker (1999)), such that a differential of π_o can be written as:

$$\partial\pi_o = -\varepsilon_t^{'}\Sigma^{-1}\partial\varepsilon_t + tr(J^{-1}\partial J)$$

where ∂J is the differential of J and can be written as:

$$\partial J = \sum_{i=1}^{n_o}\frac{\partial J}{\partial Y^o_{t,i}}\partial Y^o_{t,i} = \sum_{i=1}^{n_o} F_i \partial Y^o_{t,i}$$

Therefore, we can write:

$$tr(J^{-1}\partial J) = tr(J^{-1}\sum_{i=1}^{n_o} F_i\partial Y^o_{t,i}) = \sum_{i=1}^{n_o} tr(J^{-1}F_i)\partial Y^o_{t,i} = C\partial Y_t^o$$

where: $\partial Y_t^o = (\partial Y^o_{t,1}, Y^o_{t,2}, \dots Y^o_{t,n_o})^{'}$.
Therefore, $\partial\pi_o$ can be written as:

$$\partial\pi_o = -\varepsilon_t^{'}\Sigma^{-1}\partial\varepsilon_t + C\partial Y_t^o$$

From the definition of Jacobian J, we have that $\partial\varepsilon_t = J\partial Y_t^o$. Therefore, we can write:

$$\partial\pi_o = -\varepsilon_t^{'}\Sigma^{-1}J\partial Y_t^o + C\partial Y_t^o$$

which gives the result that proves (2.12):

$$\frac{\partial\pi_o}{\partial(Y_t^o)'} = -\varepsilon_t^{'}\Sigma^{-1}J + C = -(\hat{m}_o(X_t, Y_t^o, \sigma))'\Sigma^{-1}J + C.$$

Define $\pi_1 = \frac{\partial \pi_o}{\partial (Y_t^o)'}$ and take a differential:

$$\partial \pi_1 = -\partial \varepsilon_t^1 \Sigma^{-1} J - \varepsilon_t^{'} \Sigma^{-1} \partial J + \partial C$$

As before, we have $\partial \varepsilon_t = J \partial Y_t^o$. So, we can write:

$$\partial \pi_1 = -(\partial Y_t^o)^{'} J^{'} \Sigma^{-1} J - \varepsilon_t^{'} \Sigma^{-1} \partial J + \partial C \tag{2.16}$$

From $\partial J = \sum_{j=1}^{n_o} F_j \partial Y_{t,j}^o$, we can write:

$$-\varepsilon_t^{'} \Sigma^{-1} \partial J = -\sum_{j=1}^{n_o} \varepsilon_t^{'} \Sigma^{-1} F_j \partial Y_{t,j}^o = -(\partial Y_t^o)^{'} \begin{pmatrix} \varepsilon_t^{'} \Sigma^{-1} F_1 \\ \varepsilon_t^{'} \Sigma^{-1} F_2 \\ \vdots \\ \varepsilon_t^{'} \Sigma^{-1} F_{n_o} \end{pmatrix} = (\partial Y_t^o)^{'} \begin{pmatrix} V_1^{'} \\ V_2^{'} \\ \vdots \\ V_{n_o}^{'} \end{pmatrix} = (\partial Y_t^o)^{'} V^{'}$$

Therefore, from (2.16) we can write $\partial \pi_1$ as:

$$\partial \pi_1 = (\partial Y_t^o)^{'} (-J^{'} \Sigma^{-1} J + V) + \partial C \tag{2.17}$$

where we have used the fact that V is symmetric. In order to calculate ∂C, first note that:

$$\partial (tr(J^{-1} F_i)) = tr(\partial J^{-1} F_i + J^{-1} \partial F_i)$$

and also that (e.g. Magnus and Neudecker (1999)):

$$\partial J^{-1} = -J^{-1} \partial J J^{-1}$$

such that, using $\partial J = \sum_{j=1}^{n_o} \frac{F_j}{\partial Y_{t,j}^o}$, we have that:

$$\partial (tr(J^{-1} F_i)) = \sum_{j=1}^{n_o} tr(-J^{-1} F_j J^{-1} F_i) \partial Y_{t,j}^o + tr(J^{-1} \partial F_i)$$

Because $C = (tr(J^{-1} F_1) \ldots tr(J^{-1} F_{n_o}))$ we have that:

$$\partial C = -(\partial Y_t^o)^{'} \partial F_i$$

Now using that $\partial F_i = \sum_{j=1}^{n_o} F_{ij} \partial Y_{t,j}^o$ we can write:

$$\partial C = -(\partial Y_t^o)^{'} A + (\partial Y_t^o)^{'} F \tag{2.18}$$

Combining (2.17) and (2.18) we get:

$$\partial \pi_1 = (\partial Y_t^o)^{'} (-J^{'} \Sigma^{-1} J + V - A + F)$$

which implies that the Hessian in $(-J^{'}\Sigma^{-1}J + V - A + F)$, as we wanted to prove.

□

Proposition 2. *Assume that* $\varepsilon_t = \tilde{m}_{o,2}(X_t, Y_t^o, \sigma)$ *is given by*

$$\varepsilon_t = \tilde{G}_o^{0,0} + \tilde{G}_o^{1,0} Y_t^o + \tilde{G}_o^{0,1} X_t + \tilde{G}_o^{2,0}(Y_t^o \otimes Y_t^o) + \tilde{G}_o^{0,2}(X_t \otimes X_t) + \tilde{G}_o^{1,1}(Y_t^o \otimes X_t) \tag{2.19}$$

where $\tilde{G}_o^{0,0}, \ldots, \tilde{G}_o^{1,1}$ *are comformable matrices.*

Then the Jacobian is:

$$J = \frac{\partial \tilde{m}_{o,2}(X_t, Y_t^o, \sigma)}{\partial (Y_t^o)'} = \tilde{G}_o^{1,0} + 2\tilde{G}_o^{2,0}(I_{n_o} \otimes Y_t^o) + \tilde{G}_o^{1,1}(I_{n_o} \otimes X_t) \tag{2.20}$$

where I_{n_o} *is the identity matrix of dimension* n_o*.*

Let i_j *denote the* j^{th} *column of the identity matrix, such that* $I_{n_o} = (i_1, ..., i_{n_o})$*. Define* C *as the* $n_o \times n_o$ *matrix whose* j^{th} *column is equal to* $2tr(J^{-1}\tilde{G}_o^{2,0}(I_{n_o} \otimes i_j))$*, such that:*

$$C = 2(tr(J^{-1}\tilde{G}_o^{2,0}(I_{n_o} \otimes i_1)), ..., tr(J^{-1}\tilde{G}_o^{2,0}(I_{n_o} \otimes i_{n_o}))) \tag{2.21}$$

The gradient of $\log(\tilde{\pi}_{(y,2)}(Y_t^o|X_t))$ *with respect to* Y_t^o *is:*

$$\frac{\partial \log(\pi_y(Y_t^o|X_t))}{\partial (Y_t^o)'} = -(\tilde{m}_{o,2}(X_t, Y_t^o, \sigma))'\Sigma^{-1} J + C \tag{2.22}$$

Let A *be a* $n_o \times n_o$ *matrix defined as:*

$$A = (a_{ij}), \text{ where } a_{ij} = 4tr(J^{-1}\tilde{G}_o^{2,0}(I_{n_o} \otimes i_i) J^{-1}\tilde{G}_o^{2,0}(I_{n_o} \otimes i_j)) \tag{2.23}$$

Let V be a $n_o \times n_o$ matrix defined as:

$$V = \begin{pmatrix} V_1 & \ldots & V_{no} \end{pmatrix}, \text{ where } V_i = -(2\tilde{G}_o^{2,0}(I_{n_o} \otimes i_i))'\Sigma^{-1}(\tilde{m}_{o,2}(X_t, Y_t^o, \sigma)) \tag{2.24}$$

The Hessian of $\log(\tilde{\pi}_{(y,2)}(Y_t^o|X_t))$ *with respect to* Y_t^o *is:*

$$H = -J'\Sigma^{-1}J + V - A \tag{2.25}$$

Proof. To find the Jacobian let us take the differential of (2.19) which is given by:

$$\partial\varepsilon_t = \tilde{G}_o^{1,0}\partial Y_t^o + \tilde{G}_o^{2,0}(\partial Y_t^o \otimes Y_t^o) + \tilde{G}_o^{2,0}(Y_t^o \otimes \partial Y_t^o) + \tilde{G}_o^{1,1}(\partial Y_t^o \otimes X_t) \tag{2.26}$$

Note that $\tilde{G}_o^{2,0}$ contains second-order derivatives. In particular, each row of $\tilde{G}_o^{2,0}$ is the vectorized version of a Hessian matrix, which is symmetric. From this we have that:

$$\tilde{G}_o^{2,0}(\partial Y_t^o \otimes Y_t^o) = \tilde{G}_o^{2,0}(Y_t^o \otimes \partial Y_t^o),$$

so that (2.26) can be written as:

$$\begin{aligned} \partial\varepsilon_t &= \tilde{G}_o^{1,0}\partial Y_t^o + 2\tilde{G}_o^{2,0}(\partial Y_t^o \otimes Y_t^o) + \tilde{G}_o^{1,1}(\partial Y_t^o \otimes X_t) = \\ &= \tilde{G}_o^{1,0}\partial Y_t^o + 2\tilde{G}_o^{2,0}(I_{n_o} \otimes Y_t^o)\partial Y_t^o + \tilde{G}_o^{1,1}(I_{n_o} \otimes X_t)\partial Y_t^o \\ &= \big(\tilde{G}_o^{1,0} + 2\tilde{G}_o^{2,0}(I_{n_o} \otimes Y_t^o) + \tilde{G}_o^{1,1}(I_{n_o} \otimes X_t)\big)\partial Y_t^o \end{aligned} \tag{2.27}$$

This shows that the Jacobian is the expression given in (2.20):

$$J = \tilde{G}_o^{1,0} + 2\tilde{G}^{2,0}(I_{n_o} \otimes Y_t^o) + \tilde{G}_o^{1,1}(I_{n_o} \otimes X_t)$$

To calculate $(F_1, \ldots, F_{n_o})$ let us obtain the differential of J as:

$$\partial J = 2\tilde{G}^{2,0}(I_{n_o} \otimes \partial Y_t^o),$$

which shows that $F_j = 2\tilde{G}^{2,0}(I_{n_o} \otimes i_j)$ for $j = 1, \ldots, n_o$. Using (2.11) and (2.12) in Proposition 1 we obtain (2.21) and (2.22). The expressions for A and V in (2.23) and (2.24) were obtained by using the expression for F_j in equation (2.13) and (2.14) of Proposition 1. Finally, (2.25) is obtained from (2.15) by noting that F_j does not depend on Y_t^o, and, thus, $F_{ij} = 0$.

□

Proposition 3. *Let the Taylor Polynomials* $\varepsilon_t = \tilde{m}_{o,2}(X_t, Y_t^o, \sigma)$ *and* $Y_t^n = \tilde{m}_{n,2}(X_t, Y_t^o, \sigma)$ *be given by:*

$$\begin{aligned} \varepsilon_t &= \tilde{G}_o^{0,0} + \tilde{G}_o^{1,0} Y_t^o + \tilde{G}_o^{0,1} X_t + \tilde{G}_o^{2,0}(Y_t^o \otimes Y_t^o) + \tilde{G}_o^{0,2}(X_t \otimes X_t) + \tilde{G}_o^{1,1}(Y_t^o \otimes X_t) \\ Y_t^n &= \tilde{G}_n^{0,0} + \tilde{G}_n^{1,0} Y_t^o + \tilde{G}_n^{0,1} X_t + \tilde{G}_n^{2,0}(Y_t^o \otimes Y_t^o) + \tilde{G}_n^{0,2}(X_t \otimes X_t) + \tilde{G}_n^{1,1}(Y_t^o \otimes X_t) \end{aligned} \tag{2.28}$$

where $\tilde{G}_o^{0,0}, \ldots, \tilde{G}_o^{1,1}$, *and* $\tilde{G}_n^{0,0}, \ldots, \tilde{G}_n^{1,1}$ *are comformable matrices. The* $\tilde{G}$ *matrices in (2.28) can be obtained from the* G *matrices in (2.4) as follows.*

$$\begin{cases} \tilde{G}_o^{1,0} = \Big(G_o^{1,0} - 2G_o^{2,0}\Big(G_o^{1,0} \otimes G_o^{1,0} + 2(G_o^{2,0} \otimes G_o^{0,0})\Big)^{-1}(G_o^{0,0} \otimes G_o^{1,0})\Big)^{-1} \\ \\ \tilde{G}_n^{1,0} = \Big(G_n^{1,0} - 2G_n^{2,0}\Big(G_o^{1,0} \otimes G_o^{1,0} + 2(G_o^{2,0} \otimes G_o^{0,0})\Big)^{-1}({G_o}^{0,0} \otimes {G_o}^{1,0})\Big)\tilde{G}_o^{1,0} \end{cases} \tag{2.29}$$

$$\begin{cases} \tilde{G}_o^{2,0} = -\tilde{G}_o^{1,0} G_o^{2,0}\Big(G_o^{1,0} \otimes G_o^{1,0} + 2(G_o^{2,0} \otimes G_o^{0,0})\Big)^{-1} \\ \\ \tilde{G}_n^{2,0} = \Big({G_n}^{2,0} - \tilde{G}_n^{1,0}{G_o}^{2,0}\Big)\Big(G_o^{1,0} \otimes G_o^{1,0} + 2(G_o^{2,0} \otimes G_o^{0,0})\Big)^{-1} \end{cases} \tag{2.30}$$

$$\begin{cases} \tilde{G}_o^{1,1} = -\Big(\tilde{G}_o^{1,0} G_o^{1,1} + 2\tilde{G}_o^{2,0}(G_o^{1,0} \otimes G_o^{0,1}) + 2\tilde{G}_o^{2,0}(G_o^{1,1} \otimes {G_o}^{0,0})\Big)\Big((G_o^{1,0})^{-1} \otimes I_{n_K}\Big) \\ \\ \tilde{G}_n^{1,1} = \Big(G_n^{1,1} - \Big(\tilde{G}_n^{1,0} G_o^{2,0} + 2\tilde{G}_n^{2,0}({G_o}^{1,0} \otimes {G_o}^{0,1})\Big) + 2\tilde{G}_n^{2,0}(G_o^{1,1} \otimes {G_o}^{0,0})\Big)\Big(({G_o}^{1,0})^{-1} \otimes I_{n_K}\Big) \end{cases} \tag{2.31}$$

$$\begin{cases} \tilde{G}_o^{0,2} = -\Big(\tilde{G}_o^{1,0} G_o^{0,2} + \tilde{G}_o^{2,0}(G_o^{0,1} \otimes G_o^{0,1}) + \tilde{G}_o^{1,1}(G_o^{0,1} \otimes I_{n_K})\Big) - 2\tilde{G}_o^{2,0}\Big({G_o}^{0,0} \otimes {G_o}^{0,2}\Big) \\ \\ \tilde{G}_n^{0,2} = G_n^{0,2} - \Big(\tilde{G}_n^{1,0} G_o^{0,2} + \tilde{G}_n^{2,0}(G_o^{0,1} \otimes G_o^{0,1}) + \tilde{G}_n^{1,1}(G_o^{0,1} \otimes I_{n_K})\Big) - 2\tilde{G}_n^{2,0}\Big(G_o^{0,0} \otimes G_o^{0,2}\Big) \end{cases} \tag{2.32}$$

$$\begin{cases} \tilde{G}_o^{0,1} = -\Big(\tilde{G}_o^{1,0}G_o^{0,1} + \tilde{G}_o^{1,1}(G_o^{0,0} \otimes I_{n_K})\Big) - 2\tilde{G}_o^{2,0}\Big(G_o^{0,0} \otimes G_o^{0,1}\Big) \\ \\ \tilde{G}_n^{0,1} = G_n^{0,1} - \Big(\tilde{G}_n^{1,0}G_o^{0,1} + \tilde{G}_n^{1,1}(G_o^{0,0} \otimes I_{n_K})\Big) - 2\tilde{G}_n^{2,0}\Big(G_o^{0,0} \otimes G_o^{0,1}\Big) \end{cases} \tag{2.33}$$

$$\begin{cases} \tilde{G}_o^{0,0} = -\Big(\tilde{G}_o^{1,0}G_o^{0,0} + \tilde{G}_o^{2,0}(G_o^{0,0} \otimes G_o^{0,0})\Big) \\ \\ \tilde{G}_n^{0,0} = G_n^{0,0} - \Big(\tilde{G}_n^{1,0}G_o^{0,0} + \tilde{G}_n^{2,0}(G_o^{0,0} \otimes G_o^{0,0})\Big) \end{cases} \tag{2.34}$$

Proof. Here we use $\varepsilon_t = m_o(X_t, Y_t^o, \sigma)$ as the local inverse of the policy function $Y_t^o = g_o(X_t, \varepsilon_t, \sigma)$, which exists provided that the Jacobian is different from 0. Similarly, $Y_t^n = m_n(X_t, Y_t^o, \sigma)$ is the local inverse of $Y_t^n = g_n(X_t, \varepsilon_t, \sigma)$. From the properties of the inverse function, we have that $\varepsilon_t = m_o(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma)$, and also that $Y_t^n = m_n(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma) = g_n(X_t, \varepsilon_t, \sigma)$. Therefore, a second-order Taylor expansion of the composition functions $m_o(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma)$, and $m_n(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma)$ gives the following:

$$\begin{aligned} m_o(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma) = F_o^{0,0} + F_o^{1,0}\varepsilon_t + F_o^{0,1}X_t + F_o^{2,0}(\varepsilon_t \otimes \varepsilon_t) + \\ F_o^{0,2}(X_t \otimes X_t) + F_o^{1,1}(\varepsilon_t \otimes X_t) = \varepsilon_t \end{aligned} \tag{2.35}$$

$$\begin{aligned} m_n(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma) = F_n^{0,0} + F_n^{1,0}\varepsilon_t + F_n^{0,1}X_t + F_n^{2,0}(\varepsilon_t \otimes \varepsilon_t) + \\ F_n^{0,2}(X_t \otimes X_t) + F_n^{1,1}(\varepsilon_t \otimes X_t) = Y_n = g_n(X_t, \varepsilon_t, \sigma) \end{aligned} \tag{2.36}$$

Equations (2.35)-(2.36) imply the following restrictions on the F matrices:

$$\begin{aligned} &F_o^{0,0} = 0, \quad F_o^{1,0} = I, \quad F_o^{0,1} = 0, \quad F_o^{2,0} = 0, \quad F_o^{0,2} = 0, \quad F_o^{1,1} = 0, \\ &F_n^{0,0} = G_n^{0,0}, \quad F_n^{1,0} = G_n^{1,0}, \quad F_n^{0,1} = G_n^{0,1}, \quad F_n^{2,0} = G_n^{2,0}, \\ &F_n^{0,2} = G_n^{0,2}, \quad F_n^{1,1} = G_n^{1,1} \end{aligned} \tag{2.37}$$

Let $\tilde{g}_{o,s}(X_t, \varepsilon_t, \sigma)$ and $\tilde{g}_{n,s}(X_t, \varepsilon_t, \sigma)$ be the Taylor approximations of order s to the policy functions $g_o(X_t, \varepsilon_t, \sigma)$ and $g_n(X_t, \varepsilon_t, \sigma)$, respectively. The second-order Taylor approximation of the composition functions $m_o(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma)$, and $m_n(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma)$ can be obtained by calculating the composition function of the corresponding Taylor polynomials and keeping the terms up to 2^{nd} order.

Therefore, using the Taylor approximations in (2.4) and in (2.28) we can calculate the compositions $m_o(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma)$, and $m_n(X_t, g_o(X_t, \varepsilon_t, \sigma), \sigma)$, and obtain the coefficients for $(\varepsilon_t \otimes \varepsilon_t)$, (ε_t), $(\varepsilon_t \otimes X_t)$, $(X_t \otimes X_t)$, (X_t) and for the constant term. Equating these coefficients with those in (2.37) gives the following 12 equations with 12 unknowns.

for $(\varepsilon_t \otimes \varepsilon_t)$

$$\begin{cases} \tilde{G_o}^{2,0}(G_o^{1,0} \otimes G_o^{1,0}) + 2\tilde{G_o}^{2,0}(G_o^{0,0} \otimes G_o^{2,0}) + \tilde{G_o}^{1,0} G_o^{2,0} = F_o^{2,0} = 0 \\ \\ \tilde{G_n}^{2,0}(G_n^{1,0} \otimes G_n^{1,0}) + 2\tilde{G_n}^{2,0}(G_n^{0,0} \otimes G_n^{2,0}) + \tilde{G_n}^{1,0} G_n^{2,0} = F_n^{2,0} = G_n^{2,0} \end{cases} \tag{2.38}$$

for (ε_t)

$$\begin{cases} \tilde{G_o}^{1,0} G_o^{1,0} + 2\tilde{G_o}^{2,0}(G_o^{0,0} \otimes G_o^{1,0}) = F_o^{1,0} = I \\ \\ \tilde{G_n}^{1,0} G_o^{1,0} + 2\tilde{G_n}^{2,0}(G_o^{0,0} \otimes G_o^{1,0}) = F_n^{1,0} = G_n^{1,0} \end{cases} \tag{2.39}$$

for $(\varepsilon_t \otimes X_t)$

$$\begin{cases} \tilde{G_o}^{1,1}(G_o^{1,0} \otimes I_K) + \tilde{G_o}^{1,0} G_o^{1,1} + 2\tilde{G_o}^{2,0}(G_o^{1,0} \otimes G_o^{0,1}) + 2\tilde{G_o}^{2,0}(G_o^{1,1} \otimes G_o^{0,0}) = F_o^{1,1} = 0 \\ \\ \tilde{G_n}^{1,1}(G_o^{1,0} \otimes I_K) + \tilde{G_n}^{1,0} G_o^{1,1} + 2\tilde{G_n}^{2,0}(G_o^{1,0} \otimes G_o^{0,1}) + 2\tilde{G_n}^{2,0}(G_o^{1,1} \otimes G_o^{0,0}) = F_n^{1,1} = G_n^{1,1} \end{cases} \tag{2.40}$$

for $(X_t \otimes X_t)$

$$\begin{cases} \tilde{G_o}^{0,2} + \tilde{G_o}^{1,0} G_o^{0,2} + \tilde{G_o}^{2,0}(G_o^{0,1} \otimes G_o^{0,1}) + \tilde{G_o}^{1,1}(G_o^{1,0} \otimes I_K) + 2\tilde{G_o}^{2,0}(G_o^{0,0} \otimes G_o^{0,2}) = F_o^{0,2} = 0 \\ \\ \tilde{G_n}^{0,2} + \tilde{G_n}^{1,0} G_o^{0,2} + \tilde{G_n}^{2,0}(G_o^{0,1} \otimes G_o^{0,1}) + \tilde{G_n}^{1,1}(G_o^{1,0} \otimes I_K) + 2\tilde{G_n}^{2,0}(G_o^{0,0} \otimes G_o^{0,2}) = F_n^{0,2} = G_n^{0,2} \end{cases} \tag{2.41}$$

for (X_t)

$$\begin{cases} \tilde{G}_o^{\ 0,1} + \tilde{G}_o^{\ 1,0} G_o^{0,1} + \tilde{G}_o^{\ 1,1}(G_o^{0,0} \otimes I_K) + 2\tilde{G}_o^{\ 2,0}(G_o^{0,0} \otimes G_o^{0,1}) = F_o^{0,1} = 0 \\ \\ \tilde{G}_n^{\ 0,1} + \tilde{G}_n^{\ 1,0} G_o^{0,1} + \tilde{G}_n^{\ 1,1}(G_o^{0,0} \otimes I_K) + 2\tilde{G}_n^{\ 2,0}(G_o^{0,0} \otimes G_o^{0,1}) = F_n^{0,1} = G_n^{0,1} \end{cases} \tag{2.42}$$

for the constant term:

$$\begin{cases} \tilde{G}_o^{\ 0,0} + \tilde{G}_o^{\ 1,0} G_o^{0,0} + \tilde{G}_o^{\ 2,0}(G_o^{0,0} \otimes G_o^{0,0}) = F_o^{0,0} = 0 \\ \\ \tilde{G}_n^{\ 0,0} + \tilde{G}_n^{\ 1,0} G_o^{0,0} + \tilde{G}_n^{\ 2,0}(G_o^{0,0} \otimes G_o^{0,0}) = F_n^{0,0} = G_n^{0,0} \end{cases} \tag{2.43}$$

Thus, solving the system of equations in (2.38)-(2.43) through substituting and collecting terms we obtain the proposed solutions for matrices of the second-order approximation of the inverses. □

2.5 Simulation from the Laplace Approximated DSGE Model

Once the model has been solved by perturbation methods, equation (2.4) can be used to simulate directly values for Y_t. For a given value of X_1 this can be done by repeating the following two steps for $t = 1, ..., T$:

1. Simulate ε_t from the appropriate distribution and use equation (2.4) to obtain Y_t.

2. Obtain X_{t+1} as the appropriate subvector of Y_t.

We use this approach to obtain the generalized Impulse Response Functions (IRFs) presented in Section 2.6 (see e.g. Dynare (2021) for an explanation of how to use simulation to construct IRFs).

However, it is also possible to simulate Y_t using the likelihood of the Laplace approximated DSGE model that we have proposed in Section 2.4. Specifically, for a given initial value of X_1 this can be done by repeating the following 2 steps for $t = 1, ..., T$:

1. Simulate Y_t^o using the Laplace approximated density $\hat{\pi}_{(y,2)}(Y_t^o|X_t)$.

2. Obtain Y_t^n using the inverted Taylor polynomial $Y_t^n = \tilde{m}_{n,s}(X_t, Y_t^o, \sigma)$. Obtain X_{t+1} as the appropriate subvector of Y_t.

Note that perturbation methods give an approximation of the policy functions around the steady state, and that the quality of the approximation deteriorates as we get further from the steady state. For this reason, we cannot expect the perturbation methods to be informative about the tails of the distribution of Y_t. We can expect that several distributions will be consistent with the local properties of the true distribution around the steady state. The Laplace approximated model uses the local derivatives to construct a density which is consistent with the local properties around the steady state, and yet can be calculated easily numerically. In contrast, as argued in previous sections, the exact likelihood for the solution obtained using only perturbation methods is not available, except in limited cases.

However, if perturbation methods are accurate in approximating the true policy functions, we should expect both solutions to give similar results. We can evaluate this by comparing the IRFs obtained from both approaches. To obtain IRFs in the Laplace approximated model, we should introduce intervention dummies d_t containing the impulses. Hence, the structural errors of this economy become $\varepsilon_t + d_t$, which we introduce in the inverse Taylor polynomials $\varepsilon_t = \tilde{m}_{o,2}(X_t, Y_t^o, \sigma, d_t)$ and $Y_t^n = \tilde{m}_{n,2}(X_t, Y_t^o, \sigma)$ that were presented in equation (2.28) as follows:

$$\begin{aligned} \varepsilon_t &= \tilde{G}_o^{0,0} - d_t + \tilde{G}_o^{1,0} Y_t^o + \tilde{G}_o^{0,1} X_t + \tilde{G}_o^{2,0}(Y_t^o \otimes Y_t^o) + \tilde{G}_o^{0,2}(X_t \otimes X_t) + \tilde{G}_o^{1,1}(Y_t^o \otimes X_t) \\ Y_t^n &= \tilde{G}_n^{0,0} + \tilde{G}_n^{1,0} Y_t^o + \tilde{G}_n^{0,1} X_t + \tilde{G}_n^{2,0}(Y_t^o \otimes Y_t^o) + \tilde{G}_n^{0,2}(X_t \otimes X_t) + \tilde{G}_n^{1,1}(Y_t^o \otimes X_t) \end{aligned} \tag{2.44}$$

These equations are the same as in (2.28) except we now write $(\varepsilon_t + d_t)$ instead of (ε_t) or equivalently $(\tilde{G}_o^{0,0} - d_t)$ instead of $(\tilde{G}_o^{0,0})$. The simulation then can be carried out as explained above but using $\varepsilon_t = \tilde{m}_{o,2}(X_t, Y_t^o, \sigma, d_t)$ instead of $\varepsilon_t = \tilde{m}_{o,2}(X_t, Y_t^o, \sigma)$, and choosing the vector d_t according to the IRF that needs to be calculated.

In figures 2.1 to 2.8 we plot the IRFs obtained through these two approaches for the application in Section 2.6. We find that the IRFs are very close to each other in all cases. There are only some cases in which there seems to be a difference in the shape of the IRFs: d, μ_z, μ_l, μ_A and phi. However, in all these cases the IRFs are of the order of 10^{-17} or smaller in the first approach, while in the second approach they are also negligible. It is reasonable therefore to conclude that the two solutions are very similar.

2.6 An application

2.6.1 Model description

We present the benchmark neoclassical growth DSGE model of Fernández-Villaverde (2010) to demonstrate the proposed likelihood-based approach. The model comprises five structural errors and initially are estimated for the US economy. Using the same country data, we first estimate the log-linearized version of the model employing a first-order perturbation method. To capture nonlinearity in the model, we estimate the same model with a second-order perturbation method.

However, empirical studies provide compelling evidence for the presence of uncertainty shock in US aggregate time series, as documented by Fernández-Villaverde and Rubío-Ramirez (2007), Justiniano and Primiceri (2008), Bloom (2009), and Fernández-Villaverde and A.Guerrón-Quintana (2020). Moreover, these time series exhibit a time-varying variance component, which violates the homoscedasticity assumption that a variance of a stochastic process is constant over time. In that regard, the assumption of homoscedasticity, which refers to the assumption that the variance of a stochastic process is constant over time, no longer holds. It is also worth noting that heteroscedasticity disappears in the linear approximation of the model, implying that models with time-varying uncertainty should be solved nonlinearly. To address this issue, we modify the benchmark model by introducing Generalized Autoregressive Conditional Heteroscedasticity (GARCH)[1] and solve it with the second-order approximation method.

The basic structure of this neoclassical growth model is as follows. There is a representative household which consumes, saves, holds money, supplies labor, and sets its own wages subject to a demand curve and Calvo's pricing. The final output is produced by a final good firm, which uses as inputs a continuum of intermediate goods manufactured by monopolistic competitors. The intermediate good producers rent capital and labor to produce their good. They face the constraint that they can only change prices following a Calvo's rule. Finally, there is a monetary authority that fixes the one-period nominal interest rate through open market operations with public debt.

The equilibrium conditions of the model are in Table 2.1, the definition of variables is in Table 2.2.

The five shocks ($\varepsilon_{d,t}$, $\varepsilon_{\varphi,t}$, $\varepsilon_{\mu_I,t}$, $\varepsilon_{A,t}, m_t$) are assumed to be normally distributed with zero mean and standard deviations ($\exp\sigma_d$, $\exp\sigma_\varphi$,$\exp\sigma_\mu$, $\exp\sigma_A$, $\exp\sigma_m$), respectively.

[1]Palm (1996)

They enter in the model through equations (2.24)- (2.27) and equation (2.15) in Table 2.1, respectively. In the GARCH version of the model, the shocks continue to have the same variance, but they are multiplied by a time-varying variance with an expected value of one:

$$\log \tilde{d}_t = \rho^d \log d_{t-1} + \sqrt{\tilde{\sigma}_{d,t}}\varepsilon_{d,t}$$

$$\tilde{\sigma}_{d,t} = \rho_1^d \tilde{\sigma}_{d,t-1} + \rho_2^d \tilde{\sigma}_{d,t-1} \tfrac{(\varepsilon_{d,t-1})^2}{\exp(2\sigma_d)} + (1 - \rho_1^d - \rho_2^d)$$

$$\log \varphi_t = \rho^\varphi \log \varphi_{t-1} + \sqrt{\tilde{\sigma}_{\varphi,t}}\varepsilon_{\varphi,t}$$

$$\tilde{\sigma}_{\varphi,t} = \rho_1^\varphi \tilde{\sigma}_{\varphi,t-1} + \rho_2^\varphi \tilde{\sigma}_{\varphi,t-1} \tfrac{(\varepsilon_{\varphi,t-1})^2}{\exp(2\sigma_\varphi)} + (1 - \rho_1^\varphi - \rho_2^\varphi)$$

$$\log \mu_{I,t} = \Lambda_\mu + \sqrt{\tilde{\sigma}_{\mu,t}}\varepsilon_{\mu_I,t}$$

$$\tilde{\sigma}_{\mu,t} = \rho_1^\mu \tilde{\sigma}_{\mu,t-1} + \rho_2^\mu \tilde{\sigma}_{\mu,t-1} \tfrac{(\varepsilon_{\mu,t-1})^2}{\exp(2\sigma_\mu)} + (1 - \rho_1^\mu - \rho_2^\mu)$$

$$\log \mu_{A,t} = \Lambda_A + \sqrt{\tilde{\sigma}_{A,t}}\varepsilon_{A,t}$$

$$\tilde{\sigma}_{A,t} = \rho_1^A \tilde{\sigma}_{A,t-1} + \rho_2^A \tilde{\sigma}_{A,t-1} \tfrac{(\varepsilon_{A,t-1})^2}{\exp(2\sigma_A)} + (1 - \rho_1^A - \rho_2^A)$$

$$\tfrac{R_t}{R} = \left(\tfrac{R_{t-1}}{R}\right)^{\gamma_R} \left(\left(\tfrac{\Pi_t}{\Pi}\right)^{\gamma_\Pi} \left(\frac{\frac{\tilde{y}_t^d}{\tilde{y}_{t-1}^d} \frac{z_t}{z_{t-1}}}{\Lambda_{y^d}} \right)^{\gamma_y} \right)^{1-\gamma_R} \exp\left(\sqrt{\tilde{\sigma}_{m,t}} m_t\right)$$

$$\tilde{\sigma}_{m,t} = \rho_1^m \tilde{\sigma}_{m,t-1} + \rho_2^m \tilde{\sigma}_{m,t-1} \tfrac{(\varepsilon_{m,t-1})^2}{\exp(2\sigma_m)} + (1 - \rho_1^m - \rho_2^m)$$

We assume that $\rho_1^i > 0, \rho_2^i > 0$ and that $(\rho_1^i + \rho_2^i) < 1$, for $i = d, \varphi, \mu, A, m$. Under these restrictions the time-varying variances $(\tilde{\sigma}_{d,t},\ \tilde{\sigma}_{\varphi,t},\ \tilde{\sigma}_{\mu,t},\ \tilde{\sigma}_{A,t},\ \tilde{\sigma}_{m,t})$ have expected values equal to one, so that the long-run variances are $(\exp 2\sigma_d,\ \exp 2\sigma_\varphi,\ \exp 2\sigma_\mu,\ \exp 2\sigma_A,\ \exp 2\sigma_m)$. The GARCH version of the model has 10 extra parameters: ρ_1^d, ρ_2^d, ρ_1^φ, ρ_2^φ, ρ_1^μ, ρ_2^μ, ρ_1^A, ρ_2^A, ρ_1^m, ρ_2^m. We specify independent beta priors for each of these parameters. The prior mean and standard deviation for ρ_1^i are 0.7 and 0.046, respectively, for $i = d, \varphi, \mu, A, m$. The prior mean and standard deviation for ρ_2^i are 0.2 and 0.12, respectively, for $i = d, \varphi, \mu, A, m$.

We also consider a restricted GARCH model in which the variances evolve according to a common multiplicative factor $\widetilde{\sigma}_t$. Here we assume that for every t this restriction holds: $\widetilde{\sigma}_t = \tilde{\sigma}_{d,t} = \tilde{\sigma}_{\varphi,t} = \tilde{\sigma}_{\mu,t} = \tilde{\sigma}_{A,t} = \tilde{\sigma}_{m,t}$. The common factor $\widetilde{\sigma}_t$ responds to past values of the structural shocks as in a GARCH model:

$$\widetilde{\sigma}_t = \rho_1 \widetilde{\sigma}_{t-1} + \rho_2 \widetilde{\sigma}_{t-1} \frac{(\widetilde{\varepsilon}_{t-1})^2}{var(\widetilde{\varepsilon}_t)} + (1 - \rho_1 - \rho_2)$$

$$\widetilde{\varepsilon}_t = \delta_d \frac{\varepsilon_{d,t}}{\exp(\sigma_d)} + \delta_\varphi \frac{\varepsilon_{\varphi,t}}{\exp(\sigma_\varphi)} + \delta_\mu \frac{\varepsilon_{\mu,t}}{\exp(\sigma_\mu)} + \delta_A \frac{\varepsilon_{A,t}}{\exp(\sigma_A)} + \delta_m \frac{\varepsilon_{m,t}}{\exp(\sigma_m)}$$

$$var(\widetilde{\varepsilon}_t) = \delta_d^2 + \delta_\varphi^2 + \delta_\mu^2 + \delta_A^2 + \delta_m^2$$

In this model all structural shocks contribute to the time variation of the common factor. We can measure the relative contributions of the shocks to such time variation by the following proportions:

$$p_d = \frac{\delta_d^2}{var(\widetilde{\varepsilon}_t)},\ p_\varphi = \frac{\delta_\varphi^2}{var(\widetilde{\varepsilon}_t)},\ p_\mu = \frac{\delta_\mu^2}{var(\widetilde{\varepsilon}_t)},\ p_A = \frac{\delta_A^2}{var(\widetilde{\varepsilon}_t)},\ p_m = \frac{\delta_m^2}{var(\widetilde{\varepsilon}_t)},$$

where $p_d + p_\varphi + p_\mu + p_A + p_m = 1$. For example, p_m is the proportion of the time-variation in uncertainty driven by the monetary policy shock.

When it comes to estimating the common factor GARCH model, we have to realize that we have to normalize the vector $\delta = (\delta_d, \delta_\varphi, \delta_\mu, \delta_A, \delta_m)$ because it is not identified. We normalize it by the restriction $\delta_d^2 + \delta_\varphi^2 + \delta_\mu^2 + \delta_A^2 + \delta_m^2 = 1$, such that $var(\widetilde{\varepsilon}_t) = 1$. Regarding the prior, we specify a beta prior for ρ_1, with mean 0.7 and standard deviation 0.046. We then define $\widetilde{\delta} = \sqrt{\rho_2}\delta$ such that $\rho_2 = \widetilde{\delta}'\widetilde{\delta}$, and specify a normal prior for $\widetilde{\delta}$ with 0 mean and var-cov matrix equal to $(0.2/5)I_5$, where I_5 is the identity matrix. This implies that the prior for ρ_2 is a chi-squared distribution with 5 degrees of freedom and mean equal to 0.2.

We, therefore, estimate and compare four models:

- M_1: Log-linearized model with homoscedastic shocks.
- M_2: Model resulting from second-order approximation with homoscedastic shocks.
- $M_{2,G}$: Model resulting from second-order approximation with unrestricted GARCH in the shocks.

- $M_{2,fG}$: Model resulting from second-order approximation with a common factor in the GARCH processes.

2.6.2 Data and prior specifications

As in Fernández-Villaverde (2010), we used the following five time series for the U.S. economy: 1) the relative price of investment with respect to the price of consumption, 2) real output per capita growth, 3) real wages per capita growth, 4) the consumer price index growth and 5) the federal funds rate. We use the same data sources as Fernández-Villaverde (2010), but have a more extended sample period that runs for 1959 Q1—2019 Q4. Table 2.5 describes the data sources and the transformations to obtain the variables in the model $Y_t^o = (\log \mu_{I,t}, y_t^o, \omega_t^o, \log \Pi_t, \log R_t)'$.

We specify priors as in Fernández-Villaverde (2010). The prior means and distribution for estimated parameters are summarized in Table 2.3 and Table 2.4. Unlike Fernández-Villaverde (2010), we additionally estimate δ and γ_2 parameters to facilitate the estimation. Following his work, we restrict some parameters to their calibrated values: $\varepsilon = 10$, $\eta = 10$, $\phi = 0$ since they are difficult to identify in the data.

2.6.3 Results and analysis

We compare the models with the value of the marginal likelihood (e.g., Koop (2003), p. 4), predictive likelihood (Geweke and Amisano (2010)), and posterior probabilities. The marginal likelihood is the probability of observing given data under a specific model, averaging over all possible values of parameters of the model, and it explains how well the model fits the data (Koop et al. (2007)). Regarding predictive likelihood, existing literature has long recognized predictive likelihood as a valid Bayesian approach to model selection. The predictive distribution provides a foundation for robustness checks (Warne et al. (2013)). Gelfand and Dey (1994) emphasize that even if models have improper priors, the predictive likelihood can still be used for model selection, given a large enough sample to train the prior to a proper one. In addition, the difference in the predictive likelihood can be evaluated using posterior probabilities (presented in Table 2.13).

We use a Metropolis-Hasting algorithm with a random walk proposal to obtain draws from the posterior distribution (e.g., Koop (2003), p. 97). The marginal likelihood is calculated following Gelfand and Dey (1994) and Geweke (1999). This study calculates the predictive likelihood for observations from 45 to 244. The average computation time for estimation of models is 2–3 hours with 300000 iterations using a computer with processor Intel(R) Core (TM) i7-10700, CPU 2.90GHz, and RAM 16.0 GB, which would be faster than with relatively small models estimated with particle filters. Model comparison results are summarized in Table 2.13.

Table 2.13 shows that $M_{2,fG}$ is the best-performing model according to log marginal and predictive likelihood. Particularly, the log marginal likelihood for $M_{2,fG}$ is around 4711, while for the second best model, $M_{2,G}$, is 4639. The predictive likelihood for $M_{2,fG}$ is 3859, and for $M_{2,G}$ is 3792. In addition, the posterior probability for $M_{2,fG}$ is close to one compared to the other three models. Overall, all values indicated in the above mentioned table convey the same statement: the nonlinear solution method fits the data better than a linear approximation. According to Jeffreys (1998), a difference of 4.6 or greater in log marginal likelihoods suggests that one hypothesis is more than 100 times more probable than the other. Applying it to our model, we obtain the difference value of 72, which provides strong evidence in favor of nonlinear estimation, as it is well beyond the threshold. This is also identified as a Bayes factor, where the data strongly supports the nonlinear common factor GARCH model, $M_{2,fG}$, over the linearized model - M_1, with a value of $\exp(72)$ for the ratio of marginal likelihoods (Koop et al. (2007), p. 61).

Thus, the above result implies that taking into account the nonlinearities in the DSGE improves the empirical performance. Among the two models with incorporated GARCH processes, the above results show that the common factor GARCH model is much superior to the unrestricted GARCH model. This suggests that the data strongly support the common factor restriction.

The results of the posterior estimation of parameters of linear, M_1, and nonlinear, M_2, models are reported in Tables 2.6 and 2.7; for unrestricted GARCH, $M_{2,G}$, and common factor GARCH, $M_{2,fG}$, models - in Tables 2.8, 2.10 and 2.9. Posterior analysis of common factor GARCH model reveals that the posterior mean of p_m is 0.95 (Table 2.15), which implies that the monetary policy shock in a nonlinear heteroscedastic model is the main driving force of uncertainty in the economy (95%). The standard deviation of the parameters is estimated to be much higher for the common factor GARCH model than for the unrestricted GARCH model.

Bayesian impulse response analyses show that the positive monetary policy shock depicted in Figure 2.9 has a contractionary impact on aggregate output, consumption, real wages, and investment for all four estimated models. This finding is line with existing linearized studies. The impulse response functions for model M_2 differ from the other three models since the posterior estimates for parameters, such as h, γ, κ, and χ are different for model M_2. Since these parameters appear in equilibrium equations of consumption (h), investment (κ), output (χ), and wages (γ), it implies different steady-state values for consumption, investment, output, and wages (Table 2.11 and 2.12), which in turn induces slightly different impulse response functions. We analyzed the impulse response functions of model M_2 using estimated parameter values derived from the linear model M_1 (Figure 2.16). These impulse response functions are similar to those obtained with linear approximation and other models. Hence, the reason for the difference in impulse response functions is that the second-order approximation gives different estimated values. This difference can account for the nonlinearity inherent in economic relationships (via the quadratic terms).

In terms of overall fit of the four estimates models, Figures 2.10-2.13 display observed series used for estimation and their posterior estimates for models M_1, M_2, $M_{2,G}$, and $M_{2,fG}$. Figures indicate a good fit of some observed data, such as inflation and investment growth. In order to check the reliability of the model, we have also analysed posterior distribution of latent variables. In particular, we constructed the smoothed posterior means for productivity growth and marginal cost. Figures 2.14 and 2.15 show the posterior distribution of these latent variables implied by estimated M_1, M_2, $M_{2,G}$, and $M_{2,fG}$ models. According to these Figures, the model resulting from second-order approximation and the model with a common factor in GARCH processes show higher volatility than other two models.

The likelihood function depicted in Figure 2.17 measures the probability of observing the data for each quarter based on our estimated parameters. The figure captures several main recessions, mainly the economic downturn in the US followed by the oil price shock of the 1970s or Early 80s Recession; and the Great Recession of 2008 to 2009. During both periods, the log-likelihood for models with heteroscedastic shocks decreases less than for the linear and quadratic models. The reason for this might be that in heteroscedastic models the conditional variance in periods of crisis increases, and therefore the likelihood decreases less when there is a large shock.

Figure 2.18 represents the cumulative likelihood function for the whole data period. According to these figures, the common-factor GARCH model ($M_{2,fG}$ in cyan) outperforms other models and better captures the characteristics of the data.

2.7 Conclusions

This chapter presents the development of a new likelihood-based procedure for nonlinear DSGE models that allows the estimation of some important models that cannot be estimated with the current likelihood methods in the literature. In particular, this method neither requires that the number of shocks to be greater than the number of observed variables, nor that some shocks enter linearly in the model. In addition, the method allows for the presence of unobserved non-stochastic state variables, such as capital.

This method implicitly uses an invertible approximation of the policy function, according to which the shocks can be recovered uniquely from some of the control variables. It then uses the inverse of the policy function with respect to the shocks to calculate a second-order Taylor approximation of the likelihood. Once this likelihood function has been obtained it can be used either in a maximum likelihood estimation framework or in a Bayesian posterior simulation algorithm. We opted for Bayesian posterior simulation algorithm: computationally the method is much faster than previous methods, mainly because it does not require simulation to evaluate the likelihood, and does not solve systems of polynomial equations.

To demonstrate the efficacy of the technique, we apply it to the well-known neoclassical growth model of Fernández-Villaverde (2010) using US data from 1959 Q1–2019 Q4. In addition, to estimate the models resulting from first and second-order approximation, we modify and estimate the growth model by introducing heteroscedasticity through GARCH processes, including restricted and unrestricted versions of GARCH model (to capture uncertainty in shocks).

Our findings suggest that the nonlinear model performs better than the linear model based on computed log marginal, predictive likelihood, and posterior probability. The common factor GARCH model solved with second-order approximations is the best-performing model among the four estimated models, based on above mentioned measures. Furthermore, posterior analyses of a common factor GARCH model reveal that monetary policy shock is a primary driving force of uncertainty in the economy. Also, the standard deviation of the parameters is estimated to be much higher for the common factor GARCH model relative to the unrestricted GARCH model.

However, the method proposed is not applicable in all cases, as it requires the policy function to be monotonic near the steady state. Furthermore, in this chapter, the analysis is restricted to a) a case in which the number of shocks equals the number of observed variables and to b) a second-order approximation. Future research could analyze the effect of relaxing these two restrictions. For instance, it could explore cases where the number of shocks differs from the number of observed variables, or those where higher-order approximations are used, which could contribute to a more comprehensive understanding of the relationship between shocks and policy functions, and provide even more accurate estimation.

Name	**Model equation**
FOC consumption:	$$\mathrm{d}_t\left(\tilde{c}_t - h\tilde{c}_{t-1}\tfrac{z_{t-1}}{z_t}\right)^{-1} - h\beta E_t d_{t+1}\left(\tilde{c}_{t+1}\tfrac{z_{t+1}}{z_t} - h\tilde{c}_t\right)^{-1} = \tilde{\lambda}_t \quad (2.1)$$
Euler equation:	$$\tilde{\lambda}_t = \beta E_t\{\tilde{\lambda}_{t+1}\frac{z_t}{z_{t+1}}\frac{R_t}{\Pi_{t+1}}\} \quad (2.2)$$
FOC capital utilization:	$$\tilde{r}_t = \gamma_1 + \gamma_2(u_t - 1) \quad (2.3)$$
FOC capital:	$$\begin{aligned}\tilde{q}_t = \beta E_t\Big\{&\frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t}\frac{z_t}{z_{t+1}}\frac{\mu_t}{\mu_{t+1}}((1-\delta)\tilde{q}_{t+1} + \tilde{r}_{t+1}u_{t+1} - \\ &-(\gamma_1(u_{t+1}-1) + \frac{\gamma_2}{2}(u_{t+1}-1)^2))\Big\}\end{aligned} \quad (2.4)$$
FOC investment:	$$\begin{aligned}1 = \tilde{q}_t\Bigg(&1 - S\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right] - S'\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right]\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\frac{\tilde{z}_t}{\tilde{z}_{t-1}}\Bigg) + \\ &+\beta E_t\tilde{q}_{t+1}\frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t}\frac{z_t}{z_{t+1}}S'\left[\frac{\tilde{x}_{t+1}}{\tilde{x}_t}\frac{z_{t+1}}{z_t}\right]\left(\frac{\tilde{x}_{t+1}}{\tilde{x}_t}\frac{\tilde{z}_{t+1}}{\tilde{z}_t}\right)^2\end{aligned} \quad (2.5)$$
	where: $$S\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right] = \frac{\kappa}{2}\left(\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\frac{z_t}{z_{t-1}} - \exp\Lambda_z\right)^2$$ $$S'\left[\frac{\tilde{x}_{t+1}}{\tilde{x}_t}\frac{z_{t+1}}{z_t}\right] = \kappa\left(\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\frac{z_t}{z_{t-1}} - \exp\Lambda_z\right)$$

Wage setting 1:	$$f_t = \frac{\eta - 1}{\eta}(\tilde{\omega}_t^*)^{1-\eta}\tilde{\lambda}_t(\tilde{\omega}_t)^{\eta} l_t^d + \\ + \beta\theta_\omega E_t \left(\frac{\Pi_t^{\chi_\omega}}{\Pi_{t+1}}\right)^{1-\eta}\left(\frac{\tilde{\omega}_{t+1}^*}{\tilde{\omega}_t^*}\frac{z_{t+1}}{z_t}\right)^{\eta-1} f_{t+1} \quad (2.6)$$
Wage setting 2:	$$f_t = \psi d_t \varphi_t (\Pi_t^{*\omega})^{-\eta(1+\gamma)}(l_t^d)^{1+\gamma} + \\ + \beta\theta_\omega E_t \left(\frac{\Pi_t^{\chi_\omega}}{\Pi_{t+1}}\right)^{\eta(1+\gamma)}\left(\frac{\tilde{\omega}_{t+1}^*}{\tilde{\omega}_t^*}\frac{z_{t+1}}{z_t}\right)^{\eta(1+\gamma)} f_{t+1} \quad (2.7)$$
Firm price setting 1:	$$g_t^1 = \tilde{\lambda}_t mc_t \tilde{y}_t^d + \beta\theta_p E_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1 \quad (2.8)$$
Firm price setting 2:	$$g_t^2 = \tilde{\lambda}_t \Pi_t^* \tilde{y}_t^d + \beta\theta_p E_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{1-\varepsilon}\left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2 \quad (2.9)$$
Firm price setting 3:	$$\varepsilon g_t^1 = (\varepsilon - 1) g_t^2 \quad (2.10)$$
Optimal capital labor ratio:	$$\frac{u_t \tilde{k}_{t-1}}{l_t^d} = \frac{\alpha}{(1-\alpha)}\frac{\tilde{\omega}_t}{\tilde{r}_t}\frac{z_t}{z_{t-1}}\frac{\mu_t}{\mu_{t-1}} \quad (2.11)$$
Marginal costs:	$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}(\tilde{\omega}_t)^{1-\alpha}\tilde{r}_t^{\alpha} \quad (2.12)$$

Law of motion wages:	$$1 = \theta_\omega \left(\frac{\Pi_{t-1}^{\chi_\omega}}{\Pi_t} \right)^{1-\eta} \left(\frac{\tilde{\omega}_{t-1}}{\tilde{\omega}_t} \frac{z_{t-1}}{z_t} \right)^{1-\eta} + (1-\theta_\omega) \left(\Pi_t^{*\omega}\right)^{1-\eta} \quad (2.13)$$
Law of motion prices:	$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t} \right)^{1-\varepsilon} + (1-\theta_p) \Pi_t^{*1-\varepsilon} \quad (2.14)$$
Taylor Rule:	$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\gamma_\Pi} \left(\frac{\frac{\tilde{y}_t^d}{\tilde{y}_{t-1}^d} \frac{z_t}{z_{t-1}}}{\Lambda_{y^d}} \right)^{\gamma_y} \right)^{1-\gamma_R} \exp(m_t) \quad (2.15)$$
Resource constraint:	$$\tilde{y}_t^d = \tilde{c}_t + \tilde{x}_t + \frac{z_{t-1}}{z_t} \frac{\mu_{t-1}}{\mu_t} \left(\gamma_1 (u_t - 1) + \frac{\gamma_2}{2} (u_t - 1)^2 \right) \tilde{k}_{t-1} \quad (2.16)$$
Aggregate production:	$$\tilde{y}_t^d = \frac{\frac{A_t}{A_{t-1}} \frac{z_{t-1}}{z_t} \left(u_t \tilde{k}_{t-1} \right)^\alpha \left(l_t^d \right)^{1-\alpha} - \phi}{v_t^p} \quad (2.17)$$
Aggregate labor market:	$$l_t = v_t^\omega l_t^d \quad (2.18)$$
LOM Price dispersion term:	$$v_t^p = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t} \right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p) \Pi_t^{*-\varepsilon} \quad (2.19)$$

LOM Wage dispersion term:	$$v_t^w = \theta_w \left(\frac{\tilde{\omega}_{t-1}}{\tilde{\omega}_t} \frac{\Pi_{t-1}^{\chi_w}}{\Pi_t} \right)^{-\eta} v_{t-1}^w + (1-\theta_w)(\Pi_t^{*w})^{-\eta} \quad (2.20)$$
Law of motion for capital:	$$t \tfrac{z_t}{z_{t-1}} \tfrac{\mu_t}{\mu_{t-1}} - (1-\delta)\tilde{k}_{t-1} - \tfrac{z_t}{z_{t-1}} \tfrac{\mu_t}{\mu_{t-1}} \left(1 - S\left[\tfrac{\tilde{x}_t}{\tilde{x}_{t-1}} \tfrac{z_t}{z_{t-1}}\right]\right) \tilde{x}_t = 0. \quad (2.21)$$
Profits:	$$F_t = \tilde{y}_t^d - \frac{1}{(1-\alpha)} \tilde{\omega}_t l_t^d \quad (2.22)$$
Definition optimal wage inflation:	$$\Pi_t^{*\omega} = \frac{\omega_t^*}{\omega_t}, \text{ where } \omega_t^* = \tilde{\omega}_t^* z_t, \ \omega_t = \tilde{\omega}_t z_t \quad (2.23)$$
Preference Shock:	$$\log d_t = \rho^d \log d_{t-1} + \varepsilon_{d,t} \quad (2.24)$$
Labor disutility Shock:	$$\log \varphi_t = \rho^\varphi \log \varphi_{t-1} + \varepsilon_{\varphi,t} \quad (2.25)$$
Investment specific technology:	$$\log \mu_{I,t} = \Lambda_\mu + \varepsilon_{\mu_I,t}, \text{ where } \mu_{I,t} = \frac{\mu_t}{\mu_{t-1}} \quad (2.26)$$
Neutral technology:	$$\log \mu_{A,t} = \Lambda_A + \varepsilon_{A,t}, \text{ where } \mu_{A,t} = \frac{A_t}{A_{t-1}} \quad (2.27)$$

Definition composite technology:	$$\mu_{z,t} = \mu_{A,t}^{\frac{1}{(1-\alpha)}} \mu_{I,t}^{\frac{\alpha}{(1-\alpha)}}, \text{ where } \mu_{z,t} = \frac{z_t}{z_{t-1}} \quad (2.28)$$
and	$$\gamma_1 = \mu_{z,t} \frac{\mu_{I,t}}{\beta} - (1-\delta) \quad (2.29)$$
and	$$\Lambda_x = \exp \Lambda_z \quad (2.30)$$
and	$$\Lambda_z = \frac{\Lambda_A + \alpha \Lambda_\mu}{1-\alpha} \quad (2.31)$$
observation equation 1	$$y_t^o = \log(\tilde{y}_t^d) - \log(\tilde{y}_{t-1}^d) + \log(\mu_{z,t}) \quad (2.32)$$
observation equation 2	$$\omega_t^o = \log(\tilde{\omega}_t) - \log(\tilde{\omega}_{t-1}) + \log(\mu_{z,t}) \quad (2.33)$$

Table 2.1: Model equilibrium

Variable	**Description**	**Variable**	**Description**
$\tilde{c}_t$	consumption	Π	Inflation
d_t	shock to intertemporal preferences	$\tilde{\lambda}_t$	Lagrange multiplier
$\mu_{z,t}$	trend growth rate of the economy	$\mu_{I,t}$	growth rate of investment-specific technology growth
$\mu_{A,t}$	growth rate of neutral technology	R_t	Nominal Interest rate
$\tilde{r}_t$	rental rate of capital	$\tilde{x}_t$	investment
u_t	capacity utilization	$\tilde{q}_t$	Tobin marginal q
f_t	recursive formulation of wage setting	l_t^d	aggregate labor demand
$\tilde{\omega}_t$	real wage	$\tilde{\omega}_t^*$	optimal real wage
Π_t^*	optimal price inflation	$\Pi_t^{\omega *}$	optimal wage inflation
$\tilde{y}_t^d$	aggregate output	mc_t	marginal costs
k_t	capital	l_t	aggregate labor bundle
g_t^1	variable 1 for recursive formulation of price setting	g_t^2	variable 2 for recursive formulation of price setting
v_t^p	price dispersion term	v_t^ω	wage dispersion term
φ_t	labor disutility shock	F_t	firm profits
ω_t	non-detrended real wage	ω_t^*	non-detrended optimal real wage

Table 2.2: Main variables

Parameter	**Description**	**Distr.**	**Mean**	**St.Dev**
$100(1/\beta - 1)$	β is discount factor	Gamma	0.25	0.1
$100(\Pi - 1)$	Target inflation	Gamma	0.95	0.1
$100\Lambda_\mu$	Long-run growth investment specific	Normal	0.34	0.1
$100\Lambda_A$	Long-run growth productivity	Normal	0.178	0.075
h	Habit persistence	Beta	0.7	0.1
ψ	Labor disutility parameter	Normal	9	3
θ_p	Fraction of firms with fixed prices	Beta	0.5	0.1
χ	Price indexation	Beta	0.5	0.15
θ_ω	Fraction of firms with fixed wages	Beta	0.5	0.1
χ_ω	Wage indexation	Beta	0.5	0.1
γ_R	Taylor rule coefficient past rates	Beta	0.75	0.1
γ_Y	Taylor rule coefficient demand	Normal	0.12	0.05
γ_π	Taylor rule coefficient inflation	Normal	1.5	0.125
γ	The inverse of the Frisch labor supply elasticity	Normal	1	0.25
κ	Investment adjustment cost	Normal	4	1.5
α	Cobb–Douglas labor	Normal	0.3	0.025
ρ_d	Persistence demand shock	Beta	0.5	0.2
ρ_ϕ	Persistence labor supply shock	Beta	0.5	0.2
$\exp \sigma_A$	SD long-run productivity	IG	0.1	2
$\exp \sigma_d$	SD demand shock innovation	IG	0.1	2
$\exp \sigma_\varphi$	SD labor supply shock innovation	IG	0.1	2
$\exp \sigma_\mu$	SD investment productivity shock innovation	IG	0.1	2
$\exp \sigma_m$	SD monetary shock	IG	0.1	2
δ	depreciation rate	Beta	0.025	0.015
γ_2	capital utilization, quadratic term	Beta	0.01	0.03

Table 2.3: The prior distribution

Parameter	Description	Distr.	Mean	St.Dev
ρ_1^d	GARCH parameter	Beta	0.7	0.046
ρ_2^d	unrestricted GARCH parameter	Beta	0.2	0.12
ρ_1^φ	unrestricted GARCH parameter	Beta	0.7	0.046
ρ_2^φ	unrestricted GARCH parameter	Beta	0.2	0.12
ρ_1^μ	unrestricted GARCH parameter	Beta	0.7	0.046
ρ_2^μ	unrestricted GARCH parameter	Beta	0.2	0.12
ρ_1^A	unrestricted GARCH parameter	Beta	0.7	0.046
ρ_2^A	unrestricted GARCH parameter	Beta	0.2	0.12
ρ_1^m	unrestricted GARCH parameter	Beta	0.7	0.046
ρ_2^m	unrestricted GARCH parameter	Beta	0.2	0.12
δ^d	factor GARCH parameter	Normal	0	0.2
δ^φ	factor GARCH parameter	Normal	0	0.2
δ^μ	factor GARCH parameter	Normal	0	0.2
δ^A	factor GARCH parameter	Normal	0	0.2
δ^m	factor GARCH parameter	Normal	0	0.2

Table 2.4: The prior distribution for parameters of GARCH models

Database name	**Transformation**	**Model notation**	**Description**
Relative Price of Investment Goods (PIRIC)	$-\Delta\log(x)$	$\log(\mu_{I,t})$	log of growth rate of investment specific technology growth
Real gross domestic product per capita (A939RX0Q 048SBEA)	$\Delta\log(x)$	y_t^o	real output per capita growth
Nonfarm Business Sector: Real Compensation Per Hour (COMPRNFB)	$\Delta\log(x)$	ω_t^o	real wages per capita growth
Gross Domestic Product: Implicit Price Deflator (GDPDEF)	$\Delta\log(x)$	Π_t	log of gross inflation
Effective Federal Funds Rate (FEDFUNDS)	$\log(1 + x/400)$	R_t	log of gross nominal interest rate

Table 2.5: US Data description

Note: All variables were obtained from Federal Reserve Bank of St. Louis' FRED database. The column 'Transformation' indicates how we transformed the original data in the database to match the model variable in the column 'Model notation'. y_t^o and ω_t^o are defined in the observation equations (2.32)-(2.33) in Table 2.1.

	Distr.	Prior distribution		Posterior distribution			
				M_1 - **First-order**		M_2 - **Second-order**	
		Mean	**St.Dev.**	**Mean**	**Credible interval**	**Mean**	**Credible interval**
β	Gamma	0.998	0.1	0.9983	[0.9972, 0.9992]	0.9990	[0.9983, 0.9996]
h	Beta	0.7	0.1	0.5523	[0.4448, 0.6488]	0.8382	[0.8149, 0.8592]
ψ	Normal	9	3	9.0389	[3.4819, 14.8717]	9.8073	[5.7470, 15.1635]
γ	Normal	1	0.25	0.0761	[-0.0775, 0.3174]	1.6953	[1.2575, 2.1099]
κ	Normal	4	1.5	5.8466	[3.6256, 8.3106]	0.2005	[0.1432, 0.2709]
α	Normal	0.3	0.025	0.2937	[0.2521, 0.3351]	0.3069	[0.2643, 0.3501]
θ_p	Beta	0.5	0.1	0.6466	[0.5714, 0.7114]	0.5948	[0.5033, 0.6539]
χ	Beta	0.5	0.1	0.1228	[0.0433, 0.2403]	0.3335	[0.1624, 0.5760]
θ_ω	Beta	0.5	0.1	0.3361	[0.2166, 0.5528]	0.3138	[0.2693, 0.3598]
χ_ω	Beta	0.5	0.1	0.5408	[0.3551, 0.7189]	0.4496	[0.2732, 0.6178]
γ_R	Beta	0.75	0.1	0.6830	[0.6276, 0.7297]	0.7086	[0.6600, 0.7516]
γ_Y	Normal	0.120	0.05	0.1791	[0.0962, 0.2662]	0.2971	[0.2376, 0.3557]
γ_π	Normal	1.5	0.125	1.6409	[1.4678, 1.8212]	1.7790	[1.6430, 1.9509]
$\bar{\Pi}$	Gamma	1.01	0.1	1.0089	[1.0075, 1.0103]	1.0082	[1.0071, 1.0095]
ρ_d	Beta	0.5	0.2	0.9140	[0.8672, 0.9572]	0.7451	[0.6844, 0.8066]
ρ_ϕ	Beta	0.5	0.2	0.9948	[0.9892, 0.9989]	0.9931	[0.9860, 0.9984]

Table 2.6: Prior and Posterior distribution for structural parameters of M_1 and M_2

	Distr.	Prior distribution		Posterior distribution			
		Mean	St.Dev.	M_1 - **First-order**		M_2 - **Second-order**	
				Mean	**Credible interval**	**Mean**	**Credible interval**
Λ_μ	Normal	0.34	0.1	0.0056	[0.0049, 0.0063]	0.0059	[0.0051, 0.0065]
Λ_A	Normal	0.178	0.075	0.0007	[-0.0002, 0.0016]	0.0010	[0.0001, 0.0018]
γ_2	Beta	0.01	0.03	0.2704	[0.1168, 0.4904]	0.2768	[0.1075, 0.4830]
δ	Beta	0.025	0.015	0.0652	[0.0344, 0.1108]	0.0396	[0.0331, 0.0517]
$\exp \sigma_A$	IG*	0.1	2	-4.5647	[-4.5647, -4.3140]	-4.5792	[-4.7789, -4.3977]
$\exp \sigma_d$	IG	0.1	2	-3.6454	[-3.9580, -3.2126]	-2.5250	[-2.7005, -2.3576]
$\exp \sigma_\varphi$	IG	0.1	2	-3.6866	[-3.9529, -3.4014]	-2.4428	[-2.6186, -2.2635]
$\exp \sigma_\mu$	IG	0.1	2	-5.0890	[-5.1792, -4.9960]	-5.0900	[-5.1741, -5.0024]
$\exp \sigma_m$	IG	0.1	2	-5.7512	[-5.8581, -5.6345]	-5.7818	[-5.8811, -5.6742]

Table 2.7: Prior and Posterior distribution for structural parameters of M_1 and M_2

Note:* - Inverse Gamma Distribution. 2. ** - The posterior values for the shocks is in $\log((\sigma_x))$

	Distr.	Prior distribution		Posterior distribution			
		Mean	St.Dev.	$M_{2,G}$ - **GARCH**		$M_{2,fG}$ - **fGARCH**	
				Mean	**Credible interval**	**Mean**	**Credible interval**
β	Gamma	0.998	0.1	0.9987	[0.9977, 0.9994]	0.9986	[0.9977, 0.9984]
h	Beta	0.7	0.1	0.5525	[0.4622, 0.6492]	0.5645	[0.4695, 0.6619]
ψ	Normal	9	3	9.0195	[3.4322, 14.8874]	9.0756	[3.3670, 14.8716]
γ	Normal	1	0.25	-0.0382	[-0.0653, -0.0061]	0.0665	[-0.0866, 0.3564]
κ	Normal	4	1.5	5.5269	[3.0775, 8.2445]	6.2402	[3.9788, 8.7965]
α	Normal	0.3	0.025	0.2730	[0.2322, 0.3119]	0.2846	[0.2425, 0.3257]
θ_p	Beta	0.5	0.1	0.5598	[0.5059, 0.6004]	0.5577	[0.4911, 0.6144]
χ	Beta	0.5	0.1	0.2181	[0.1669, 0.2582]	0.1574	[0.0584, 0.2929]
θ_ω	Beta	0.5	0.1	0.3353	[0.2936, 0.3809]	0.2898	[0.1424, 0.6048]
χ_ω	Beta	0.5	0.1	0.5171	[0.3225, 0.7110]	0.4805	[0.2884, 0.6749]
γ_R	Beta	0.75	0.1	0.6872	[0.6468, 0.7187]	0.8038	[0.7687, 0.8349]
γ_Y	Normal	0.120	0.05	0.1688	[0.1094, 0.2217]	0.1770	[0.0937, 0.2617]
γ_π	Normal	1.5	0.125	1.6800	[1.5929, 1.7591]	1.6974	[1.5432, 1.8576]
$\bar{\Pi}$	Gamma	1.01	0.1	1.0073	[1.0063, 1.0083]	1.0064	[1.0057, 1.0071]
ρ_d	Beta	0.5	0.2	0.9313	[0.9172, 0.9401]	0.9152	[0.8830, 0.9436]
ρ_ϕ	Beta	0.5	0.2	0.9954	[0.9912, 0.9989]	0.9954	[0.9911, 0.9987]

Table 2.8: Prior and Posterior distribution for structural parameters of $M_{2,G}$ and $M_{2,fG}$

	Distr.	Prior distribution		Posterior distribution			
				$M_{2,G}$ - **GARCH**		$M_{2,fG}$ - **fGARCH**	
		Mean	**St.Dev.**	**Mean**	**Credible interval**	**Mean**	**Credible interval**
Λ_μ	Normal	0.34	0.1	0.0054	[0.0048, 0.0060]	0.0058	[0.0051, 0.0064]
Λ_A	Normal	0.178	0.075	0.0006	[-0.0002, 0.0014]	0.0003	[-0.0005, 0.0010]
ρ_1^d	Beta	0.7	0.046	0.6533	[0.6101, 0.6886]	0.6019	[0.4979, 0.6550]
ρ_2^d	Beta	0.2	0.12	0.1377	[0.1243, 0.1552]	-	-
ρ_1^φ	Beta	0.7	0.046	0.6933	[0.5972, 0.7795]	-	-
ρ_2^φ	Beta	0.2	0.12	0.0568	[0.0159, 0.1048]	-	-
ρ_1^μ	Beta	0.7	0.046	0.6610	[0.5711, 0.8102]	-	-
ρ_2^μ	Beta	0.2	0.12	0.1377	[0.0843, 0.1956]	-	-
ρ_1^A	Beta	0.7	0.046	0.6974	[0.5934, 0.7877]	-	-
ρ_2^A	Beta	0.2	0.12	0.0810	[0.0251, 0.1448]	-	-
ρ_1^m	Beta	0.7	0.046	0.7194	[0.6216, 0.8128]	-	-
ρ_2^m	Beta	0.2	0.12	0.0483	[0.0321, 0.0680]	-	-
δ^d	Normal	0	0.2	-	-	-0.0392	[0.0274, 0.0517]
δ^φ	Normal	0	0.2	-	-	-0.0064	[-0.0021, 0.0160]
δ^μ	Normal	0	0.2	-	-	-0.1302	[0.1007, 0.1716]
δ^A	Normal	0	0.2	-	-	0.0217	[-0.0493, 0.0029]
δ^m	Normal	0	0.2	-	-	0.6096	[-0.6770, -0.5622]
γ_2	Beta	0.01	0.03	0.3002	[0.1550, 0.4980]	0.3381	[0.1654, 0.5544]
δ	Beta	0.025	0.015	0.1137	[0.0618, 0.1897]	0.0900	[0.0494, 0.1503]

Table 2.9: Prior and Posterior distribution for structural parameters of $M_{2,G}$ and $M_{2,fG}$

	Distr.	Prior distribution		Posterior distribution			
		Mean	St.Dev.	$M_{2,G}$ - **GARCH**		$M_{2,fG}$ - **fGARCH**	
				Mean	**Credible interval**	**Mean**	**Credible interval**
$\exp \sigma_A$	IG*	0.1	2	-4.7088**	[-4.8680, -4.5454]	-2.9338	[-3.5957, -2.2367]
$\exp \sigma_d$	IG	0.1	2	-3.9732	[-4.0799, -3.8386]	-2.1175	[-2.7748, -1.4216]
$\exp \sigma_\varphi$	IG	0.1	2	-3.8355	[-4.000, -3.6370]	-1.9639	[-2.6691, -1.2032]
$\exp \sigma_\mu$	IG	0.1	2	-5.0391	[-5.1912, -4.8591]	-3.3121	[-3.9670, -2.6109]
$\exp \sigma_m$	IG	0.1	2	-5.7833	[-5.9118, -5.6464]	-4.2818	[-4.9142, -3.5850]

Table 2.10: Prior and Posterior distribution for structural parameters of $M_{2,G}$ and $M_{2,fG}$

Note:* - Inverse Gamma Distribution. 2. ** - The posterior values for the shocks is in $\log((\sigma_x))$

	Distr.	Prior distribution		Posterior distribution			
		Mean	St.Dev.	M_1 - **First-order**		M_2 - **Second-order**	
				Mean	**Credible interval**	**Mean**	**Credible interval**
h	Beta	0.7	0.1	0.5523	[0.4448, 0.6488]	0.8382	[0.8149, 0.8592]
γ	Normal	1	0.25	0.0761	[-0.0775, 0.3174]	1.6953	[1.2575, 2.1099]
κ	Normal	4	1.5	5.8466	[3.6256, 8.3106]	0.2005	[0.1432, 0.2709]
χ	Beta	0.5	0.1	0.1228	[0.0433, 0.2403]	0.3335	[0.1624, 0.5760]

Table 2.11: Prior and Posterior distribution for structural parameters of M_1 and M_2

Model variables	**Model** M_1	**Model** M_2	**Deviations**
Consumption, c	-2.068	-0.5128	-1.555
Investment, x	-3.123	-1.5038	-1.619
Labor demand, ld	-2.285	-0.9476	-1.338
Output, yd	-1.769	-0.1971	-1.572
Capital, k	-0.5121	1.5109	-2.022
Aggregate labor, l	-2.2851	-0.9473	-1.3378
Firm profits, F	-4.073	-2.4990	-1.5727

Table 2.12: Steady-state values for models M_1 and M_2

Model	Number of parameters	Log Marginal Likelihood	Predictive Likelihood	Pr(M)
M_1	25	4596.30	3782.58	0
M_2	25	4602.10	3794.59	0
$M_{2,G}$	35	4639.30	3792.19	0
$M_{2,fG}$	31	4711.60	3859.99	1

Table 2.13: Model Performance Values for M_1, M_2, $M_{2,G}$ and $M_{2,fG}$

Note: Log marginal likelihood is estimated by importance sampling (method of Geweke) using a normal distribution. The predictive likelihood is calculated for observations 45 to 244. $Pr(M)$ is the posterior probability of model M using the predictive likelihood.

Model	Number of parameters	Log Likelihood	Log prior
M_1	25	4702.30	-78.80
M_2	25	4720.90	-79.90

Table 2.14: Model Performance Values for M_1 and M_2

Note: Log likelihood and Log prior are calculated for M_1 and M_2 at a posterior mean.

Parameter	Estimated values
p_d	0.0039
p_φ	0.0001
p_μ	0.0434
p_A	0.0012
p_m	0.9514

Table 2.15: Proportions for $M_{2,fG}$

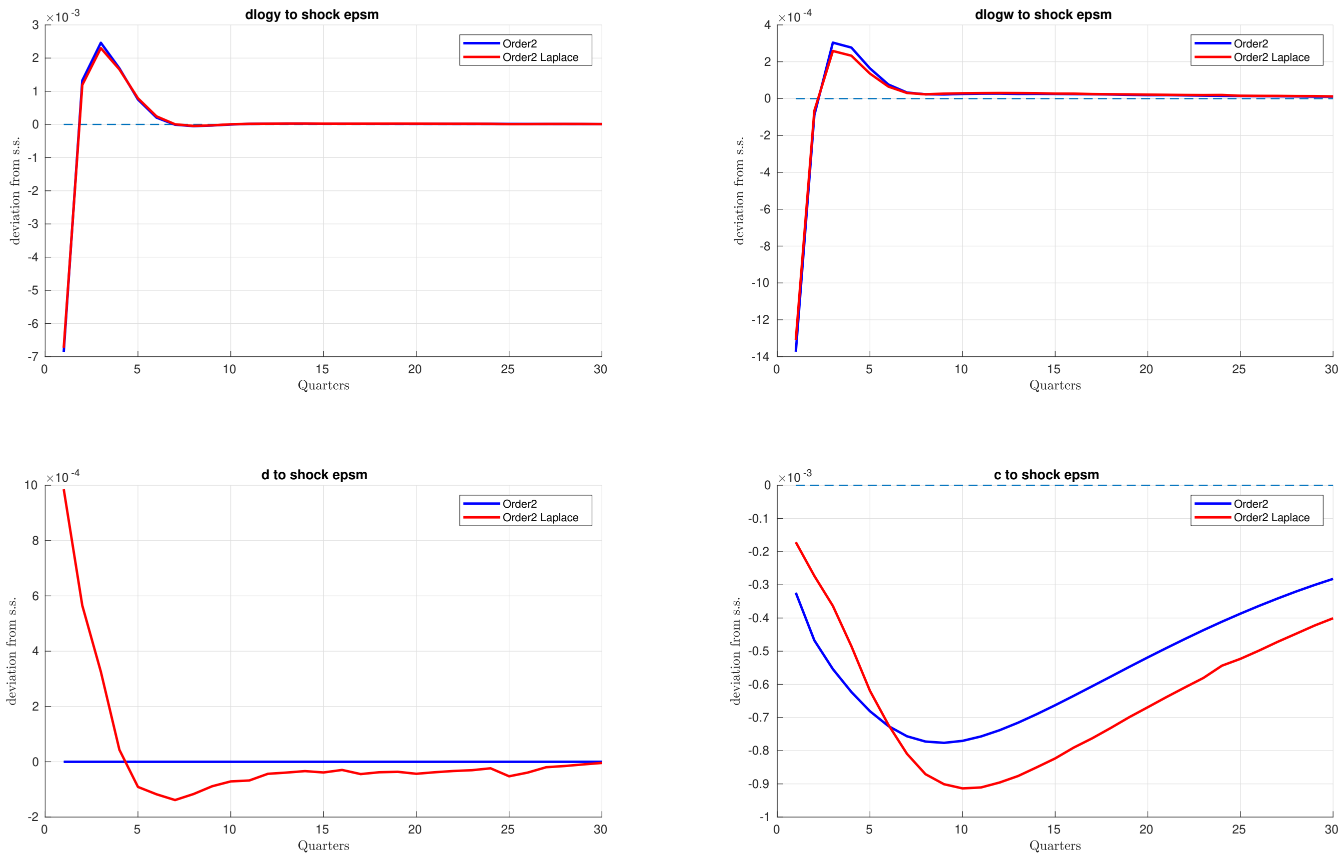

Figure 2.1: IRFs to a monetary policy shock: second-order perturbation methods versus Laplace approximated model

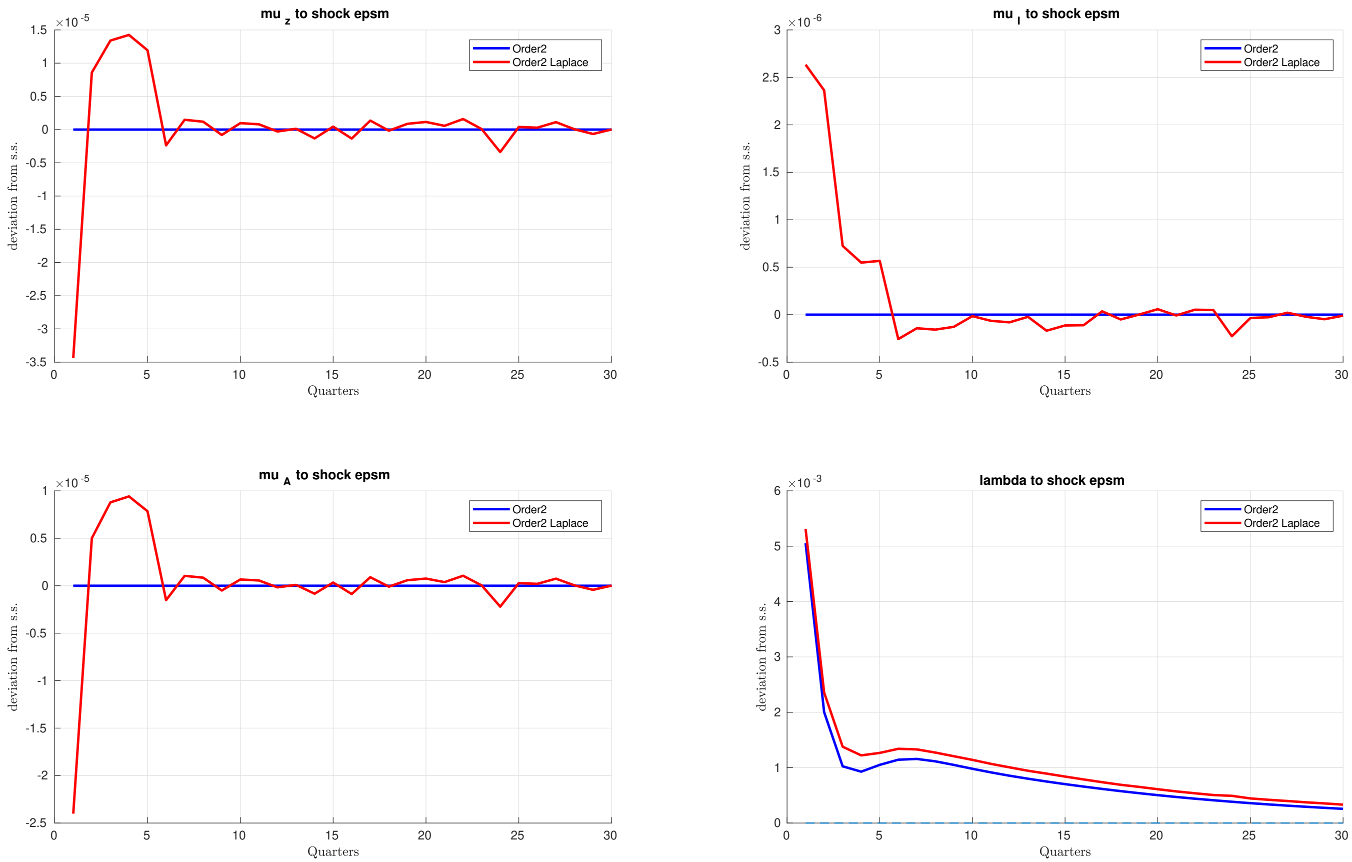

Figure 2.2: IRFs to a monetary policy shock: second-order perturbation methods versus Laplace approximated model

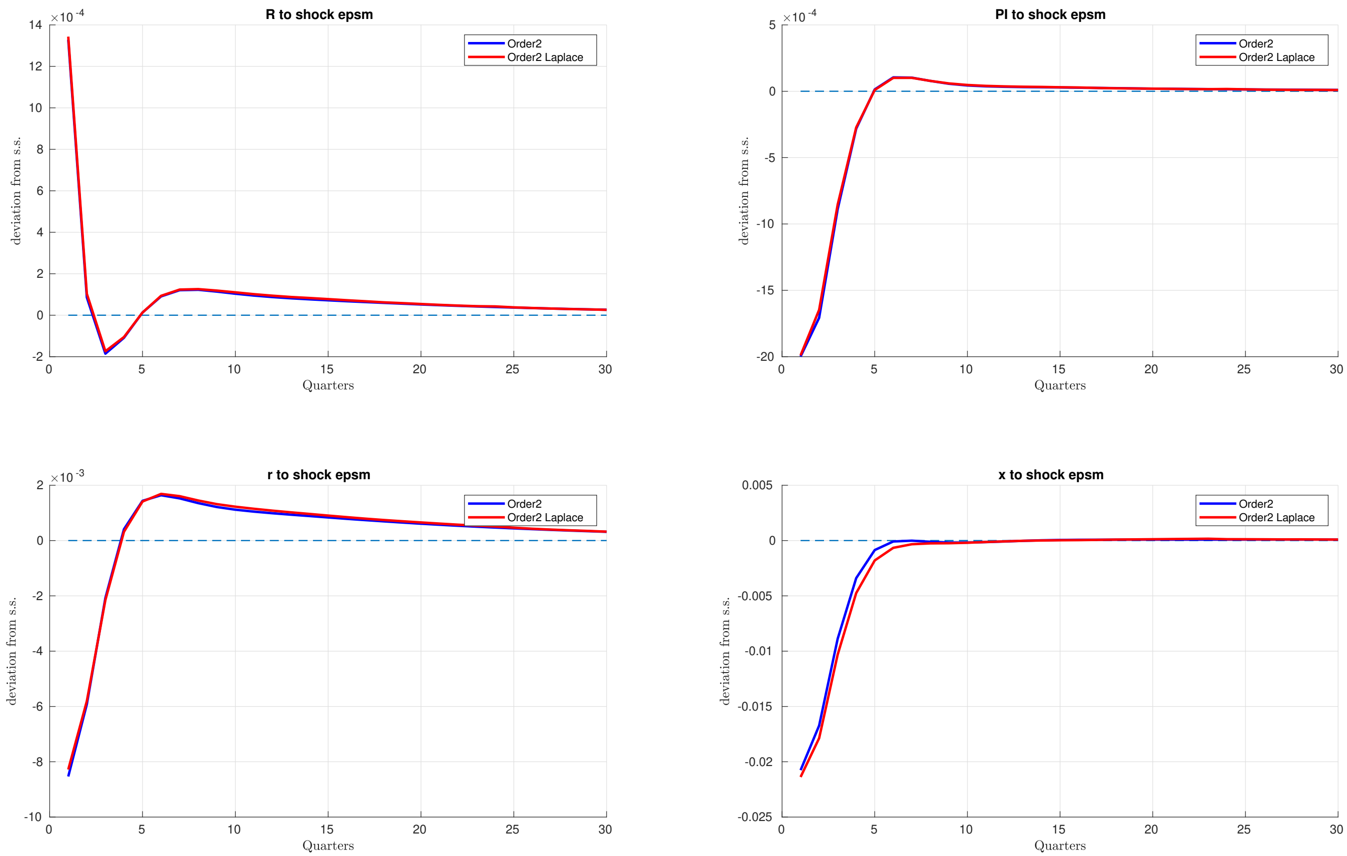

Figure 2.3: IRFs to a monetary policy shock: second-order perturbation methods versus Laplace approximated model

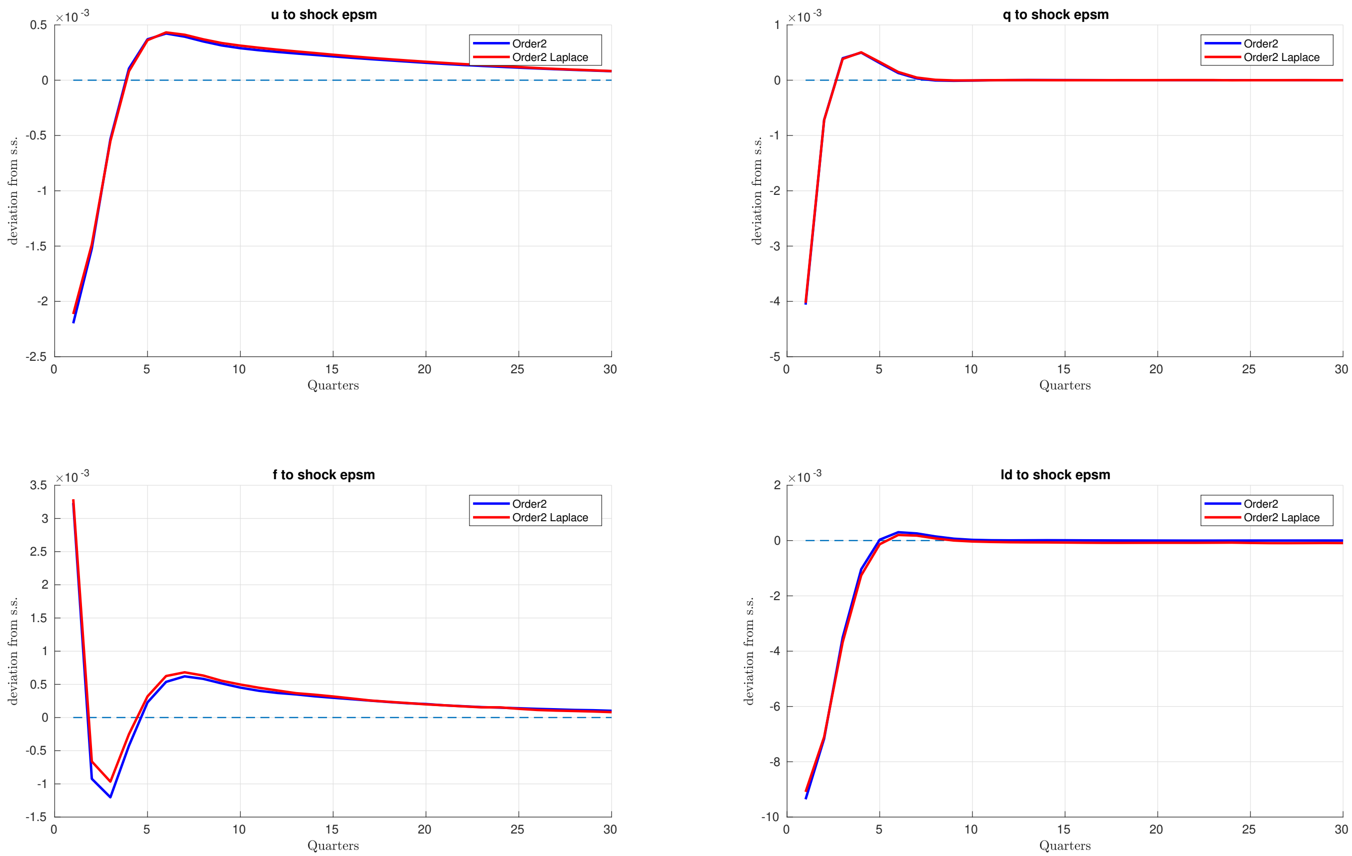

Figure 2.4: IRFs to a monetary policy shock: second-order perturbation methods versus Laplace approximated model

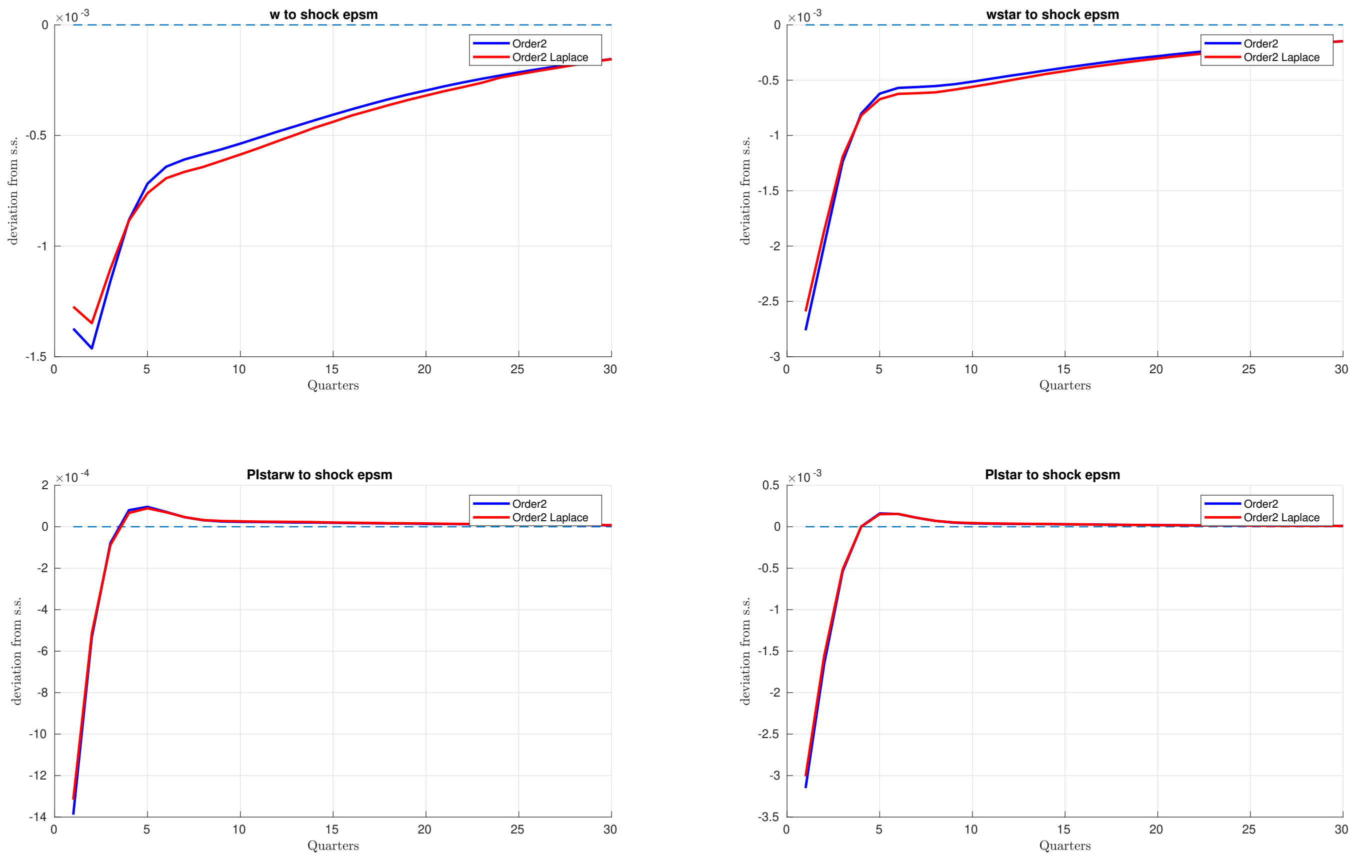

Figure 2.5: IRFs to a monetary policy shock: second-order perturbation methods versus Laplace approximated model

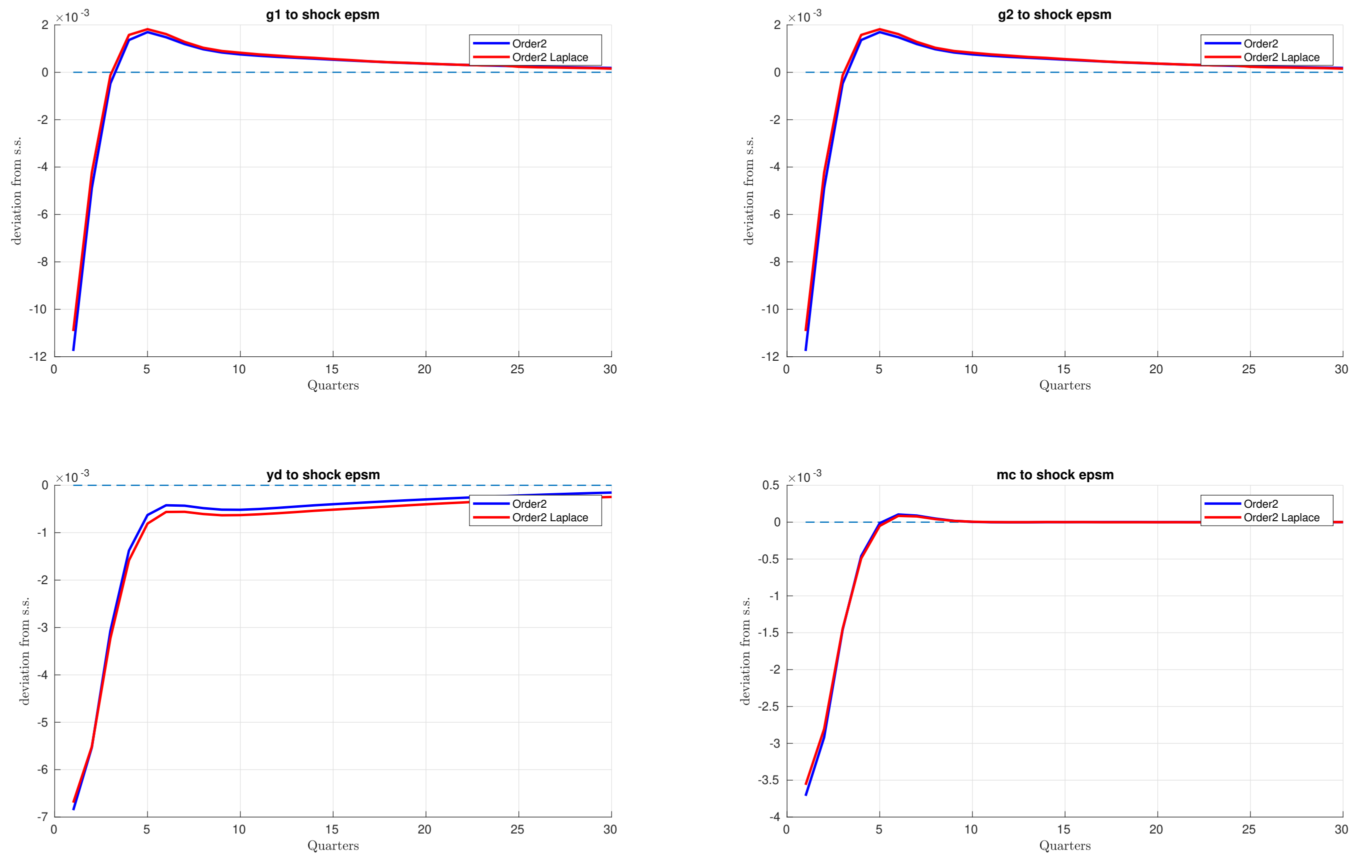

Figure 2.6: IRFs to a monetary policy shock: second-order perturbation methods versus Laplace approximated model

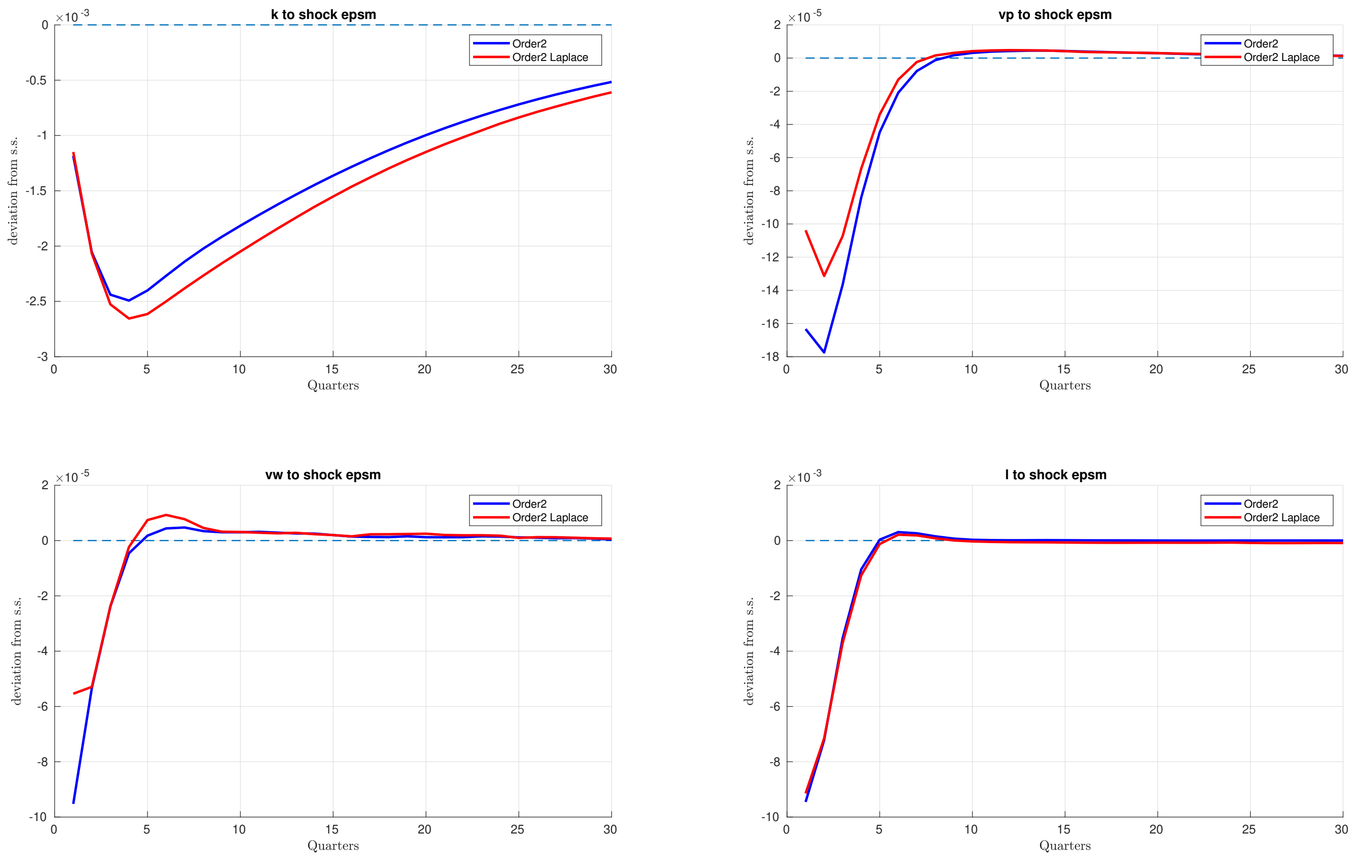

Figure 2.7: IRFs to a monetary policy shock: second-order perturbation methods versus Laplace approximated model

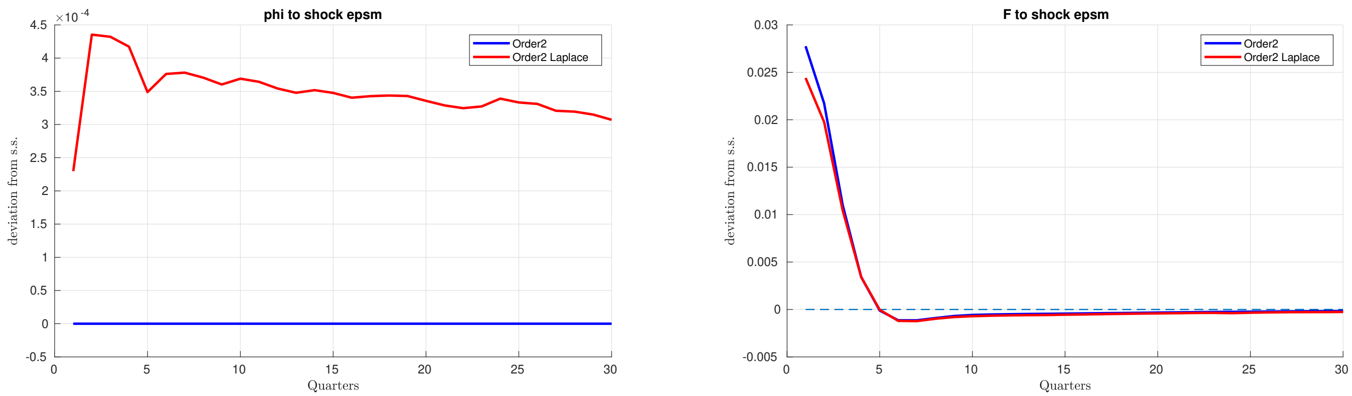

Figure 2.8: IRFs to a monetary policy shock: second-order perturbation methods versus Laplace approximated model

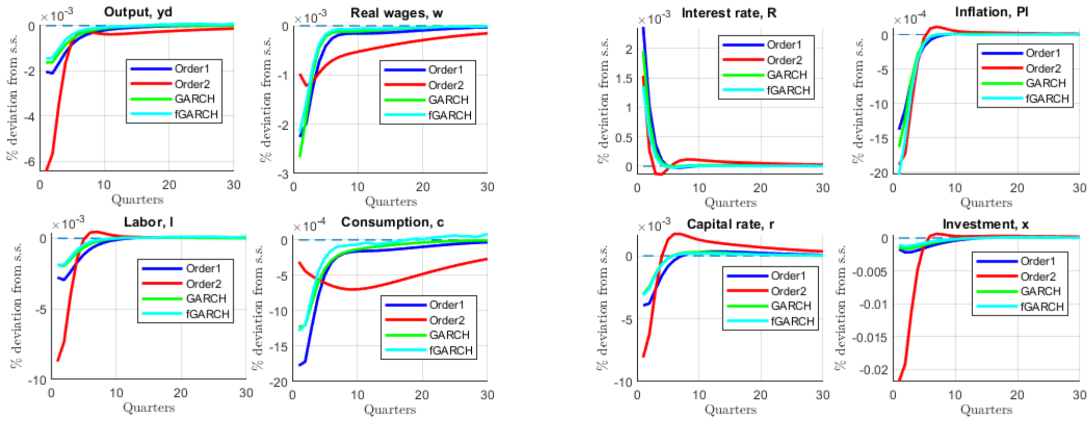

Figure 2.9: IRFs to monetary policy shock (positive)

Note: Response of macroeconomic variables to positive monetary policy shock (the plots are the posterior mode).

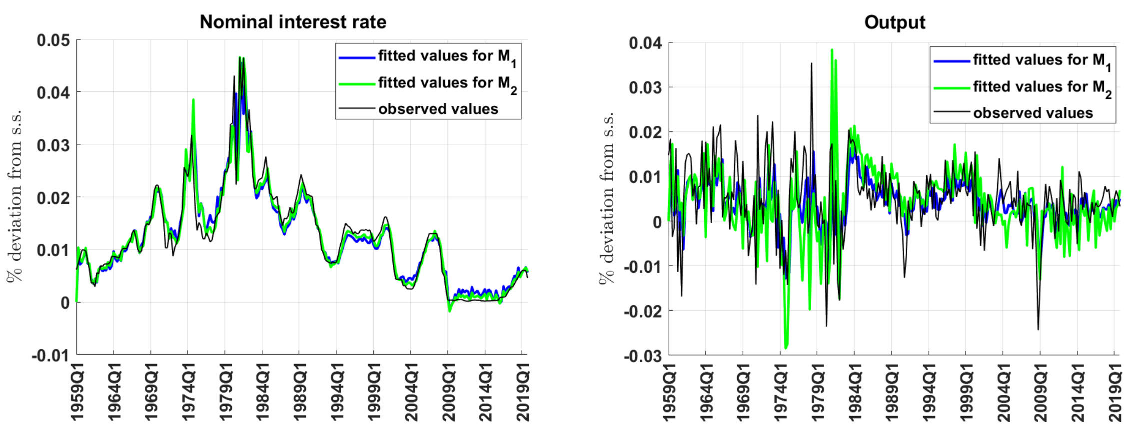

Figure 2.10: Fitted vs observed values for M_1 and M_2

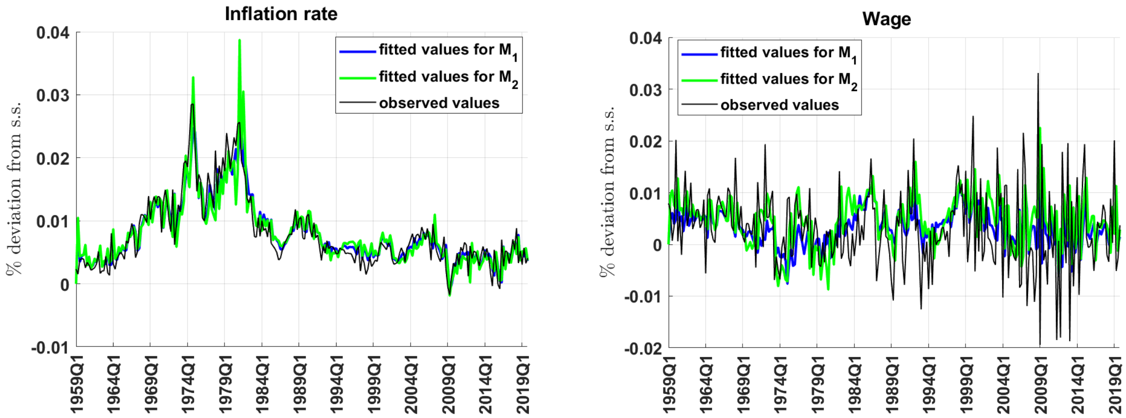

Figure 2.11: Fitted vs observed values for M_1 and M_2

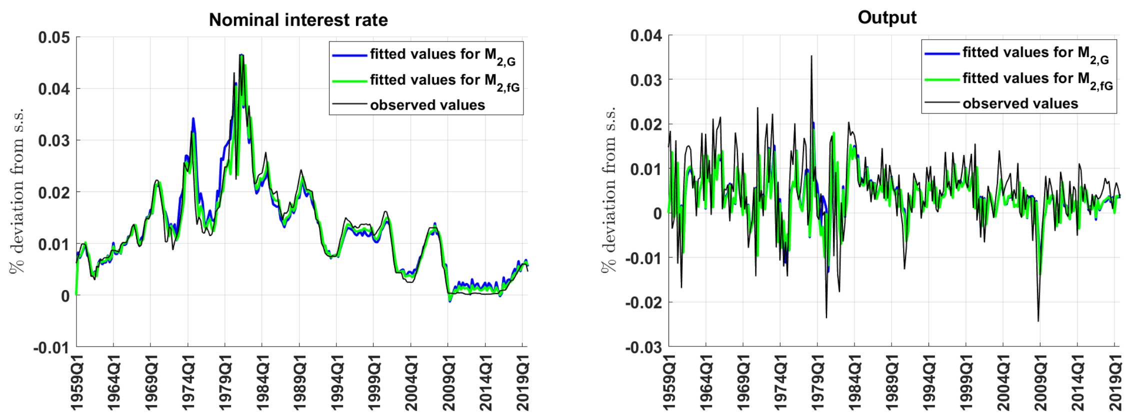

Figure 2.12: Fitted vs observed values for $M_{2,G}$ and $M_{2,fG}$

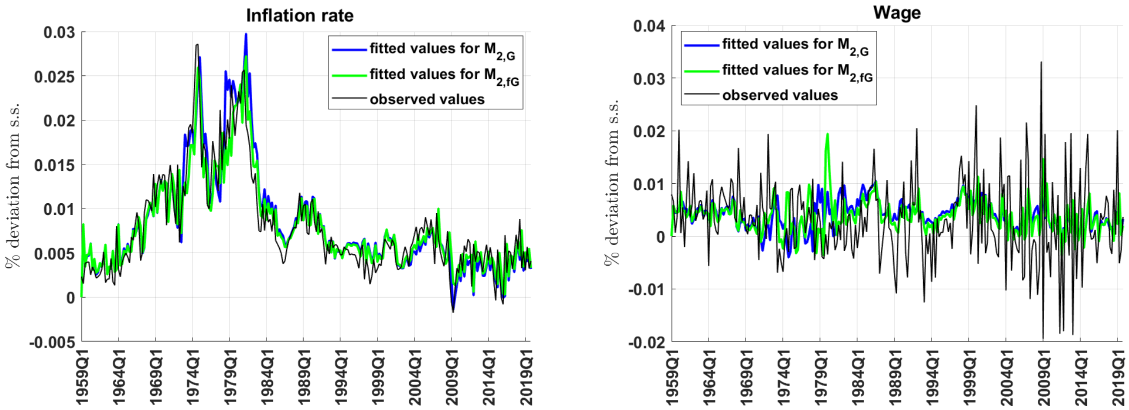

Figure 2.13: Fitted vs observed values for $M_{2,G}$ and $M_{2,fG}$

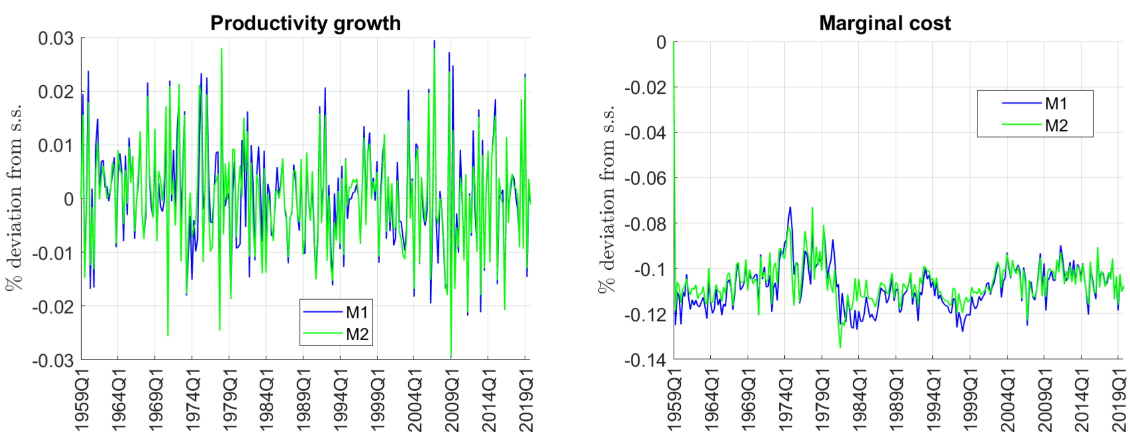

Figure 2.14: Posterior estimates of productivity growth and marginal cost for M_1 and M_2

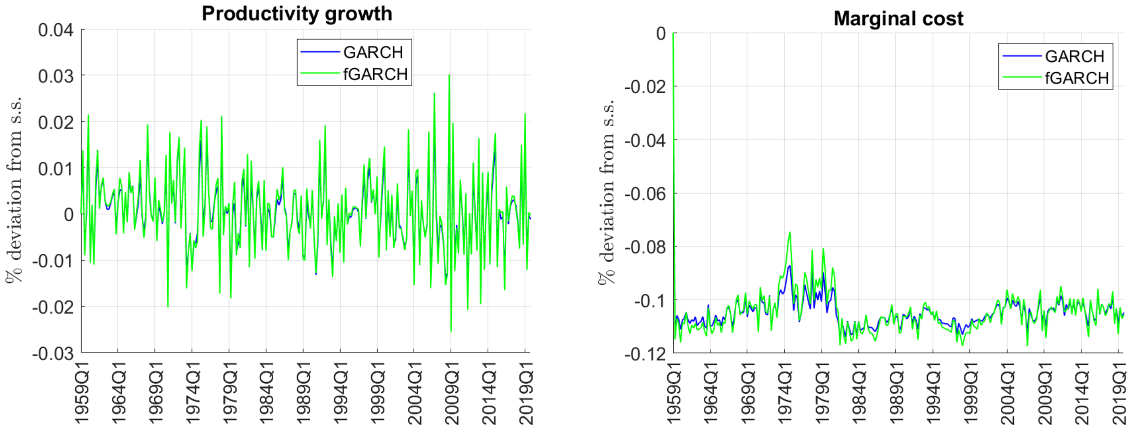

Figure 2.15: Posterior estimates of productivity growth and marginal cost for $M_{2,G}$ and $M_{2,fG}$

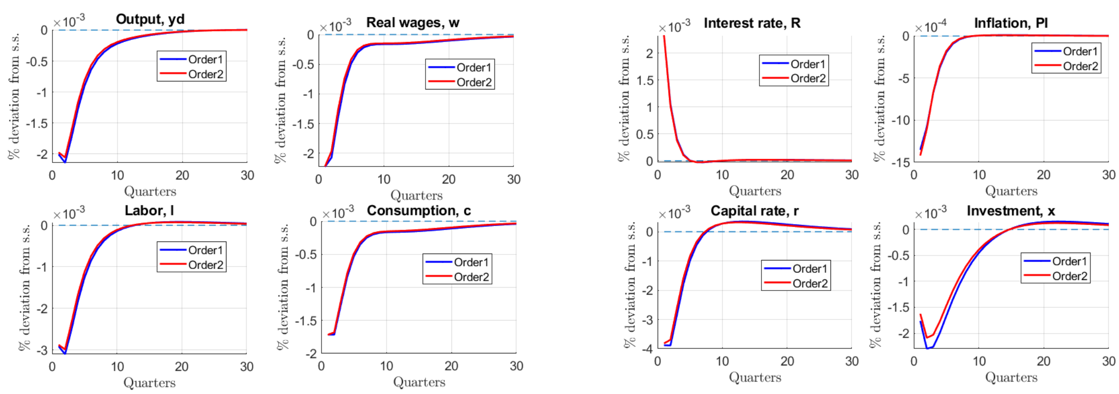

Figure 2.16: IRFs to monetary policy shocks for M_1 and M_2 using common posteriors

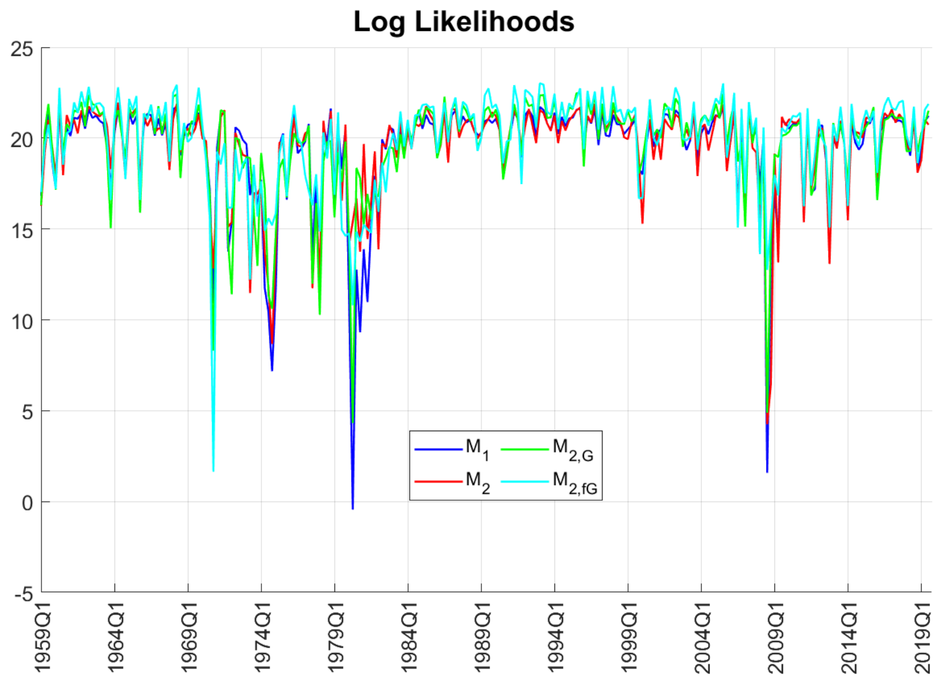

Figure 2.17: Log Likelihood for M_1, M_2, $M_{2,G}$ and $M_{2,fG}$

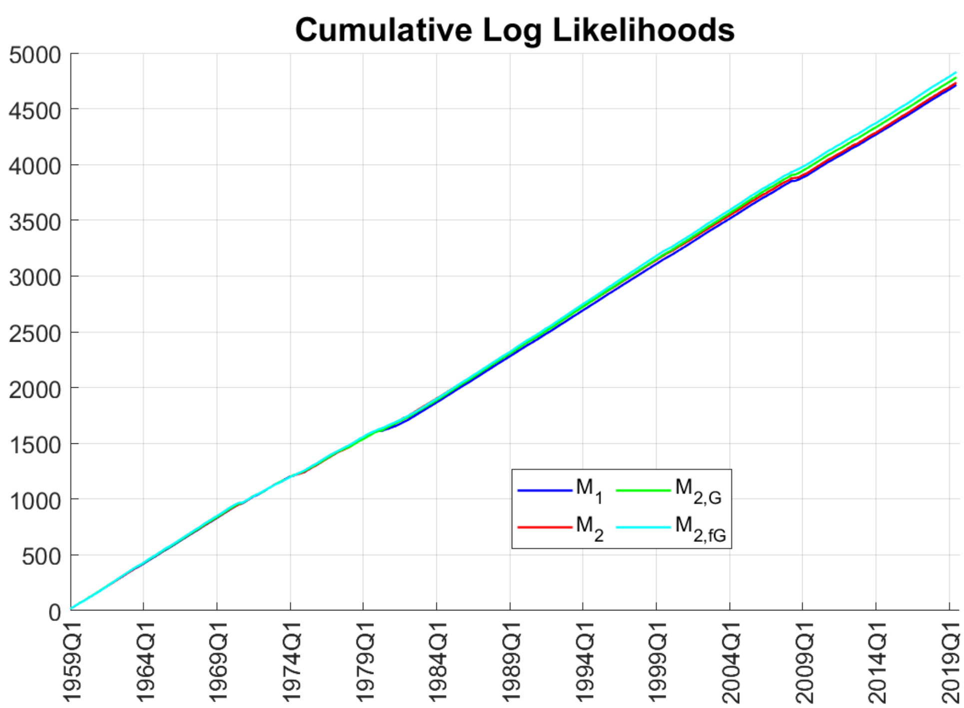

Figure 2.18: Cumulative Log Likelihoods for M_1, M_2, $M_{2,G}$ and $M_{2,fG}$

Chapter 3 Nonlinear analysis and estimation of Two Agent New Keynesian DSGE model

3.1 Introduction

A rapidly growing macroeconomics literature embraces various types of DSGE models, including models with heterogeneous agents, since heterogeneity is widespread in macroeconomic data. The data show that households vary substantially in terms of income, wealth, and consumption; and that firms do so in productivity, investment, and technology used.

Accordingly, numerous studies have sought to enhance knowledge of monetary and fiscal policy and their effect on aggregate economic performance; those efforts led to the development of Heterogeneous Agent New Keynesian (HANK) models. In contrast to the well-known Representative Agent New Keynesian model (RANK), HANK models incorporate market incompleteness and heterogeneity (Ahn et al. (2018)), Kaplan et al. (2018)), as well as the forward guidance puzzle (McKay et al. (2016)). Thus, these models feature a distribution of households with different levels of income, wealth, and borrowing constraints. A number of studies have adopted that new approach to modeling the New Keynesian framework to improve our understanding of the transmission of monetary and fiscal policy (Kaplan and Violante (2014), McKay and Reis (2016), Kaplan et al. (2018), Auclert and Rognlie (2018), Auclert (2019), Alves et al. (2020)).

Notably, Kaplan et al. (2018) developed a quantitative analytic HANK model with price rigidity, uninsurable income shocks, and liquid and illiquid assets holdings by households. They show that changes in the interest rate due to monetary policy shock influence consumption of households through direct (intertemporal substitution) and indirect effects (expansion of labor demand). Their result indicates that, in contrast to the RANK model, the HANK model's direct effect of interest rate shock is small, while the indirect effect is substantial. This finding indicates that monetary policy can be effective through the latter channel, i.e. general equilibrium response in household disposable income. Concerning fiscal role, Kaplan et al. (2018) found that transitory but significant interest rate cuts can effectively expand aggregate consumption because they lead to an immediate reduction in interest payments on government debt, which that translates to additional fiscal incentives for households. Overall, since the monetary transmission mechanism of HANK models differs from that of RANK, this changes the way authorities should conduct monetary policy.

Nevertheless, it must be acknowledged that solving HANK models is computationally expensive since the aggregate state of the economy contains the distribution of agents, which is generally infinite-dimensional, and since most algorithms approximate the distribution with a finite number of moments (Winberry (2018)). Due to this complication, in the literature HANK models are primarily calibrated rather than, for example, performing a full-information estimation, i.e. applying the Bayesian inference method. It is well known that calibration may not accurately capture the underlying economic mechanisms and would result, for example, in overfitting the model to the data. Although several studies have attempted to evaluate a HANK DSGE model, their implementations have met with varying degrees of success.

Thus, Winberry (2018), in his heterogeneous firms model, estimates parameters of aggregate shocks of only two aggregate shock processes. All other parameters that determine the steady state of the model are calibrated.

Hasumi and Iiboshi (2019) conduct Bayesian estimation of the one-asset HANK model of the U.S. and Japan using aggregate variables: real GDP, inflation, and real interest rate. First, the authors solved the model in a manner similar to continuous-time solution methods of Achdou et al. (2022) and Ahn et al. (2018), which involve approximating the nonlinear HANK model around a steady state. Then they employed the Sequential Monte Carlo (SMC) method with a Kalman filter with parallel computing to evaluate the likelihood of the model. As discussed in the previous chapter, this approach can be computationally intensive, particularly when the number of particles representing the posterior distribution is large. In addition, the accuracy of their computation depends on the specific characteristics of their model and the data being analyzed.

Adopting a more advanced approach, Kase et al. (2022) develop a solution and estimation method based on neural networks that do not require approximation of the *true nonlinear dynamics of the model* by disregarding aggregate uncertainty or assuming that all agents are identical. They also employ a particle filter to obtain a likelihood of a nonlinear HANK model. To do so, they trained an additional neural network as a surrogate model to approximate the results of the particle filter and estimate the joint distribution of the agent characteristics and the model's endogenous variables. The authors applied this method to HANK DSGE with a zero lower bound (ZLB) constraint for the nominal interest rate. Even though it is common to expect high accuracy from neural networks, this method can lead to a problem of overfitting the data. In addition, their approach may require significant computational resources and would need to be more practical for large-scale models.

Since existing heterogeneous agent models are built on micro-data, monetary policy shock is often characterized by a consumption response to transitory income shock in the short run, which is consistent with micro-moments (Kaplan et al. (2018)). Nonetheless, HANK models fail to match the macroeconomic dynamics of RANK models, where the latter are based on macro data (Auclert et al. (2020)). This mismatch makes heterogeneous agent models unsuitable for estimation using macro data. To address this issue, Auclert et al. (2020) built and estimated a model of monetary transmission mechanism with some degree of heterogeneity, combining insights from both models. In the first step, they estimated the parameters governing the distribution of idiosyncratic income and employment shocks using microeconomic data on household income dynamics. In the subsequent step, they estimated the remaining model parameters using the Bayesian approach. In the final step, they used those estimated parameters to simulate the model and compare its impulse responses to actual macroeconomic data. As a result, they were able to match macro and micro-moments simultaneously. Nevertheless, since their technique relies on linearization to estimate the model, it may generate inaccurate results since the model is highly nonlinear.

Overall, applying Bayesian statistical methods to estimation of HANK DSGE models is challenging and computationally intensive. As we specified, one of the main reasons for that difficulty is that HANK models incorporate a large number of parameters and high-dimensional data and need to match an extensive set of moments from the data. Additionally, wealth distribution plays a critical role in HANK models, and obtaining accurate estimates of wealth distribution from survey data is challenging. Hence, heterogeneous agent models require analysis due to their lack of analytical tractability, which induces difficulties and limits our understanding of the forces driving business cycles (Alves et al. (2020)).

In this study, to preserve analytical tractability in the model and for further estimation, we employ a New Keynesian model with limited heterogeneity featuring two agent properties, referred to in macroeconomic literature as the Two-Agent New Keynesian (TANK) DSGE model. These models incorporate agents' limited asset market participation (LAMP) in the New-Keynesian DSGE model, where a time-invariant fraction of agents does not hold assets, which means that unlike in the RANK model, these agents do not smooth consumption. Conversely, another fraction of agents is endowed with all assets and smooth consumption. They are also referred to as "constrained" and "unconstrained" (or "non-Ricardian" and "Ricardian") households in the literature. These types of New Keynesian models have a long history; they were first introduced by Campbell and Mankiw (1989) and were studied by many macroeconomists, including Galí et al. (2004), Bilbiie (2008).

Regarding the ability of above the models to approximate the impact of heterogeneity, one interesting finding from the analysis of TANK models is that they can capture the fluctuations in consumption heterogeneity in response to aggregate shocks between constrained and unconstrained households. In that light, Debortoli and Galí (2021) compared HANK, TANK, and RANK models by tracing responses of those models to aggregate shock. Using the previously discussed two-agent structure of financially constrained and unconstrained households, they identified three forms of heterogeneity: variation in the average consumption gap between constrained and unconstrained households; variations in consumption dispersion within unconstrained households; and changes in the share of constrained households. In that approach, these features are inherent to heterogeneous models and absent in representative household models. These heterogeneity forms were then used to quantify and explain differences between HANK and RANK models in terms of aggregate responses. Debortoli and Galí (2021) found that much of the cyclicality is due to the first factor—a cyclical component of the average consumption gap between constrained and unconstrained households.

Meanwhile the other two factors mutually offset each other. In the TANK model, constrained households do not adjust their consumption in response to aggregate fluctuations; that leads to a widening consumption gap between the two agents. As a result, the authors show that similar to the HANK model, the TANK model captures the average consumption gap between two households and can generate equilibrium responses of aggregate output to monetary, technology, and preference shocks — demonstrating that for plausible specification of heterogeneity, TANK models can closely track HANK model dynamics.

Furthermore, the macroeconomic literature has employed TANK DSGE models to explain the aggregate shocks, including the Great Recession of 2008 and the recent coronavirus pandemic. These shocks caused severe economic downturns in many countries, increasing unemployment and inflation rates. To address the adverse effects of shocks, monetary agencies took various measures, including lowering interest rates, providing liquidity to financial markets through lending facilities and asset purchases, and engaging in quantitative easing. Fiscal agencies' response was to provide stimulus packages, increase government spending, and implement tax cuts to stimulate demand and support the economy.

Regarding the estimation of TANK DSGE models, related studies were either calibrated or estimated using the linearized methods, which, as discussed in the previous chapter, might affect the model's dynamics and predictions. The main difference between our work and that of previous studies (our contribution) is our nonlinear approach in estimating the TANK DSGE model with monetary and fiscal policy rules, including real and nominal rigidities and distortionary taxation. Moreover, we provide nonlinear first-order wage and price-setting conditions for the two-agent model. Considering this issue, we opted for a higher-order solution to preserve the model's nonlinearity feature. First, we solved the model using the perturbation method of order two via the Dynare software platform. Then we estimated model parameters using a likelihood-based approach for the second-order approximation to capture nonlinear features of related macroeconomic data. Within this study, we also conducted first-order estimation of the model using the same method to compare the main estimation results, Bayesian impulse responses, and other outcomes.

We employed quarterly aggregate Korean data for 1999 Q4–2021 Q4. We analyzed this time range since the Bank of Korea (the central bank) became more proactive in controlling inflation and adopted an inflation-targeting rule in 1998, after the Asian financial crisis. This allowed the central bank to focus on keeping inflation under control in contrast to the previous period when it implemented a loose monetary policy by expanding the money supply (Jung (2022)).

Previewing our results from the second study, we find that the linear estimation of the linear representation of the TANK model generates a better fit of the model to the data as measured by log-likelihood at the posterior mode. However, nonlinear estimation of the nonlinear representation of the model reveals better result in terms of the marginal and predictive likelihoods. Similar to the literature, we find that government expenditure shock has an expansionary effect on consumption and output, whereas distortionary taxation has a crowding-out effect. Contractionary monetary policy shock restrains inflationary pressures in the economy. We also find that the estimated share of financially excluded households is relatively high, 0.42 for the linear model, and 0.53 for the nonlinear model.

The rest of this chapter is organized as follows: section 2.2 describes the model; section 2.3 explains the Bayesian estimation procedure and the data used; section 2.4 discusses the results; and Section 2.5 presents conclusions and policy implications.

3.2 Review of the literature: TANK DSGE

As mentioned above, one of the first calibrated studies of two-agent models to examine the role of constrained households in conducting monetary policy was Galí et al. (2004). The authors argue that including financially constrained households (Non-Ricardian) in the RANK model with sticky price properties can change the attributes of the contemporaneous interest rate rule (simple Taylor rule) in the formulation of a monetary policy. Particularly, they show that if the weight of constrained consumers is sufficiently large, the response to inflation needs to be significantly greater than unity to ensure the uniqueness of equilibrium. As for the forward-looking interest rate rule, they argue that in order to guarantee unique equilibrium, it is required that the central bank respond to inflation changes less than unity, which violates the Taylor principle (Taylor (1993)).

Bilbiie (2008) elaborates monetary policy analysis using an analogous two-agent structure: Ricardian and Non-Ricardian. Notably, he develops an analytical framework to capture the influence of limited asset participation on aggregate dynamics through the elasticity of aggregate demand to real interest rates, which depends nonlinearly on the degree of asset participation. Interest rate changes alter intertemporal consumption and labor supply of asset holders, which in turn affect the real wage and hence the demand of agents with no assets, which can change the standard theoretical prediction between output and real interest rate. This relationship becomes inverted when the share of non-asset holders is high enough and/or the elasticity of labor supply is low enough. In this situation, the slope of the well-known IS curve changes from negative to positive, which Bilbiie characterizes as *an inverted Taylor principle*. As a result, the central bank should follow a passive policy rule whereby it increases the nominal interest rate by less than inflation. Unlike Galí et al. (2004), this principle does not depend on whether the Taylor rule is specified in terms of current or expected future inflation.

In a subsequent study, Bilbiie and Straub (2013) estimated the two-agent DSGE model using the Bayesian estimation technique on US data for two prominent periods in the US economy: Great Inflation and Great Moderation. Their empirical results show that the monetary authorities implemented a passive policy approach during the Great Inflation period; and changed to an active strategy during the Great Moderation era. They explain this switching in terms of changes in the sign of the IS curve, which in turn came from the transformation of US financial systems around the 1980s and shifts in business cycle shocks. During that period, the share of agents participating in the asset market increased. Overall, their estimation results support the notion of *an inverted Taylor principle*; they find that most of the changes can be accounted for by structural economic changes. Regarding the shock process, to explain the fall in US output and volatility, shifts in the shock process should also be considered.

Colciago (2011) incorporates nominal-wage stickiness and capital accumulation into the TANK model of Galí et al. (2004), and finds that under plausible parameterization of the model, a small degree of wage stickiness restores the standard Taylor principle, which is a necessary and sufficient condition for equilibrium determinacy (contrary to Galí et al. (2004) and Bilbiie (2008)). Specifically, nominal wage stickiness dampens the variations in the real wage associated with productivity shocks; it averts a large increase in labor income and prevents a strong movement in the consumption of constrained households.

The above mechanism differs from the results of Galí et al. (2004),where higher fluctuations in real wage lead to a higher increase in aggregate demand, followed by increased consumption of constrained agents. Reflecting the result of findings of Colciago (2011), the central bank rules out any sunspot shocks and can manage aggregate demand. To explore this notion, Ascari et al. (2017) further examine the implication of LAMP in the TANK models for the design of monetary policy, arguing that the result of Bilbiie (2008) relies on nominal wage flexibility. Ascari et al. (2017) show that a small amount of wage rigidity restricts the parameter space where the Taylor principle is inverted. Hence, in their analysis, wage stickiness prevents the inversion of the sign of the elasticity of aggregate demand with respect to the real interest rate. Since LAMP uniquely affects the demand side, the authors demonstrate that the trade-off a monetary policy faces marginally depends on LAMP when wages are sticky. Thus, incorporating the constrained agents into the model does not affect the design of optimal monetary policy, which contrasts with the findings of Bilbiie (2008).

By considering a simple two-agent model, Bartolomeo and Rossi (2007) investigate the earlier ideas of the New Keynesian transmission mechanism, where they differentiate two different channels (the direct and indirect effects) for the effectiveness of monetary policy. They argue that monetary policy pertains to its effectiveness even if the degree of asset market participation falls. The authors show that in the LAMP economy, a change in interest rate stimulates the revision of consumption of both unconstrained and constrained agents because of changes in disposable income (Keynesian effect). Also, since aggregate marginal propensity to consume increases with a fraction of constrained agents, it compensates for the reduction of direct effects of monetary policy where now fewer agents can intertemporally smooth their consumption.

Referring to the effects of fiscal shocks, Galí et al. (2007) examine the impact of government spending on aggregate variables using a two-agent model with sticky prices. They argue that consumption and output can rise in response to government spending shock. In their model, constrained households can partly insulate aggregate demand from the negative effect of raised lump-sum taxes since real wages and labor income increased. Moreover, as one implication of the model of Galí et al. (2007) is that generating a positive response of aggregate consumption and output requires a specific combination of fiscal parameters: a sufficiently high response of taxes to debt and a sufficiently low response of taxes to current government spending. Thus, their finding suggests that a careful balance between addressing debt and avoiding the negative effect of higher taxes is required to achieve the desired outcome.

Bilbiie and Straub (2004) argue that obtaining positive consumption response to government spending shock is challenging. Using the calibrated model with a two-agent structure with distortionary taxation, they show that aggregate consumption increases in response to the above shock in the presence of low persistent fiscal shock, passive monetary rule, and high price stickiness. Similar, to the findings of Galí et al. (2007), strong response of consumption is driven by strong positive response of real wages.

Following Galí et al. (2007), Furlanetto (2011) examined the impact of expanding public spending in an economy with a two-agent structure incorporating sticky prices and wages; he argues that nominal wage rigidity implies that wage inflation is much lower, and the reaction of real wages is thus also low. Therefore, the constrained agents increase their consumption less than in a model with flexible wages. At the same time, another factor comes to light: lower wage inflation. This aspect indicates a lower pronounced impact on marginal cost, lower price inflation, and a much lower increase in interest rate by the central bank. A lower increase in interest rate affects unconstrained agents and investment, both of which decrease less than in the flexible wage case. Thus, the expansionary effects of government spending shock on output and consumption are preserved. As mentioned, earlier monetary policy analysis studies by Colciago (2011) and Ascari et al. (2017) confirm this result. In particular, after experimenting with different monetary rules, they conclude that consumption crowds in upon a government-spending shock only when the monetary policy rule is characterized by interest rate smoothing and a moderately anti-inflationary stance.

One of the earlier estimation of fiscal shocks with limited assets participation models was a study of Coenen and Straub (2005). Following Smets and Wouters (2003), they used Bayesian inference method to estimate the model using the Bayesian inference method for the euro area with real and nominal rigidities and lump-sum taxation and endogenously set distortionary taxes. They find that the estimated share of constrained agents is relatively low, between 25% and 37%, which they explain by the financial deregulation period in the Eurozone. Even though government spending shock positively stimulates consumption of constrained agents, the high asset participation rate hinders a positive aggregate consumption and output response to that shock. Unlike the calibrated model of Galí et al. (2007), the model of Coenen and Straub (2005) does not result in a sharp increase in real wages that would help curb the negative wealth effect generated by government spending shock.

During the recession of 2008 and 2009, many countries implemented a large-scale spending program to revive the economy. During that time, the US enforced the Economic Stimulus Act of 2008 and the American Reconstruction and Reinvestment Act (ARRA) of 2009. These laws aimed to provide economic stimulus and support the US economy. The European Commission announced the European Economic Recovery Plan of 2008, and other OECD countries reported their domestic stimulus fiscal actions. Analysis of these measures is reflected in many studies, for example Coenen et al. (2012a) simulated policy and academic DSGE models with constrained households to study the domestic effects of stimulus packages. They found that temporary fiscal stimulus mitigated the economic downturn in the aftermath of the 2008 financial crisis, and show that spending and targeted transfer can generate sizable output multipliers along with an accommodative monetary policy rule, where interest rates remain constant for several years.

There is also a strand of the literature on the estimation of TANK DSGE models incorporating endogenously driven distortionary taxation. For example, Forni et al. (2009) estimated the effects of fiscal policy using TANK DSGE and applying Bayesian inference methods on euro area data as in Christiano et al. (2005) and Smets and Wouters (2003), where the latter studies investigated the estimation of the linearized RANK model. The model of Forni et al. (2009) considers distortionary taxation on labor, capital, and income, where government spending is partitioned between goods and services. They find a significant share of constrained agents in the euro area, between 30% and 40%, and observe that a positive shock to government purchases has small and short-lived expansionary effects on consumption. The decreasing of consumption tax has a positive effect on output and consumption, while a reduction in capital tax has an expansionary effect on investment and output.

Moreover, several other analyses show that fiscal policy shocks imply an expansionary effect on consumption and output (Linnemann (2004), Iwata (2009), Coenen and Straub (2012b), and Drygalla et al. (2020)). Drygalla et al. (2020) investigate the effects of the stimulus packages adopted by the German government during the Great Recession by developing an open-economy DSGE model with a two-agent structure with discretionary fiscal policy effects distinguished from automatic stabilizers. They demonstrate that discretionary fiscal policy had a positive but small impact on the cyclical output component during the Recession based on German data, which amounted to a 0.6 percentage point increase in output growth. In the presence of a 5 percentage point drop in GDP, fiscal measures helped to offset this decline to some degree. Among fiscal policy instruments, the largest positive impact is contributed to government consumption, government investments in the future periods, and government transfers. Drygalla et al. (2020)) also find that reduction of labor tax rates, including social security contributions and capital tax rates, had a slightly smaller positive effect on output growth in Germany. On the other hand, consumption tax rates had positive and negative effects, which were neutralized over time. Thus, according to their results, public consumption, investment, and transfers prevented a sharp and prolonged decline in German output at the beginning of the Great Recession. Their results align with those of Coenen and Straub (2012b).

Nevertheless, some studies reveal the contractionary effect of fiscal shocks. Thus, Ratto et al. (2009) developed a TANK DSGE model with an open economy structure and subsequently estimated it on the euro area using the Bayesian estimation method. They find that the aggregate multiplier of government consumption is negative. These findings are consistent with the findings of Cogan et al. (2010), where the estimated

LAMP model for US data shows the crowding-out effect of consumption and investment goods in response to increased government spending shock. Bhattarai and Trzeciakiewicz (2017) obtained similar results for the UK economy, although results are at odds with the results of studies discussed earlier.

Thus, unlike the RANK models, which imply a substantial wealth-effect mechanism where the agent's consumption falls due to government expenditure shock (higher current or future taxes) and increased labor supply, the TANK models show better outcomes and are better suited for fiscal analysis as interaction with monetary policy since they model more realistic behavior of agents. Nevertheless, estimated TANK DSGE models can feature various outcomes in response to aggregate shocks, depending on various market frictions and rigidities, including the share of constrained households and the persistence of habit, price, and wage rigidity. Henceforth, the implication of the two-agent model relies on model-specific details, and is subject to the calibration of the model.

In this study, we revisit the effects of aggregate shocks on macroeconomic variables by developing our TANK DSGE model. In particular, we analyze the two-agent model and the role of the constrained agent in the propagation of government spending, distortionary taxation, and monetary policy shocks and constrained agent's accounting for observed fluctuations in output and consumption. We investigate the limited participation assumption for the Korean economy. Some studies have analyzed Korea's economy and examined the driving forces of business cycles in that country. For example, Jung (2022) examines the main shock processes using a two-agent model to analyze Korean data, and finds that during the high economic growth era, a significant portion of constrained households played a substantial role in aggregate fluctuations in Korea. Monetary policy innovation contributes much more than any other shock to explanation of the behavior of the main economic variables: output and inflation. During the inflation targeting regime, cost-push and productivity shocks played significant roles in economic fluctuations. Also, during the Great Recession, a fraction of constrained households, which previously surged during the Asian financial crisis, played an essential role in economic turbulence. Jung (2022) also compared the TANK DSGE model with the RANK model for the Korean economy, and found that the two-agent model outperforms the simple representative model in generating the co-movements between output and selected financial variables. Jung (2022) used the maximum likelihood method to estimate the model's parameters and concluded that models featuring LAMP are necessary for analysis of Korea's economy.

As emphasized earlier, we contribute to the literature by developing a TANK DSGE model with a richer structure and estimating it using first and second-order perturbation methods, where the latter requires a nonlinear solution of the model. As a result, we are able to analyze how well our model aligns with macroeconomic data; and investigate Bayesian impulse responses to various shocks. Unlike existing studies on TANK DSGE models, we resort to the full estimation technique using nonlinear approximation.

3.3 The Model

The model we use in this study builds on the earlier literature on two-agent New Keynesian DSGE models. It is a standard cashless dynamic general equilibrium model with two agent properties. There are two types of consumers: "constrained" and "unconstrained" (or Non-Ricardian and Ricardian). We assume the fraction of constrained households to be exogenous. In addition, the model features a single perfectly competitive final-good producer and monopolistic competitive intermediate-good producers determining prices in a staggered fashion. The model comprises fiscal and monetary authority, where the latter pursues its objectives by setting its nominal interest rate, and the former enforces taxes and subsidies. Real frictions include nominal rigidities – price and wage rigidities. Since the model essentially builds on the work of Smets and Wouters (2007), Galí et al. (2007), Bilbiie (2008), we focus on the additional features.

3.3.1 Households

Both types of households have preferences represented by the expected utility function:

$$E_0 \sum_{k=0}^{\infty} \beta^k U_{t+k}^i,$$

where $\beta^k \in (0,1)$ – discount factor. The non-separable in consumption and leisure identical instantaneous utility functional form is given by:

$$U_t^i = \left(\frac{\left(C_{j,t}^i\right)^{1-\sigma}}{1-\sigma} - \frac{\left(L_{j,t}^i\right)^{1+\varphi}}{1+\varphi} \right),$$

where $C_{j,t}^i$ is consumption, $i \in \{R, NR\}$, where R stands for Ricardian households, and NR for Non-Ricardian. $L_{j,t}^i$ is labor supplied to the market, σ the coefficient of relative risk aversion, so, $1/\sigma$ – the intertemporal elasticity of substitution, φ is the elasticity of marginal disutility of labor, where $1/\varphi$ is the Frisch elasticity of labor supply at the household level. The Ricardian households own all the firms in the economy and labor markets are incomplete. Since we consider here TANK model the households do not face any form of idiosyncratic uncertainty.

3.3.1.1 Ricardian Households

The Ricardian households solve the following problem

$$\max_{C^R_{j,t}, K^R_{j,t+1}, B^R_{j,t+1}, I^R_{j,t}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{(C^R_{j,t})^{1-\sigma}}{1-\sigma} - \frac{(L^R_{j,t})^{1+\varphi}}{1+\varphi} \right\}$$

subject to:

$$\begin{aligned} & P_t C^R_{j,t}(1+\tau^c_t) + P_t I^R_{j,t} + \frac{B^R_{j,t+1}}{R^B_t} = \\ & P_t W_{j,t} L^R_{j,t}(1-\tau^l_t) - P_t T^R_t + B^R_{j,t} + D^R_{j,t} + R_t P_t K^R_{j,t}. \end{aligned} \tag{3.1}$$

In each time period t, Ricardian agents can purchase any desired single type of nominally riskless bond, B^R, in period $t+1$ at nominal returns R^B_t, $I^R_{j,t}$ are investments. The expression $R_t P_t K^R_{j,t}$ is capital income from renting the capital stock to firms at the nominal rental rate R_t. Nominal dividends received for the ownership of firms are denoted by $D^R_{j,t}$. Ricardian household taxed by consumption and labor taxes, τ^c_t, τ^l_t. Finally, T^R_t represents nominal lump-sum taxes. The household's stock of physical capital evolves according to:

$$K^R_{j,t+1} = (1-\delta)K^R_{j,t} + \phi\left(\frac{I^R_{j,t}}{K^R_{j,t}}\right) K^R_{j,t}. \tag{3.2}$$

When solving the model in a nonlinear setting, we need to specify the capital adjustment cost function - for the loglinear approximation, a single elasticity was enough. Thus, following Jermann (1998) we specify the functional form of $\phi\left(\frac{I^R_{j,t}}{K^R_{j,t}}\right)$ as:

$$\phi\left(\frac{I^R_{j,t}}{K^R_{j,t}}\right) = \frac{b}{1-a}\left(\frac{I^R_{j,t}}{K^R_{j,t}}\right)^{(1-a)} + c, \tag{3.3}$$

where δ denotes the physical rate of depreciation, the coefficients a, b are set so as to yield the same steady-state properties, and where a replicates the elasticity parameter.

Capital adjustment costs are introduced via the term $\phi\left(\frac{I^R_{j,t}}{K^R_{j,t}}\right)K^R_{j,t}$ which determines the change in capital stock induced by investment spending $I^R_{j,t}$. The function ϕ satisfies the following properties: $\phi^{'}(\cdot)$, $\phi^{''}(\cdot) \geq 0$, $\phi^{'}(\delta) = 0$, $\phi(\delta) = \delta$. That is, adjustment costs are proportional to the investment rate per unit of installed capital.

Ricardian households maximize the expected discounted sum of instantaneous utility subject to constraints (3.1), (3.2), and (3.3) along with transversality conditions ruling out Ponzi games be satisfied: $\sum_{t=0}^{\infty}\beta^t\zeta_t B_{t+1} = 0$, whereas it mentioned before β is the discount factor, B_{t+1} is the expected value of the nominally riskless bond at time t. The sum represents the present value of all future nominally riskless bonds.

Regarding the labor choice, in the presence of wage stickiness, both type of households do not determine wages. They supply differentiated labor in a market structure of monopolistic competition. Further in this Chapter, we discuss about it.

The firs-order conditions (expressed in relative prices) of Ricardian households with respect to $C^R_{j,t}, K^R_{j,t+1}, B^R_{j,t+1}, I^R_{j,t}$:

$$\zeta^R_{j,t} = \frac{(C^R_{j,t})^{-\sigma}}{(1+\tau^c_t)} \tag{3.4}$$

$$\zeta j, t^R = \beta R^B_t E_t \frac{\zeta^R_{j,t+1}}{\Pi_{t+1}}, \tag{3.5}$$

where $\zeta^R_{j,t}$ is a Lagrange multiplier on the flow budget constraint of Ricardian households in period (t).

$$Q_{j,t} = \frac{1}{b\left(\frac{I^R_{j,t}}{K^R_{j,t}}\right)^{-a}}, \tag{3.6}$$

where $Q_{j,t}$ is the (real) shadow value of capital in place, namely, Tobin's marginal q.

$$\begin{aligned} Q_{j,t} = \beta E_t \Bigg\{ \frac{\zeta^R_{t+1}}{\zeta^R_t}\Bigg[R_{t+1} + Q_{t+1}\Bigg((1-\delta) - \\ \left(\frac{I^R_{t+1}}{K^R_{t+1}}\right) b\left(\frac{I^R_{t+1}}{K^R_{t+1}}\right)^{-a} + \frac{b}{1-a}\left(\frac{I^R_{t+1}}{K^R_{t+1}}\right)^{1-a} + c\Bigg)\Bigg]\Bigg\}. \end{aligned} \tag{3.7}$$

3.3.1.2 Non-Ricardian Households

The Non-Ricardian households solve following problem

$$\max_{C_{j,t}^{NR}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{(C_{j,t}^{NR})^{1-\sigma}}{1-\sigma} - \frac{(L_{j,t}^{NR})^{1+\varphi}}{1+\varphi} \right\}$$

subject to:

$$P_t C_{j,t}^{NR}(1+\tau_t^c) = P_t W_{j,t} L_{j,t}^{NR}(1-\tau_t^l) - P_t T_t^{NR} \tag{3.8}$$

Non-Ricardian households do not have an access to financial market, do not hold physical capital nor do they receive profits in the form of dividend income. The first-order condition of Non-Ricardian households with respect to $C_{j,t}^{NR}$:

$$\zeta_{j,t}^{NR} = \frac{(C_{j,t}^{NR})^{-\sigma}}{(1+\tau_t^c)}, \tag{3.9}$$

where $\zeta_{j,t}^{NR}$ is a Lagrange multiplier on the flow budget constraint of Non-Ricardian households in period (t).

3.3.1.3 Labor setting

The first-order condition with respect to labor and wages is more involved than other variables. The labor used by intermediate good producers is supplied by a representative competitive firm that hires the labor supplied by each household j. This labor supplier aggregates the differentiated labor of households with the following production function (technology):

$$L_t^d = \left(\int_0^1 L_{j,t}^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dj \right)^{\frac{\varepsilon_w}{\varepsilon_w - 1}} \tag{3.10}$$

where $\varepsilon_w > 1$ is the elasticity of substitution among different types of labor and L_t^d is the aggregate labor demand. The labor "packer" maximizes profits subject to the production function (technology),taking as given all differentiated wages $W_{j,t}$ and wage W_t. Therefore, the maximization problem is:

$$\max_{L_{j,t}} W_t L_t^d - \int_0^1 W_{j,t} L_{j,t} dj$$

whose first-order conditions are:

$$L_{j,t} = \left(\frac{W_{j,t}}{W_t}\right)^{-\varepsilon_w} L_t^d \tag{3.11}$$

which represents demand equation for differentiated labor j (input demand function). To find the aggregate wage, we use the zero profit condition $W_t L_t^d = \int_0^1 W_{j,t} L_{j,t} dj$ and plug-in the input demand functions:

$$W_t L_t^d = \int_0^1 W_{j,t} L_{j,t} dj = \int_0^1 W_{j,t} \left(\frac{W_{j,t}}{W_t}\right)^{-\varepsilon_w} L_t^d dj$$

to obtain:

$$W_t = \left(\int_0^1 W_{j,t}^{1-\varepsilon_w} dj\right)^{\frac{1}{1-\varepsilon_w}}. \tag{3.12}$$

3.3.1.4 Wage setting

Nominal wage rigidities are modeled according to the Calvo (1983), Coenen and Straub (2005) and Junior (2016). As mentioned above, the households supply their labor services via a continuum of monopolistically competitive firms (unions), which act as labor "packers" for the differentiated labor services, taking the aggregate labor demand of firms as given. Each period, a random fraction $(1-\theta_w)$ of the continuum of monopolistically competitive firms receive permission to optimally reset their nominal wage rate in a given period t. In contrast, those firms that do not receive permission maintain previous period wages. Accordingly, the the maximization problem of labor "packer i (where $i = R, NR$) problem is:

$$\max_{W_{j,t}} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left\{ -\frac{1}{1+\varphi} [L_{j,t+s}^i]^{1+\varphi} + \zeta_{i,t+s} W_{j,t} L_{j,t+s}^i (1-\tau_{t+s}^l)) \right\}$$

where:

$$L_{j,t+s} = \left(\frac{W_{j,t}}{W_{t+s}}\right)^{-\varepsilon_w} L_{t+s}^d. \tag{3.13}$$

Substituting equation 3.13 into maximization problem we have:

$$
\begin{aligned}
&\max_{W_{j,t}} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \Bigg\{ -\frac{1}{1+\varphi} W_{j,t} \left[\left(\frac{W_{j,t}}{W_{t+s}} \right)^{-\varepsilon_w} (1-\tau^l_{t+s}) L^d_{t+s} \right]^{1+\varphi} + \\
&+ \zeta_{j,t+s} W_{j,t} \left[\left(\frac{W_{j,t}}{W_{t+s}} \right)^{-\varepsilon_w} L^d_{t+s} (1-\tau^l_{t+s}) \right] \Bigg\}
\end{aligned}
\tag{3.14}
$$

resulting in the following first-order condition:

$$
W_{j,t} = \frac{\varepsilon}{\varepsilon-1} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left[\frac{(L^{d,i}_{t+s})^{\varphi}}{\zeta^i_{j,t+s}(1-\tau^l_{t+s})} \right]. \tag{3.15}
$$

We can rewrite the above first-order condition for each household types as:

$$
W_{j,t} = \frac{\varepsilon}{\varepsilon-1} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left[\frac{(L^{d,R}_{t+s})^{\varphi}}{\zeta^R_{j,t+s}(1-\tau^l_{t+s})} \right] \tag{3.16}
$$

$$
W_{j,t} = \frac{\varepsilon}{\varepsilon-1} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left[\frac{(L^{d,NR}_{t+s})^{\varphi}}{\zeta^{NR}_{j,t+s}(1-\tau^l_{t+s})} \right] \tag{3.17}
$$

As in Fernández-Villaverde (2010), all Ricardian households set the same wage because complete markets allow them to hedge the risk of the timing of wage change. Non-Ricardian agents who reset their wages face the same problem because they do not have access to the financial market. Consequently, we can drop the jth from the choice of wages and other variables, $W_{j,t} = W^*_t$, $\zeta_{j,t} = \zeta_t$, $L_{j,t} = L^d_t$ (labor aggregation will be shown further). Unlike Galí et al. (2007), we do not assume that employment is uniformly distributed across two types of households $L^{d,R} \neq L^{d,NR}$. However, we rely on the assumption of a common wage between Ricardian and Non-Ricardian households and follow Junior (2016).

$$
W^*_t = \frac{\varepsilon}{\varepsilon-1} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left[\frac{(L^{d,R}_{t+s})^{\varphi}}{\zeta^R_{t+s}(1-\tau^l_{t+s})} \right] \tag{3.18}
$$

$$
W^*_t = \frac{\varepsilon}{\varepsilon-1} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left[\frac{(L^{d,NR}_{t+s})^{\varphi}}{\zeta^{NR}_{t+s}(1-\tau^l_{t+s})} \right] \tag{3.19}
$$

After recursively solving the equations above to obtain the optimal wage for use in nonlinear estimation, we have the following equations:

$$W_t^* = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{(L_t^{d,R})^{\varphi}}{\zeta_t^R(1-\tau_t^l)} + (\theta_w\beta)W_{t+1}^* \tag{3.20}$$

and

$$W_t^* = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{(L_t^{d,NR})^{\varphi}}{\zeta_t^{NR}(1-\tau_t^l)} + (\theta_w\beta)W_{t+1}^* \tag{3.21}$$

Defining

$$\Pi_t^{w*} = \frac{W_t^*}{W_t},$$
$$\Pi_t^{w} = \frac{W_t}{W_{t-1}}$$

and in $t+1$:

$$\Pi_{t+1}^{w*} = \frac{W_{t+1}^*}{W_{t+1}},$$
$$\Pi_{t+1}^{w} = \frac{W_{t+1}}{W_t}$$

Under a Calvo-type wage setting, the equation (3.12) can be rewritten as follows:

$$W_t = \left[(1-\theta_w)\left(W_t^*\right)^{1-\varepsilon_w} + \theta_w W_{t-1}^{1-\varepsilon_w}\right]^{\frac{1}{1-\varepsilon_w}}$$
$$1 = \theta_w\left(\Pi_t^w\right)^{\varepsilon_w-1} + (1-\theta_w)\left(\Pi_t^{w*}\right)^{1-\varepsilon_w}$$

The wage dispersion, induced by the assumed nature of wage stickiness, is inefficient and entails labor (hours) loss. Considering this labor loss, we further define the wage dispersion term. The state variable S_t^W measures the resource costs (labor costs) induced by the inefficient wage dispersion present in the Calvo-Yun model in equilibrium. Thus, we define the following wage dispersion term.

Wage dispersion term:

$$
\begin{aligned}
S_t^W &= \int_0^1 \left(\frac{W_t^j}{W_t}\right)^{-\varepsilon_w} dj = \\
&(1-\theta_w)\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w} + \theta_w \int_0^1 \left(\frac{W_{t-1}^j}{W_t}\right)^{-\varepsilon_w} dj = \\
&(1-\theta_w)\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w} + \theta_w \left(\frac{W_{t-1}}{W_t}\right)^{-\varepsilon_w} \int_0^1 \left(\frac{W_{t-1}^j}{W_{t-1}}\right)^{-\varepsilon_w} dj = \\
&(1-\theta_w)\left(\Pi_t^{w*}\right)^{-\varepsilon_w} + \theta_w \left(\frac{W_{t-1}}{W_t}\right)^{-\varepsilon_w} S_{t-1}^W
\end{aligned}
\tag{3.22}
$$

Hence,

$$
S_t^W = (1-\theta_w)\left(\Pi_t^{w*}\right)^{-\varepsilon_w} + \theta_w (\Pi_t^w)^{\varepsilon_w} S_{t-1}^W. \tag{3.23}
$$

As in Fernández-Villaverde (2010), using the zero profits condition for the labor supplier, $W_t L_t^d = \int_0^1 W_{j,t} L_{j,t} dj$, we have that the labor supply by each Ricardian households can be written as $\int_0^1 L_{j,t}^R dj = (L_t^R)^d$ and, similarly, for each Non-Ricardian households $\int_0^1 L_{j,t}^{NR} dj = (L_t^{NR})^d$. Finally, the the aggregate budget constraint of Ricardian households can be written as:

$$
\begin{aligned}
&C_t^R(1+\tau_t^c) + I_t^R + \frac{B_{t+1}^R}{R_t^B P_t} = \\
&W_t (L_t^d)^R (1-\tau_t^l) - T_t^R + \frac{B_t^R}{P_t} + \frac{D_t^R}{P_t} + R_t K_t^R
\end{aligned}
\tag{3.24}
$$

and for Non-Ricardian:

$$
C_t^{NR}(1+\tau_t^c) = W_t (L_t^d)^{NR} (1-\tau_t^l) - T_t^{NR} \tag{3.25}
$$

As a result, in a symmetric equilibrium, in every period, a fraction $1-\theta_w$ of households set W_t^* as their wage, while the remaining fraction θ_w remain with the same wages as the previous period as was shown above.

3.3.2 Firms

The demand side of market participants is presented by two distinct categories: the producers of final goods and intermediate goods. The intermediate goods producers provide inputs utilized by the final goods producers, who behave competitively, maximizing their profit. The total profits are distributed among the asset holders as dividend payments.

3.3.2.1 Final goods firm

There is one final good is produced using intermediate goods with the following production function:

$$Y_t = \left(\int_1^0 Y_{j,t}^{\frac{\psi-1}{\psi}} dj \right)^{\frac{\psi}{\psi-1}} \tag{3.26}$$

where $\psi > 1$ is the elasticity of substitution between different varieties. At the next stage we obtain demand schedule for intermediate goods:

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\psi} Y_t \tag{3.27}$$

Integrating over j and using the zero-profit condition for the final good producer, we can get a price as it follows:

$$P_t = \left(\int_0^1 P_{j,t}^{1-\psi} dj \right)^{\frac{1}{1-\psi}} \tag{3.28}$$

3.3.2.2 Intermediate goods firm

Production function for a typical intermediate goods firm is given by:

$$Y_{j,t} = A_t K_{j,t}^{\alpha} (L_{j,t}^d)^{1-\alpha}$$

where $K_{j,t}$ is the capital rented by the firm, $L_{j,t}^d$ is the amount of the"packed" labor input rented by the firm, and where A_t -productivity shocks - follows AR(1) process, such that:

$$\log A_t = (1 - \rho_A) \log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t \tag{3.29}$$

where $0 < \rho_A < 1$, and ϵ_t represents an i.i.d with constant variance σ_ϵ^2.

Cost minimization, taking the wage and the rental cost of capital as given, implies:

$$L^d_{j,t} = (1-\alpha) MC_{j,t} \frac{Y_{j,t}}{W_t} \tag{3.30}$$

$$K_{j,t} = \alpha MC_{j,t} \frac{Y_{j,t}}{R_t} \tag{3.31}$$

$$MC_{j,t} = \frac{1}{A_t} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \left(\frac{R_t}{\alpha} \right)^{\alpha} \tag{3.32}$$

After combining equations (3.30) and (3.31) we get:

$$\frac{K_{j,t}}{L^d_{j,t}} = \frac{\alpha}{(1-\alpha)} \frac{W_t}{R_t} \tag{3.33}$$

Note that the marginal cost does not depend on j: all firms receive the same technology shocks and all firms rent inputs at the same price. Thus, the equation (3.32) can be written:

$$MC_t = \frac{1}{A_t} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \left(\frac{R_t}{\alpha} \right)^{\alpha} \tag{3.34}$$

As in the literature, in the second stage, intermediate good producers choose the price that maximizes discounted real profits. They assume to set prices in staggered fashion Calvo (1983) proposed. These firms can reset their prices with probability $(1-\theta)$ each period, independently of the time elapsed since the last adjustment.Therefore, as in Galí et al. (2007) each period a measure a measure $(1-\theta)$ of producers reset their prices, while a fraction (θ) keep their prices unchanged.

3.3.2.3 Price setting

The problem of the intermediate firms is then:

$$\max_{P^*_{j,t}} E_t \sum_{i=0}^{\infty} \theta^i E_t \left\{ \Lambda_{t,t+i} Y_{j,t,t+i} \left(\left(\frac{P^*_{j,t}}{P_{t+i}} \right) - MC_{t+i} \right) \right\} \tag{3.35}$$

subject the sequence of demand constraints:

$$Y_{j,t+i} = \left(\frac{P^*_{j,t}}{P_{t+i}} \right)^{-\psi} Y_{t+i} \tag{3.36}$$

we imply that:

$$P_{j,t} = P_t^*$$

$$\Lambda_{t,t+i} = \beta^i \frac{U^R_{c,t+i}}{U^R_{c,t}} = \beta^i \frac{(C^R_{t+i})^{-\sigma}}{(C^R_t)^{-\sigma}}$$

So, we can drop j in price when solving maximization problem in (3.35).
The first-order condition for this problem is:

$$\sum_{i=0}^{\infty} \theta^i E_t \left\{ \Lambda_{t,t+i} Y_{j,t,t+i} \left(\left(\frac{P_t^*}{P_{t+i}} \right) - \frac{\psi}{(\psi - 1)} MC_{t+i} \right) \right\} = 0, \tag{3.37}$$

where $\mu_p = \frac{\psi}{\psi - 1}$ represents the markup over the price that would prevail in the absence of nominal rigidities.

Given Calvo's pricing, the price index evolves:

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta)(P_t^*)^{1-\psi} \right]^{\frac{1}{1-\psi}} \tag{3.38}$$

and we have optimal price inflation and inflation which are given by:

$$\Pi_t^* = \frac{P_t^*}{P_t}$$

$$\Pi_t = \frac{P_t}{P_{t-1}}$$

Or we can rewrite it as using the above definition:

$$1 = \theta \Pi_t^{\psi - 1} + (1-\theta)(\Pi_t^*)^{1-\psi} \tag{3.39}$$

Derivation of the recursive pricing equation

As in Schmitt-Grohé and Uribe (2007), we retain non-linearity of equilibrium equations. We start from equation (3.37). As they suggests, we rewrite this equation in a recursive fashion in order to get rid of infinite sums.

$$\sum_{i=0}^{\infty}\theta^i E_t\left\{\Lambda_{t,t+i}Y_{j,t,t+i}\left(\left(\frac{P_t^*}{P_{t+i}}\right)-\frac{\psi}{(\psi-1)}MC_{t+i}\right)\right\}=0 \tag{3.40}$$

$$\textstyle\sum_{i=0}^{\infty}\theta^i E_t\left\{\Lambda_{t,t+i}Y_{t+i}\left(\frac{P_t^*}{P_{t+i}}\right)^{-\psi}\frac{1}{P_{t+i}}\left(\left(\frac{P_t^*}{P_{t+i}}\right)-\frac{\psi}{(\psi-1)}MC_{t+i}\right)\right\}=0 \tag{3.41}$$

$$\textstyle\sum_{i=0}^{\infty}\theta^i E_t\left\{\beta^i\frac{(C_{t+i}^R)^{-\sigma}}{(C_t^R)^{-\sigma}}\left(\frac{P_t}{P_{t+i}}\right)^{-\psi}\left(\frac{P_t^*}{P_t}\right)^{-\psi}Y_{t+i}\frac{P_t}{P_{t+i}}\left(\frac{P_t^*}{P_t}-\frac{\psi}{(\psi-1)}MC_{t+i}\frac{P_{t+i}}{P_t}\right)\right\}=0 \tag{3.42}$$

$$\textstyle\sum_{i=0}^{\infty}\theta^i E_t\left\{\beta^i\frac{(C_{t+i}^R)^{-\sigma}}{(C_t^R)^{-\sigma}}\left(\frac{P_t}{P_{t+i}}\right)^{1-\psi}\left(\frac{P_t^*}{P_t}\right)^{-\psi}Y_{t+i}\left(\frac{P_t^*}{P_t}-\frac{\psi}{(\psi-1)}MC_{t+i}\frac{P_{t+i}}{P_t}\right)\right\}=0 \tag{3.43}$$

Next, we multiply by $(C_t^R)^{-\sigma}$ and using the inflation definition:

$$\textstyle\sum_{i=0}^{\infty}\theta^i E_t\left\{\beta^i(C_{t+i}^R)^{-\sigma}\left(\frac{P_t}{P_{t+i}}\right)^{1-\psi}(\Pi_t^*)^{-\psi}Y_{t+i}\left(\Pi_t^*-\frac{\psi}{(\psi-1)}MC_{t+i}\frac{P_{t+i}}{P_t}\right)\right\}=0 \tag{3.44}$$

We can write this:

$$\begin{gathered}\sum_{i=0}^{\infty}\theta^i E_t\left[\beta^i(C_{t+i}^R)^{-\sigma}\left(\frac{P_t}{P_{t+i}}\right)^{1-\psi}(\Pi_t^*)^{1-\psi}Y_{t+i}\right]=\\ \sum_{i=0}^{\infty}\theta^i E_t\left[\beta^i(C_{t+i}^R)^{-\sigma}\left(\frac{P_t}{P_{t+i}}\right)^{1-\psi}\left(\frac{P_{t+i}}{P_t}\right)(\Pi_t^*)^{-\psi}Y_{t+i}\frac{\psi}{\psi-1}MC_{t+i}\right]\end{gathered} \tag{3.45}$$

$$\begin{gathered}\sum_{i=0}^{\infty}\theta^i E_t\left[\beta^i(C_{t+i}^R)^{-\sigma}\left(\frac{P_t}{P_{t+i}}\right)^{1-\psi}(\Pi_t^*)^{1-\psi}Y_{t+i}\right]=\\ \sum_{i=0}^{\infty}\theta^i E_t\left[\beta^i(C_{t+i}^R)^{-\sigma}\left(\frac{P_t}{P_{t+i}}\right)^{-\psi}(\Pi_t^*)^{-\psi}Y_{t+i}\frac{\psi}{\psi-1}MC_{t+i}\right]\end{gathered} \tag{3.46}$$

Factoring out, we get:

$$\begin{aligned} &\sum_{i=0}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{1-\psi} (\Pi_t^*)^{1-\psi} Y_{t+i} \right] = \\ &\frac{\psi}{\psi-1} \sum_{i=0}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{-\psi} (\Pi_t^*)^{-\psi} Y_{t+i} MC_{t+i} \right] \end{aligned} \tag{3.47}$$

$$\begin{aligned} &\sum_{i=0}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{1-\psi} \Pi_t^* Y_{t+i} \right] = \\ &\frac{\psi}{\psi-1} \sum_{i=0}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{-\psi} Y_{t+i} MC_{t+i} \right] \end{aligned} \tag{3.48}$$

We define two auxiliary variables:

$$g_t^2 = \frac{\psi}{\psi-1} g_t^1 \tag{3.49}$$

$$g_t^2 = \sum_{i=0}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{1-\psi} \Pi_t^* Y_{t+i} \right] \tag{3.50}$$

$$g_t^1 = \sum_{i=0}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{-\psi} Y_{t+i} MC_{t+i} \right] \tag{3.51}$$

After recursively solving the equations above to obtain the optimal price for use in nonlinear estimation, we have the following equations:

$$g_t^1 = (C_t^R)^{-\sigma} Y_t MC_t + \beta\theta E_t \Pi_{t+1}^{\psi} g_{t+1}^1 \tag{3.52}$$

$$g_t^2 = (C_t^R)^{-\sigma} Y_t \Pi_t^* + \beta\theta E_t \Pi_{t+1}^{\psi-1} \frac{\Pi_t^*}{\Pi_{t+1}^*} g_{t+1}^2 \tag{3.53}$$

We agree with Schmitt-Grohé and Uribe (2007) on the invalidity of a restriction that the set of equilibrium conditions should include a resource constraint since the model implies relative price dispersion across varieties. The price dispersion, induced by the assumed nature of price stickiness, is inefficient and entails output loss. Considering this output loss, we further define the price dispersion term. The state variable S_t^p measures the resource costs induced by the inefficient price dispersion present in the Calvo-Yun model in equilibrium.

Price dispersion term:

$$S_t^p = \int_1^0 \left(\frac{P_{j,t}}{P_t}\right)^{-\psi} dj \tag{3.54}$$

$$S_t^p = \int_\theta^0 \left(\frac{P_{j,t-1}}{P_t}\frac{P_{t-1}}{P_{t-1}}\right)^{-\psi} dj + \int_1^\theta \left(\frac{P_t^*}{P_t}\right)^{-\psi} dj \tag{3.55}$$

$$S_t^p = \theta \left(\frac{P_{t-1}}{P_t}\right)^{-\psi} S_{t-1}^p + (1-\theta)\left(\Pi_t^*\right)^{-\psi} \tag{3.56}$$

$$S_t^p = \theta \Pi_t^\psi S_{t-1}^p + (1-\theta)\left(\Pi_t^*\right)^{-\psi} \tag{3.57}$$

3.3.3 Monetary Policy

Our interest rate setting rule is required to fully specify the dynamics of the model. Therefore, monetary policy follows standard Taylor type rule according to Taylor (1993):

$$\frac{R_t^B}{R^B} = \left(\frac{R_{t-1}^B}{R^B}\right)^{\gamma_R} \left[\left(\frac{Y_t}{Y}\right)^{\gamma_Y} \left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi}\right]^{(1-\gamma_R)} \exp(\varepsilon_t^m), \tag{3.58}$$

where γ_Y and γ_π are parameters governing the central bank's responsiveness to inflation and the output gap, respectively. R^B is steady state nominal gross return of government bond, and π is target level of inflation (we assume that it is not equal to inflation in the steady state). $\varepsilon_t^m \sim \mathcal{N}(0,1)$ is i.i.d. monetary policy shock.

3.3.4 Fiscal Policy

The government conducts fiscal policy through collecting the revenue from taxes on consumption, labor and lump-sum tax, cash returns from bonds issued in current period. The government makes expenses on interest and debt payments and public consumption. For the government budget constraint, it thus follows:

$$\frac{B_{t+1}}{R_t^B P_t} + \tau_t^c C_t + \tau_t^l W_t L_t^d + T_t = G_t + \frac{B_t}{P_t} \tag{3.59}$$

where $T_t \equiv \lambda T_t^{NR} + (1-\lambda)T_t^R$.
Consumption and labor tax are assumed to follow an exogenous process because consumption taxes are mostly excise taxes in Korea. Labor tax is assumed this way for determinacy issues and estimation purposes. Thus, these taxes do not adjust to changes in current output and government debt. Also, government expenditure is normalized by a steady-state output.

$$\frac{G_t - G}{Y} = \rho_g \frac{G_t - G}{Y} + \varepsilon_t^G \tag{3.60}$$

$$\tau_t^c = (1-\rho_{\tau^c})\tau^c + \rho_{\tau^c}\tau_{t-1}^c + \varepsilon_t^{\tau^c} \tag{3.61}$$

$$\tau_t^l = (1-\rho_{\tau^l})\tau^l + \rho_{\tau^l}\tau_{t-1}^l + \varepsilon_t^{\tau^l} \tag{3.62}$$

where $0 < \rho_{\tau^c} < 1$ and $0 < \rho_{\tau^l} < 1$, and ε_t^G, $\varepsilon_t^{\tau^c}$, $\varepsilon_t^{\tau^l}$ represent an i.i.d shocks with zero means and variances σ_G, σ_{τ^c}, σ_{τ^l}.

Lump-sum taxes are assumed to be set in reaction to the evolution of debt, output, and prices. We do not put any exogenous shock in the evolution of the lump-sum tax rule:

$$\frac{T_t - T}{Y} = \phi_b \left[\frac{B_t}{P_{t-1}} \frac{B}{P}\right] / Y + \phi_g \left[\frac{G_{t-1} - G_t}{Y}\right] \tag{3.63}$$

3.3.5 Aggregation

Our model implies relative price and wage dispersion across varieties and labor supply. These two dispersion, which are induced by the assumed nature of price and wage stickiness, are inefficient and entails output and labor loss. To see this, as usual, first, we derive an expression for aggregate demand:

$$Y_t = C_t + I_t + G_t$$

The demand for each intermediate firms j is:

$$Y_{j,t} = (C_t + I_t + G_t)\left(\frac{P_{j,t}}{P_t}\right)^{-\psi} \quad \forall j$$

$$Y_{j,t} = A_t K_{j,t}^{\alpha}(L_{j,t}^d)^{1-\alpha}$$

$$A_t K_{j,t}^{\alpha} (L_{j,t}^d)^{1-\alpha} = (C_t + I_t + G_t) \left(\frac{P_{j,t}}{P_t} \right)^{-\psi} \quad \forall j$$

By market clearing:

$$\int_0^1 (L_{j,t}^d) dj = L_t^d$$

$$\int_0^1 (K_{j,t}) dj = K_t$$

$$\int_0^1 A_t K_{j,t}^{\alpha} (L_{j,t}^d)^{1-\alpha} dj = A_t K_t^{\alpha} (L_t^d)^{1-\alpha}$$

and

$$A_t K_t^{\alpha} (L_t^d)^{1-\alpha} = (C_t + I_t + G_t) \left(\int_0^1 \frac{P_{j,t}}{P_t} \right)^{-\psi} dj$$

we defined before price dispersion term in (3.54) and expression for it in (3.57):

$$S_t^p = \left(\int_0^1 \frac{P_{j,t}}{P_t} \right)^{-\psi} dj$$

Thus, we get:

$$C_t + I_t + G_t = \frac{A_t K_t^{\alpha} (L_t^d)^{1-\alpha}}{S_t^p} \tag{3.64}$$

As in Galí et al. (2007) for each consumer type aggregate consumption and labor are given by a weighted average of the corresponding variables:

$$C_t \equiv \lambda C_t^{NR} + (1-\lambda) C_t^R \tag{3.65}$$

$$L_t^d \equiv \lambda (L_t^d)^{NR} + (1-\lambda)(L_t^d)^R. \tag{3.66}$$

Similarly, aggregate investment, bond, dividends and the capital stock are given by:

$$I_t \equiv (1-\lambda) I_t^R \tag{3.67}$$

$$B_t \equiv (1-\lambda) B_t^R \tag{3.68}$$

$$D_t \equiv (1-\lambda) D_t^R \tag{3.69}$$

and

$$K_t \equiv (1-\lambda) K_t^R \tag{3.70}$$

Following literature, we define an expression for aggregate labor demand. We know that:

$$L_{j,t} = \left(\frac{W_{j,t}}{W_t}\right)^{-\varepsilon_w} L_t^d$$

Once we integrate over all households j, we obtained next:

$$\int_0^1 L_{j,t} = L_t = \int_0^1 \left(\frac{W_{j,t}}{W_t}\right)^{-\varepsilon_w} L_t^d dj = L_t^d \int_0^1 \left(\frac{W_{j,t}}{W_t}\right)^{-\varepsilon_w} dj$$

where L_t is the aggregate labor supply of all households. As we defined in (3.22):

$$S_t^W = \int_0^1 \left(\frac{W_{j,t}}{W_t}\right)^{-\varepsilon_w} dj$$

Thus,

$$L_t^d = \frac{L_t}{S_t^W}. \tag{3.71}$$

3.3.6 Equilibrium

The equilibrium in this model of economy is standard and the symmetric equilibrium policy functions are determined by the following equations:

- The first-order conditions and the budget constraint of the Ricardian households:

$$\zeta_t^R = \frac{(C_t^R)^{-\sigma}}{(1+\tau_t^c)} \tag{3.1}$$

$$\zeta_t^R = \beta R_t^B E_t \frac{\zeta_{t+1}^R}{\Pi_{t+1}} \tag{3.2}$$

$$\begin{aligned} Q_{j,t} = \beta E_t \Bigg\{ \frac{\zeta_{t+1}^R}{\zeta_t^R} \Bigg[R_{t+1} + Q_{t+1} \Bigg((1-\delta) - \\ \left(\frac{I_{t+1}^R}{K_{t+1}^R}\right) b \left(\frac{I_{t+1}^R}{K_{t+1}^R}\right)^{-a} + \frac{b}{1-a} \left(\frac{I_{t+1}^R}{K_{t+1}^R}\right)^{1-a} + c \Bigg) \Bigg] \Bigg\}. \end{aligned} \tag{3.3}$$

$$Q_t = \frac{1}{b\left(\frac{I_t^R}{K_t^R}\right)^{-a}} \tag{3.4}$$

$$
\begin{aligned}
&C_t^R(1+\tau_t^c) + I_t^R + \frac{B_{t+1}^R}{R_t^B P_t} = \\
&W_t(L_t^d)^R(1-\tau_t^l) - T_t^R + \frac{B_t^R}{P_t} + \frac{D_t^R}{P_t} + R_t K_t^R
\end{aligned}
\tag{3.5}
$$

$$
W_t^* = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{(L_t^{d,R})^{\varphi}}{\zeta_t^R(1-\tau_t^l)} + (\theta_w \beta) W_{t+1}^* \tag{3.6}
$$

$$
W_t^* = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{(L_t^{d,NR})^{\varphi}}{\zeta_t^{NR}(1-\tau_t^l)} + (\theta_w \beta) W_{t+1}^* \tag{3.7}
$$

and we define as:

$$
\mu^w = \frac{\varepsilon_w}{(\varepsilon_w - 1)}
$$

- The first-order conditions and the budget constraint of the Non-Ricardian households

$$
\zeta_t^{NR} = \frac{(C_t^{NR})^{-\sigma}}{(1+\tau_t^c)} \tag{3.8}
$$

$$
C_t^{NR}(1+\tau_t^c) = W_t(L_t^d)^{NR}(1-\tau_t^l) - T_t^{NR} \tag{3.9}
$$

- The firms that can change prices set them to satisfy:

$$
g_t^1 = (C_t^R)^{-\sigma} Y_t MC_t + \beta\theta E_t \Pi_{t+1}^{\psi} g_{t+1}^1 \tag{3.10}
$$

$$
g_t^2 = (C_t^R)^{-\sigma} Y_t \Pi_t^* + \beta\theta E_t \Pi_{t+1}^{\psi-1} \frac{\Pi_t^*}{\Pi_{t+1}^*} g_{t+1}^2 \tag{3.11}
$$

$$
g_t^2 = \frac{\psi}{\psi - 1} g_t^1 \tag{3.12}
$$

where they rent inputs to satisfy their static minimization problem

$$
\frac{K_t}{L_t^d} = \frac{\alpha}{(1-\alpha)} \frac{W_t}{R_t} \tag{3.13}
$$

$$
MC_t = \frac{1}{A_t} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \left(\frac{R_t}{\alpha} \right)^{\alpha} \tag{3.14}
$$

- The wages evolve as:

$$
1 = \theta_w \left(\Pi_t^w\right)^{\varepsilon_w - 1} + (1-\theta_w)\left(\Pi_t^{w*}\right)^{1-\varepsilon_w} \tag{3.15}
$$

and the prices evolves as:

$$1 = \theta \Pi_t^{\psi-1} + (1-\theta)(\Pi_t^*)^{1-\psi} \tag{3.16}$$

- Government budget constraint:

$$\frac{B_{t+1}}{R_t^B P_t} + \tau_t^c C_t + \tau_t^l W_t L_t^d + T_t = G_t + \frac{B_t}{P_t} \tag{3.17}$$

where $T_t \equiv \lambda T_t^{NR} + (1-\lambda) T_t^R$.

- Monetary policy rule:

$$\frac{R_t^B}{R^B} = \left(\frac{R_{t-1}^B}{R^B}\right)^{\gamma_R} \left[\left(\frac{Y_t}{Y}\right)^{\gamma_Y} \left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi}\right]^{(1-\gamma_R)} \exp \varepsilon_t^m \tag{3.18}$$

where $\varepsilon_t^m \sim \mathcal{N}(0,1)$.

- Fiscal policy rule:

$$\frac{G_t - G}{Y} = \rho_g \frac{G_t - G}{Y} + \varepsilon_t^G \tag{3.19}$$

$$\tau_t^c = (1-\rho_{\tau^c})\tau^c + \rho_{\tau^c}\tau_{t-1}^c + \varepsilon_t^{\tau^c} \tag{3.20}$$

$$\tau_t^l = (1-\rho_{\tau^l})\tau^l + \rho_{\tau^l}\tau_{t-1}^l + \varepsilon_t^{\tau^l} \tag{3.21}$$

where $0 < \rho_{\tau^c} < 1$ and $0 < \rho_{\tau^l} < 1$, and ε_t^G, $\varepsilon_t^{\tau^c}$, $\varepsilon_t^{\tau^l}$ represent an i.i.d shocks with zero means and variances σ_G, σ_{τ^c}, σ_{τ^l}.

Lump-sum taxes are assumed to be set in reaction to the evolution of debt, output, and prices. We do not put any exogenous shock in the evolution of the lump-sum tax rule:

$$\frac{T_t - T}{Y} = \phi_b \left[\frac{B_t}{P_{t-1}} \frac{B}{P}\right] / Y + \phi_g \left[\frac{G_{t-1} - G_t}{Y}\right] \tag{3.22}$$

- Markets clear:

$$Y_t = C_t + I_t + G_t \tag{3.23}$$

$$Y_t = \frac{A_t K_t^{\alpha} (L_t^d)^{1-\alpha}}{S_t^p} \tag{3.24}$$

where:

$$L_t = S_t^W L_t^d \tag{3.25}$$

$$D_t = Y_t - W_t L_t^d \frac{1}{(1-\alpha)} \tag{3.26}$$

$$\log A_t = (1-\rho_A)\log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t \tag{3.27}$$

$$S_t^p = \theta \Pi_t^{\psi} S_{t-1}^p + (1-\theta)\left(\Pi_t^*\right)^{-\psi} \tag{3.28}$$

$$S_t^W = \theta_w (\Pi_t^w)^{\varepsilon_w} S_{t-1}^W + (1-\theta_w)\left(\Pi_t^{w*}\right)^{-\varepsilon_w}. \tag{3.29}$$

- Aggregating both types of households

$$C_t \equiv \lambda C_t^{NR} + (1-\lambda) C_t^R \tag{3.30}$$

$$L_t^d \equiv \lambda (L_t^d)^{NR} + (1-\lambda)(L_t^d)^R. \tag{3.31}$$

$$T_t \equiv \lambda T_t^{NR} + (1-\lambda) T_t^R. \tag{3.32}$$

$$I_t \equiv (1-\lambda) I_t^R \tag{3.33}$$

$$B_t \equiv (1-\lambda) B_t^R \tag{3.34}$$

$$D_t \equiv (1-\lambda) D_t^R \tag{3.35}$$

$$K_t \equiv (1-\lambda) K_t^R \tag{3.36}$$

and

$$K_{t+1}^R (1-\lambda) = (1-\delta) K_t^R (1-\lambda) + K_t^R (1-\lambda) \left(\tfrac{b}{1-a} \left(\tfrac{I_t^R}{K_t^R} \right)^{(1-a)} + c \right) \tag{3.37}$$

3.4 Solving the Model

In this study, we solve the model by log linearizing the equilibrium conditions and applying perturbation techniques of orders one and two. We implement it using version 4.6.1 of the Dynare software platform, Matlab pre-processor, which takes log-linear approximation around the deterministic steady-state. Once we have the solution of a TANK DSGE model in terms of its approximated policy functions, we can write the laws of motion of variables in a state space representation and further apply our likelihood-based approach we developed earlier to compute the likelihood function and derive the posterior estimates of the model's parameters using Bayesian inference method. Before Dynare approximates the model, we need to provide the steady-state model.

3.4.1 Steady state

Since we assume that the economy returns to its steady-state level in the long run, we must define our steady-state model. We will find steady-state values for our DSGE model variables in order to evaluate optimality conditions when shocks are shut down and assume that all variables are constant. In steady-state model we consider zero inflation with $P_t = P_{t-1}$, and consequently, $\Pi_t = \Pi_{t-1} = 1$, our steady-state price and wage terms are $S_t^p = 1$, $S_t^w = 1$. The steady-state riskless interest rate can be derived from the Euler equation of unconstrained agents $R_t^B = \frac{1}{\beta}$. We define steady-state wage and price mark-up as $\mu^w = \frac{\varepsilon_w}{\varepsilon_w - 1}$ and $\mu^p = \frac{\psi}{\psi - 1}$, and share of government consumption as γ_g of total output. As is shown in our Appendix, we follow the literature on the TANK DSGE model and assume that steady-state consumption and labor hours are the same across household types: $C_t = C_t^R = C_t^{NR}$ and $L_t^d = (L_t^R)^d = (L_t^{NR})^d$, an outcome that can always be guaranteed by an appropriate choice of T_t^R and T_t^{NR}. The full steady-state derivation is presented in Appendix.

3.5 Bayesian estimation of the model using likelihood approach

3.5.1 Methodology

There is much to be gained from a nonlinear estimation of the model, both in terms of accuracy and identification (Fernández-Villaverde (2010)). Considering this and other factors described in the first study, we estimate the model for the first and second-order approximation. We use the methodology proposed in Chapter 2 to estimate the parameters of our TANK DSGE model, presented in this chapter. According to Bayesian statistics, the likelihood function, which defines the probability the model assigns for each observation given the vector of parameter values, is our primary object. The likelihood states that all information in the sample is contained in this function. For that reason, we went into great detail about the technique of for computing the likelihood function, $\pi(Y^T|\Psi)$, where the Ψ is a vector of parameter values. Specifically, we propose a procedure for obtaining the approximation of $\pi_y(Y_t^o|X_t)$, which is the density of observed control variables Y_t^o conditional on state variables X_t.

Upon computing $\pi(Y^T|\Psi)$, we apply the Bayes rule (fundamental principle in probability theory, which combines the likelihood function of the model with a prior density, $p(\Psi)$) for the parameters to form a posterior distribution, $p(\Psi|Y^T)$:

$$\pi(\Psi|Y^T) = \frac{\pi(Y^T|\Psi)\pi(\Psi)}{\int \pi(Y^T|\Psi)\pi(\Psi)d\Psi}$$

Zellner (1988) defines the Bayes theorem as the most effective way of processing information. The Bayes theorem utilizes all the information in the data, regardless of sample size. Moreover, Bayes' theorem avoids adding any superfluous information to the analysis (Fernández-Villaverde (2010)). Since the posterior, $\pi(\Psi|Y^T)$ is the density of fundamental interest, it summarizes what we know about Ψ, after seeing the data. The posterior distribution of the parameters is proportional to the posterior density after integrating out Ψ in the the denominator of the above equation:

$$\pi(\Psi|Y^T) \propto \pi(Y^T;\Psi)\pi(\Psi).$$

We employ Metropolis-Hastings to simulate the posterior distribution with 350,000 replications for two Markov chains. We disregard 10 percent of the initially generated parameters because of an unrepresentative equilibrium distribution.

We follow standard literature on two-agent New Keynesian model in assigning values to the structural parameters of the model, which is defined in the following sections. We also calibrate the model to the Korean economy. This choice of country is primarily in response to data convenience. The time unit is meant to be a quarter of a year. Our model identifies and examines various macroeconomic shocks on the Korean economy including: technology shock, monetary policy shock, fiscal policy shock (which here comprises government expenditure shock), consumption tax shock, and labor tax shock.

3.5.2 Data

As discussed in Chapter 2, it is impossible to estimate the model with fewer shocks than observables because of stochastic singularity. Therefore, we utilize as many shocks as observable in the estimation of this model. To estimate the current TANK DSGE model, we use data from Korea for 1999Q4–2021Q4. This range of the dataset is determined based on the availability of data. The Korean data is obtained from Statistics Korea, The Bank of Korea, and the CEIC database. Model observable variables include five time series data:

1. Per capita Gross Domestic Product (GDP), Y_t;
2. Government consumption, G_t;
3. Tax Revenue from income tax, which is matched with the product of model's T_t, L_t, and W_t;
4. Tax Revenue from Goods and services, which corresponds to the product of model's T_t and C_t; and
5. Nominal interest rate, R_t^B, which is the key policy rate for the Bank of Korea to implement its monetary policy and achieve its objectives.

This macroeconomic series can capture the primary characteristics of the dynamics of the data and model much of the information that policymakers are interested in. All gathered data sets are adjusted for seasonal variation except for the nominal interest rate since it is a stationary variable. Trending variables such as output, government consumption, and tax revenues are detrended using Hodrick-Prescott one-sided filter since we define the model without a specified trend. We should note here that the information in the sample is limited (we have only 89 observations for the Korean economy), and it is not easy to obtain stable estimates otherwise.

3.5.3 Calibration

Before specifying the priors, we need to calibrate some parameters to achieve estimation convergence. Thus, after analyzing the observed data, we set the γ_g, steady-state real government expenditure to real GDP, at 0.3. In particular, we computed the long-run average ratio of real government expenditure to real GDP for the Korean economy for 1999Q4–2021Q. We implemented similar analysis and set steady-state government debt to real GDP ratio, γ_B, at 0.4. Also, we fixed the capital depreciated rate, δ, at 0.025 as in Fernández-Villaverde (2010) since it is difficult to identify δ in the data. Moreover, we calibrated the price and wage markups, μ_p and μ_w, to their steady-state levels. In our model, transfers/lump-sum tax is defined residually from the government budget constraint. The remaining parameters have been given the prior information based on previous empirical and theoretical studies on New-Keynesian DSGE models. The definitions of structural parameters are summarized in Table 3.1.

3.5.4 Prior specifications

We limit the estimation of some prior specifications to that in the literature on the TANK-DSGE model.

Some structural parameters that are difficult to identify are set according to the respective sample means or to values that are widely used in the existing literature. The elasticity of the production level with respect to capital, α, is assumed to follow normal distribution with mean 0.3 and standard deviation 0.1, as in Smets and Wouters (2007). Intertemporal discount factor, β, follows the gamma distribution with a mean 0.9975 and a standard deviation 0.1. However, we transform parameter β to get rid of the constraints on its domain.

To satisfy the Taylor principle, in which an increase in inflation eventually leads to a more than one-for-one rise in the nominal interest rate, we follow New Keynesian DSGE literature and set parameters of the Taylor rule according to well-known academic studies. Thus, we follow the literature and parameterize the monetary policy rule using the long-run reaction inflation, γ_π, which is assumed to follow a gamma distribution with mean 1.5 and standard deviation 0.05. The prior on the short-run reaction coefficient on the output gap, γ_y, is also assumed to follow a gamma distribution, with mean 0.125 and standard deviation 0.05 as in Smets and Wouters (2007). In addition, the degree of interest-rate smoothing parameter,γ_R, is assumed to follow a beta distribution with mean 0.7 and standard deviation 0.15. We are aware that under specific parameter configurations, our economy's equilibrium may be indeterminate even if the Taylor rule satisfies.

Thus, throughout, we restrict ourselves to configurations of parameter values for which the equilibrium is unique.

Since we employ a CRRA utility, the coefficient of the relative risk-aversion, σ, is an inverse elasticity of intertemporal substitution. We set the prior mean for this parameter at 2, following DSGE literature, where agents are assumed to be risk-averse with standard deviation 0.1 and following normal distribution. Regarding the inverse of the Frisch elasticity of labor supply, ϕ, it also follows normal distribution with prior means 2 and standard deviation 0.25, respectively. The parameters describing price and wage markups, ψ and ε_W, are specified by a gamma distribution with prior means

5 and standard deviations 1 for both parameters. These prior means imply that the markups at the steady-state are 25 percent. These values are broadly similar to those in the existing literature on DSGE models. The parameters describing Calvo probabilities, θ and θ_w, follow beta distribution, with means to be around 0.5, suggesting an average length of price and wage contract of half a year, a value consistent with much of the empirical evidence; the standard deviations are 0.15 for each parameter.

The prior to the share of constrained agents is set roughly in line with the specification of studies using Korean data. For example, according to Jung (2022), the estimated share of constrained households in the Korean economy is around 0.5, which is relatively high compared to that in other surveys. Park (2017) estimated the shares of hand-to-mouth households (financially constrained) using Korean Labor and Income Panel Study data over the period 2001–2013 and found it to be around about 0.32 in 2013. Song (2020) found the share of constrained agents to be around 0.3 for 2012–2017. In our study we set a prior mean for the share of constrained households, λ, at 0.4 and standard deviation 0.1. We assume that λ follows a beta distribution.

The standard errors of innovations, $\exp \sigma_x$, are assumed to follow inverse gamma distribution with mean 0.1 and standard deviation 2. Parameters governing the persistence of monetary and fiscal shocks, ρ_a, ρ_G, ρ_{τ^c}, ρ_{τ^l}, are believed to follow a beta distribution with prior means 0.5, and standard deviation 0.1 for each of the parameters. These settings are associated with moderately persistent shocks.

Parameter of adjustment cost function, a_j is assumed to follow beta distribution with mean 4 and standard deviation 0.1.

Finally, consumption and labor tax rate parameters, τ^c and τ^l, follow beta distribution with mean 0.1 and 0.125 respectively (which is equal to long-run average tax rates in South Korea); standard deviation is assumed to be equal to 0.01 for both parameters. Tables 3.2 and 3.3 summarize the information on prior means, standard deviations and distributions for all estimated parameters.

3.6 Results and analysis

Tables 3.2 and 3.3 also present the posterior distribution of the estimated parameters and structural shocks for linear and nonlinear models, represented by the first (M_1) and second-order (M_2) approximations and their respective credible intervals. The Metropolis-Hastings algorithm obtains the posterior distribution draws of the listed parameters with random walk proposal (e.g., Koop (2003), p. 97). We compare the models with the value of: log likelihoods; the marginal likelihood, which is calculated following Gelfand and Dey (1994) and Geweke (1999); and predictive likelihoods (Geweke and Amisano (2010)). The predictive likelihood is calculated for observations 30 to 89. In addition to it, we report the log priors measured for both models. Model performance results are shown in Table 3.4. In order to test the general stability of our results, we run several different simulation rounds. Estimates are quite stable for (M_1), but less so for (M_2).

Thus, the data in Table 3.4 shows that the linear model M_1 is a better performing model in terms of estimated log-likelihood. However, the nonlinear model, M_2, is preferable based on marginal and predictive likelihoods. Particularly, the log likelihood for the first model M_1 is 1469, for the second model M_2 1451. The marginal likelihood for the first model is 1247, for the second model 1282. Finally, the predictive likelihood for the first model is 883, and for the second model is about 912. The posterior probability for model M_2 is close to one compared to model M_1.

These findings suggest that the first-order approximation of our estimated model is a better fit for the data. However, it should be noted that log likelihood measures the fit of a model to the data, whereas marginal and predictive likelihoods consider both the fit to the data and complexity of the model. Since the second model has better marginal and predictive likelihood, it is more parsimonious and less likely to overfit the data, making it in our view a better alternative. In addition, the nonlinear model is preferred by the data based on the Bayes factor $\exp(5)$, where the Bayes factor is defined as a ratio of the marginal likelihoods (Koop et al. (2007), p. 61). Our nonlinear model is more likely to provide better predictions for future data points.

Finally, according to Table 3.4, log prior at posterior mean is higher for nonlinear model, which suggests that the prior distribution favors model M_2.

Figures 3.1 and 3.2 show observed series used for estimation and their posterior estimates for models M_1 and M_2. These figures indicate a good fit of some observed data, such as output, government expenditure, income, and consumption tax revenues. The fitted values of the second-order approximated model better replicate the observed variables compared to the linearized model, as shown by these figures.

Figure 3.3 illustrates the log-likelihood function of the data given estimated parameter values, where the plot for the nonlinear likelihood function has a higher maximum point for some data periods than the plot for the linear model. At the same time, the log-likelihood function of the linear model shows better results for some other regions, which suggests that M_1 model fits the Korean data better.

3.6.1 Posterior estimates

In this section we discuss the posterior estimates of the models presented in Tables 3.2 and 3.3. After transformation back to the original parameter, β, the mean of the discount factor is estimated to be around 0.9927 for M_1 and 0.9923 for M_2 for the Korean economy, slightly lower than prior mean since likelihood function aims to match a low-interest rate (Fernández-Villaverde (2010)). We should note that in some studies on DSGE models with Korean data this parameter is usually fixed (e.g. Lee (2012), Jung (2019)). In the observed data, the policy rate experiences high values at the beginning of the sample, leading to a discount factor rate lower than prior mean.

Our estimation results reveal a share of constrained agents, λ, to be around 0.42 for M_1 model; and 0.53 for M_2. These estimates are higher than those reported in previous studies (Park (2017) and Song (2020)) but are more similar to the findings of Jung (2022), a TANK DSGE model examination of the Korean economy.

The inverse of intertemporal elasticity of substitution, σ, is estimated to be 0.33 for M_1 and 0.38 for M_2, higher for the nonlinear model. Regarding the inverse of Frisch elasticity parameters, ϕ, the posterior mean is about 3.7 for the first model and much higher for the second, 5.3.

The elasticity,α, of the production level with respect to capital is estimated to be 0.06 for both models, lower than previous studies. The posterior means of elasticity of substitution across inputs, ψ, is relatively similar for linear and nonlinear models, 5.2 and 5.3 respectively, higher than the assumed prior mean. Estimates of elasticity labor demand for intermediate firms, ε_w, are higher for M_1 (4.6) than for M_2 (4.0).

The degree of both price θ and wage stickiness θ_w is estimated the be higher than the prior means of 0.5. Based on the posterior mean estimates for θ, the average duration of price contracts is estimated to be approximately one year for both M_1 and M_2 (0.8 and 0.9 respectively). Similarly, the average duration of wage contracts is estimated to be around six months for the linear model, estimated at 0.5, and slightly higher than six months for the nonlinear model, estimated at 0.6. Therefore, the results suggest that the duration of contracts varies between models, with longer-term contracts being more prevalent for prices than for wages.

In the estimates of the monetary policy parameters, the analysis demonstrates that the posterior means for the reaction coefficients in the monetary rule are somewhat different for the output gap than those in the existing literature. That is, the posterior mean for the output coefficient,γ_y, is around 0.06 for M_1, and 0.01 for M_2, which indicates the Korean central bank prioritizes the price stability over employment or growth. The inflation coefficient, γ_π, is around 1.5 for the first and second-order approximation models. This indicates that the central bank puts a relatively high weight on stabilizing inflation. The parameter γ_R, which characterizes the responsiveness of monetary authority on changes in natural interest rate, shows a moderate level for linear and nonlinear models. It is estimated to be 0.7 for M_1 and 0.8 for M_2, which is in line with results of existing studies.

The posterior mean of the steady-state consumption tax rate τ_c is about 0.15 for both models; while the estimated mean of labor tax, τ_l, is 0.24 for the linear and nonlinear models. The degree of smoothing of consumption tax rule,ρ_{τ_c}, is 0.51 for the first model, 0.48 for the second model. The degree of smoothing of labor tax rule, ρ_{τ_l}, is 0.76 for the first model, 0.70 for the second model. Government expenditure exhibits a moderate degree of smoothing, ρ_G about 0.65 and 0.68 for model M_1 and M_2 respectively.

Finally, the standard deviation of the parameters is estimated to be much higher for the second-order approximation model than for the first-order approximation relative to the prior distribution, indicating that the nonlinear likelihood is more dispersed, which is consistent with estimations using nonlinear filters (Fernández-Villaverde and Rubío-Ramirez (2005)).

3.6.2 Bayesian impulse response function

In order to assess the dynamic responses of macroeconomic variables to economic shocks, we perform Bayesian impulse response analyses based on the estimated models and one standard deviation from the steady state. We measure the responses on a quarterly basis. All variables in the model are in *logs*, so the impulse response functions can be interpreted as percentage point deviations from their deterministic steady-state.

Thus, the impulse responses are shown in Figures 3.4–3.10, illustrating dynamic responses of main macroeconomic variables as percentage-point deviations from steady state. Each figure has an impulse response function for the models M_1 (blue) and M_2 (red). We analyze the effects of various structural shocks (government spending shock, monetary policy shock, labor income tax shock, and consumption tax shocks) on macroeconomic variables, e.g. consumption and labor supply of the differentiated agents; output; public spending and inflation.

Figure 3.4 displays expansionary government spending shock for the Korean economy for the linear and nonlinear model. The size of this shock is a one percent increase in the government spending-to-output ratio. The figure shows that for both models this shock positively affects aggregate output, consumption, inflation, real wages, and hours worked on impact. For example, output increased by 0.004 for M_1 and by 0.0036 for M_2. Aggregate consumption increased by 0.0025 and 0.0011 percentage points for the linear and nonlinear models respectively. Government spending increased by 0.008 for both models.

Also, Figure 3.4 shows increased labor supply, which had been caused by a surge in labor demand following the government spending shock. The labor demand of constrained agents increased by 0.004 for M_1, and 0.003 for M_2. The government increased its spending substantially; this led to a significant increase in the general price level in the economy, including the price of goods and services, marginal cost, and the nominal wage level. This increase implies 1) higher wage inflation and 2) the real wage. However, due to the sticky wage mechanism, real wages rise by only 0.007 percentage points for the linear model and 0.003 for the nonlinear. This resulted in a crowd-in effect for the consumption of constrained agents, 0.01 percentage points for M_1 and 0.005 for M_2. This expansionary effect of public spending is consistent with the findings of linearized studies with two-agent model structure (e.g. Coenen and Straub (2005)).

The high inflation rate generated by increased government spending forced the central bank to implement anti-inflationary measures, and the nominal interest rate increased in response. A rise in the interest rate has a negative wealth effect on unconstrained households actively participating in the assets market; thus, the consumption of unconstrained households has decreased by 0.002 percentage points for M_1; and by 0.003 for M_2. Model M_1 experienced a slight decrease in aggregate investment, while model M_2 showed an increase of 0.005 percentage points.

The overall positive effect of government spending shock is subdued after subsequent quarters as higher wages from increased productivity led households to substitute work for leisure. As expected, the general public debt has increased as an after-effect of increased public spending.

Figure 3.6 shows the effects of contractionary monetary policy shock on macroeconomic outcomes: the central bank increases the nominal interest rate by 0.0012 percentage points for M_1, and 0.0017 for M_2, which effectively leads to a general price fall in the economy by 0.0007 percentage points for model M_1 and 0.0008 for M_2. Contractionary monetary policy shock implemented through the Taylor rule slows down overall economic activity. Figure 3.6 shows a decline in aggregate output and consumption, investment, and hours worked. For example, Bayesian impulse responses of M_1 model depict an immediate fall in total output by about 0.006 percentage points; and a fall in aggregate consumption by about 0.009. As for the nonlinear model, M_2, the contractionary monetary policy shock induces a slightly smaller drop in output of 0.0043, and in aggregate consumption by 0.0062. The reduction of aggregate output in both models is related to the greater investment decrease caused by interest rate increases. Aggregate investment falls by 0.001 for a linear model; and 0.005 for a nonlinear model.

In response to a contractionary monetary policy shock, consumption of constrained agents falls significantly, due to the fall in real wages, which are the main source of their income. In both models, consumption of these agents declines by around 0.01 percentage points.

Consumption of unconstrained agents falls by 0.005 for model M_1; and 0.002 for model M_2 during the first two periods. However, in the following quarters, consumption of unconstrained agents rises for both models because these agents can offset the negative effects of the increased interest rates by relying on their savings in the financial market.

The impulse responses of main macroeconomic variables to positive consumption tax shocks are shown in Figure 3.8. In the linear and nonlinear model results, the increase in consumption tax has a negative impact on aggregate output and consumption. Thus for the linear model, aggregate output and consumption falls by 0.005 and 0.008 percentage points; for the nonlinear model by 0.0045 and 0.007 percentage points. This is because the increase in the consumption tax rates raises the prices of goods and services, leading to an overall contractionary effect in the economy. At the same time, higher consumption tax leads to an increase in the cost of production for firms, which can decrease their profits. In response, the firms reduce labor costs, which decreases labor demand and real wages. As shown in the figure, aggregate labor demand decreases by about 0.005 percentage points for models M_1 and M_2. Since both households now have less disposable income, it reduces their purchasing power and negatively impacts aggregate consumption. However, as in contractionary monetary policy shock, the consumption of Ricardian agents increases in subsequent quarters. Aggregate investment decreases by 0.0005 percentage point for model M_2 due to a reduction in aggregate demand, which in turn decreases production. However, impulse response analysis shows increased capital and investment for the linear model by 0.0001.

Finally, Figure 3.10 shows the impulse responses for the positive labor income shocks. With increased labor income tax on both agents, both agents have less disposable income, which lowers their purchasing power and decreases consumption levels as expected. This reduction leads to a substantial decrease in aggregate output, aggregate consumption, investment, and hours worked. Figure 3.10 illustrates that aggregate output falls by 0.005 and 0.006 percentage points for models M_1 and M_2, respectively. Aggregate consumption decreases by 0.008 for the linear model and by 0.009 for the nonlinear model. The aggregate labor demand decreases by about 0.006 percentage points for both models. The nonlinear model shows a greater negative impact of increased labor income tax on nominal return on capital and marginal cost, which leads to a higher decreased effect on the real wages in the model M_2.

In sum, the Bayesian impulse response analyses indicate that an expansionary government spending shock has positive effects on output, consumption, inflation, real wages, and hours worked; and contractionary monetary policy shock decreases output, aggregate consumption, investment, and hours worked. Positive consumption tax shocks have an adverse impact on aggregate output and consumption, as they increase the cost of production for firms, leading to a contractionary effect on the economy. The results show that an increase in labor income tax negatively affects output and consumption.

However, we obtain different impulse responses to aggregate shocks for several variables of the linear and nonlinear TANK DSGE model. For instance: investments in response to government spending and consumption tax shocks; public debt in response to government spending shocks; and consumption of Ricardian agents in response to labor tax shocks. We investigate the reason for this difference and obtain the impulse response functions of the second-order model M_2 using the parameter estimates from the estimation of the first-order model M_1. The impulse response functions of the first and second-order are nearly identical with the same parameter values (Figures 3.5, 3.7, 3.9, and 3.11). Hence, the main reason for the difference is that we obtain different estimated values when using a second-order approximation. The nonlinear estimation uses quadratic terms to represent economic relationships better; therefore, some parameters have more impact on a second-order approximation. Understandably, the parameter estimates would differ due to this reason.

3.7 Conclusions

In this study, we develop and estimate a nonlinear TANK DSGE model using the first and second-order perturbation technique proposed in Chapter 2. The choice of this model is primarily driven by its tractability relative to the recently developed HANK model, its ability to capture important features of the economy, and its explanation of the effect of aggregate fluctuations on the economy, better than that of the RANK model.

The TANK model has a time-invariant fraction of agents who do not hold assets and rely solely on labor income and government transfers. Another subgroup of agents have access to the bonds market and smooth its consumption intertemporally. In some studies, estimation of these models is mainly done using the linearization technique. In order to achieve greater accuracy, we opt for a higher-order solution of the DSGE model and estimate the model using a likelihood-based approach. Thus, we construct the TANK DSGE model with financially constrained and unconstrained households, monopolistic competitive producers, and fiscal and monetary authority. We model real and nominal frictions nonlinearly, where the former includes investment costs, the latter price and wage frictions.

Further, the parameters of the model are estimated using a likelihood-based approach for the first and second-order approximation, so as to capture nonlinear features of related macroeconomic data. Since our model features a two-agent structure, after obtaining the posterior estimates and constructing the Bayesian impulse responses, we examine the role of the constrained agents in the propagation of government spending, distortionary taxation, and monetary policy shocks and to account for observed fluctuations in output and consumption. For data convenience, we analyzed the quarterly time series data for the Korean economy for 1999 Q4–2021 Q4.

Our post-estimation results show that the linear TANK DSGE model M_1 better fits the Korean economy in terms of the log likelihood. However, the nonlinear model M_2 performs better according to the estimated log marginal and predictive likelihoods. Thus, the marginal likelihood for the nonlinear model is higher than for the linear model by a Bayes factor of $\exp(5)$. This suggests that the nonlinear model is likely to explain new data better and to be less prone to overfitting.

The estimated share of constrained agents in the linear model is around 0.41 for the linear model and 0.53 for the nonlinear model. These results are in line with the existing literature. The estimated parameters in both models indicate that the discount factor, the elasticity of intertemporal substitution, and the Frisch elasticity of labor supply play essential roles in the Korean economy. Furthermore, the estimated parameters for both models suggest that the Korean economy exhibits moderate price and wage stickiness, with an average duration of price contracts of around one year and an average duration of wage contracts of around half a year for the linear model and around one year for the nonlinear model. Finally, posterior estimates suggest that the Korean central bank behaves is a slightly more anti-inflationary manner in the nonlinear model than in the linear model.

In addition, this study analyzes the Bayesian impulse responses of macroeconomic variables to aggregate shocks. The results provide valuable insights into a) how various shocks impact macroeconomic variables such as output, consumption, investment, labor supply and inflation, and b) the differences between the linear and nonlinear models.

The results show that the expansionary spending shock positively impacts aggregate output, consumption, and hours worked with a significant crowd-in effect for the consumption of constrained agents, particularly in the nonlinear model. However, due to the wealth effect, the consumption of financially unconstrained agents decreases. The overall positive effects of government spending shocks subside after subsequent quarters, as higher wages from increased productivity lead households to substitute work for leisure and increased price levels. Moreover, this expansionary effect increases the general price level in the economy, which raises real wages and marginal cost in subsequent periods.

At the same time, the contractionary monetary policy shock has a decreasing effect on output, aggregate consumption, investment, and hours worked. Aggregate output and consumption plummet in response to a monetary policy shock in models M_1 and M_2. In M_2, the results indicate that the contractionary monetary policy positively affects the consumption of unconstrained households because these agents can access financial markets and have savings to mitigate the negative effect of contractionary monetary policy. However, again this positive effect decreases in subsequent quarters.

The consumption tax and labor income tax shocks have a contractionary impact on the economy, decreasing aggregate output, consumption, investment, and hours worked. Regarding the differences between the linear and nonlinear models, the nonlinear model shows more pronounced effects on real wages and consumption of unconstrained agents. Bayesian impulse responses resulting from the nonlinear model show smoother results than those from the linear model.

Thus, our study conducts the linear and nonlinear estimation of DSGE model, featuring limited assets participation. The current chapter emphasizes the importance of evaluating an understanding of the dynamic effects of different shocks on macroeconomic variables in the presence of nonlinearities and constrained agents. In general, the findings of this study provide insights for policymakers and researchers interested in working towards an understanding of the dynamics of the Korean economy.

While we acknowledge that the estimated TANK model is relatively simple compared to other heterogeneous agent models, we recognize that more complex models may be necessary to explain complex subjects such as the monetary and fiscal mechanisms of the entire economy. Nonetheless, we believe that the TANK model can be expanded in various ways while maintaining its relative tractability. For instance, one possible extension is the inclusion of idiosyncratic income shocks, as suggested by Debortoli and Galí (2021). However, estimating the parameters and evaluating the likelihood of such an extended model may require a different approach.

Parameter	Description
$100(1/\beta - 1)$	β is discount factor
δ	capital depreciation
γ_g	share of government spending
μ_p	price markup
μ_w	wage markup
σ	relative risk aversion
ϕ	labor disutility parameter
a_j	parameter of adjustment cost function
γ_π	inflation parameter
γ_y	output parameter
ψ	elasticity demand for intermediate firms
ε_w	elasticity labor demand for intermediate firms
α	capital share
λ	share of constrained households
θ	Calvo's parameter for price setting
θ_w	Calvo's parameter for wage setting
γ_B	share of government bonds
ρ_a	persistence productivity shock
ρ_G	persistence government spending shock
ρ_{τ^c}	persistence consumption tax shock
ρ_{τ^l}	persistence labor tax shock
$\bar{\tau}^c$	consumption tax
γ_R	nominal interest parameter
$\bar{\tau}^l$	labor tax
σ_e	productivity shock
σ_{ϵ_m}	monetary shock
σ_G	government spending shock
σ_{τ^c}	consumption tax shock
σ_{τ^l}	labor tax shock

Table 3.1: Structural parameters

	Distr.	Prior distribution		Posterior distribution			
				M_1 - **First-order**		M_2 - **Second-order**	
		Mean	**St.Dev.**	**Mean**	**Credible interval**	**Mean**	**Credible interval**
α	Normal	0.3	0.1	0.0673	[-0.0025, 0.1484]	0.0777	[-0.2824, 0.4374]
β	Gamma	0.9975	0.1	0.9927	[0.9915, 0.9915]	0.9923	[0.9910, 0.9940]
λ	Beta	0.4	0.1	0.4170	[0.3486, 0.5106]	0.5318	[0.4450, 0.6159]
σ	Normal	2	0.1	0.3329	[0.1663, 0.5673]	0.3804	[-0.8504, 1.6150]
ϕ	Normal	2	0.25	3.7197	[2.7335, 5.1773]	5.3350	[2.1042, 8.5824]
ψ	Normal	5	1	5.1773	[4.3595, 8.3084]	5.1576	[3.6328, 7.0939]
ε_w	Normal	5	1	4.6989	[3.0299, 5.8997]	4.0559	[2.8455, 5.4619]
θ	Beta	0.5	0.15	0.8998	[0.8563, 0.9316]	0.9661	[0.9463, 0.9828]
θ_w	Beta	0.5	0.15	0.4920	[0.3206, 0.6382]	0.6642	[0.5677, 0.7601]
γ_R	Beta	0.7	0.1	0.7109	[0.5857, 0.8118]	0.8435	[0.7904, 0.8886]
γ_π	Gamma	1.5	0.05	1.5256	[1.4319, 1.6323]	1.5266	[1.4280, 1.6313]
γ_Y	Gamma	0.125	0.05	0.0663	[0.0168, 0.0168]	0.0146	[0.0099, 0.0230]
ρ_a	Beta	0.5	0.1	0.6442	[0.5526, 0.7335]	0.4408	[0.3336, 0.5519]
ρ_G	Beta	0.5	0.1	0.6514	[0.5513, 0.7488]	0.6892	[0.5713, 0.7969]
ρ_{τ_c}	Beta	0.5	0.1	0.5192	[0.4551, 0.5968]	0.4872	[0.3530, 0.6255]
ρ_{τ_l}	Beta	0.5	0.1	0.7644	[0.6692, 0.8559]	0.70098	[0.5773, 0.8135]
φ_g	Gamma	0.3	0.05	0.0712	[0.0484, 0.1284]	0.0150	[0.0015, 0.0415]
φ_b	Gamma	0.07	0.05	0.1324	[0.0727, 0.2486]	0.0573	[0.0364, 0.1034]
τ_c	Beta	0.1	0.01	0.1503	[0.1342, 0.1695]	0.1425	[0.1425, 0.1636]
τ_l	Beta	0.25	0.01	0.2450	[0.2286, 0.2627]	0.2469	[0.2279, 0.2660]
a_j	Beta	4	0.1	3.9935	[3.7900 4.2065]	3.9971	[3.9971 4.1951]

Table 3.2: Prior and Posterior distribution for structural parameters of M_1 and M_2

	Distr.	Prior distribution		Posterior distribution			
				M_1 - **First-order**		M_2 - **Second-order**	
		Mean	**St.Dev.**	**Mean**	**Credible interval**	**Mean**	**Credible interval**
$\exp \sigma_e$	IG*	0.1	2	-4.2566**	[-4.6327, -3.7853]	-3.9135	[-4.2520, -3.5694]
$\exp \sigma_{e_m}$	IG	0.1	2	-6.3548	[-6.5501, -6.1368]	-6.2811	[-6.4690, -6.0785]
$\exp \sigma_{\varepsilon_g}$	IG	0.1	2	-5.1549	[-5.7970, -4.2023]	-3.0666	[-4.7413, -3.0666]
$\exp \sigma_{\tau_c}$	IG	0.1	2	-5.6748	[-5.8507, -5.4961]	-5.6956	[-5.8997, -5.4929]
$\exp \sigma_{\tau_l}$	IG	0.1	2	-4.3996	[-4.5660, -4.2132]	-4.6137	[-4.8131, -4.4096]

Table 3.3: Prior and Posterior distribution for structural parameters of M_1 and M_2

Note:* - Inverse Gamma Distribution. 2. ** - The posterior values for the shocks is in $\log((\sigma_x))$

Model	$\boldsymbol{n}$	**Log Prior**	**Log Likelihood**	**Log Marginal Likelihood**	**Predictive Likelihood**	**Posterior probability**
M_1	26	-529.8	1469.6	1277.2	883.2	0
M_2	26	-520.6	1451.7	1281.9	911.9	1

Table 3.4: Model Performance Values

Note: M_1 - a model resulting from the first-order approximation. M_2 - a model resulting from the second-order approximation

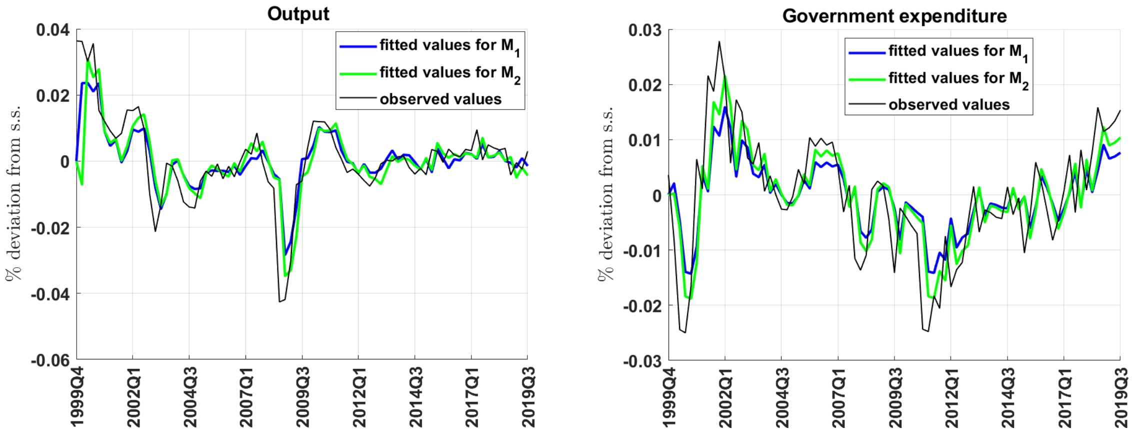

Figure 3.1: Fitted vs observed values for M_1 and M_2

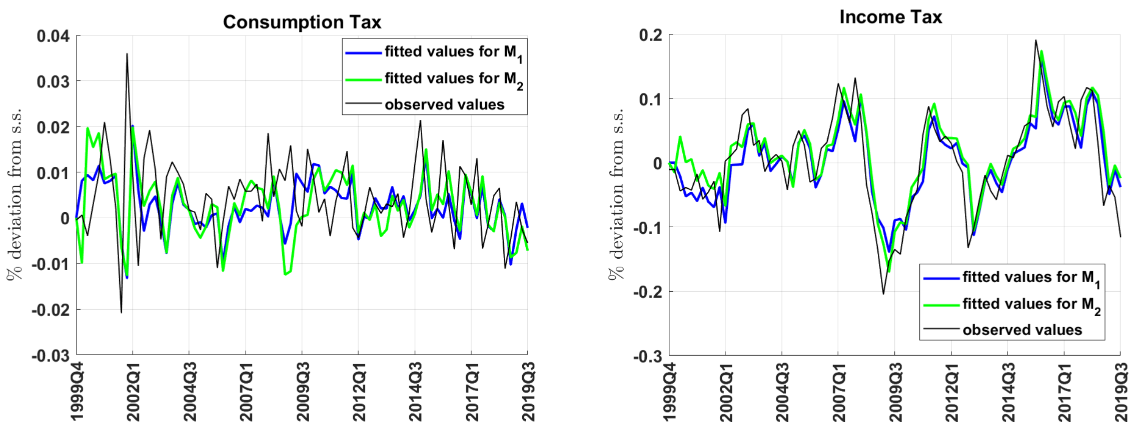

Figure 3.2: Fitted vs observed values for M_1 and M_2

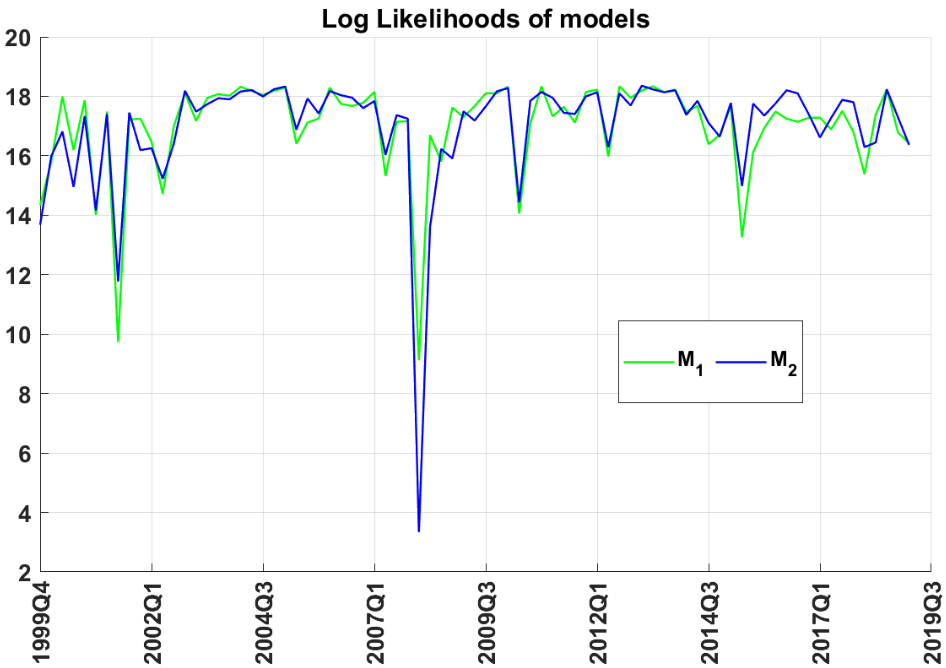

Figure 3.3: Log Likelihood of M_1 and M_2

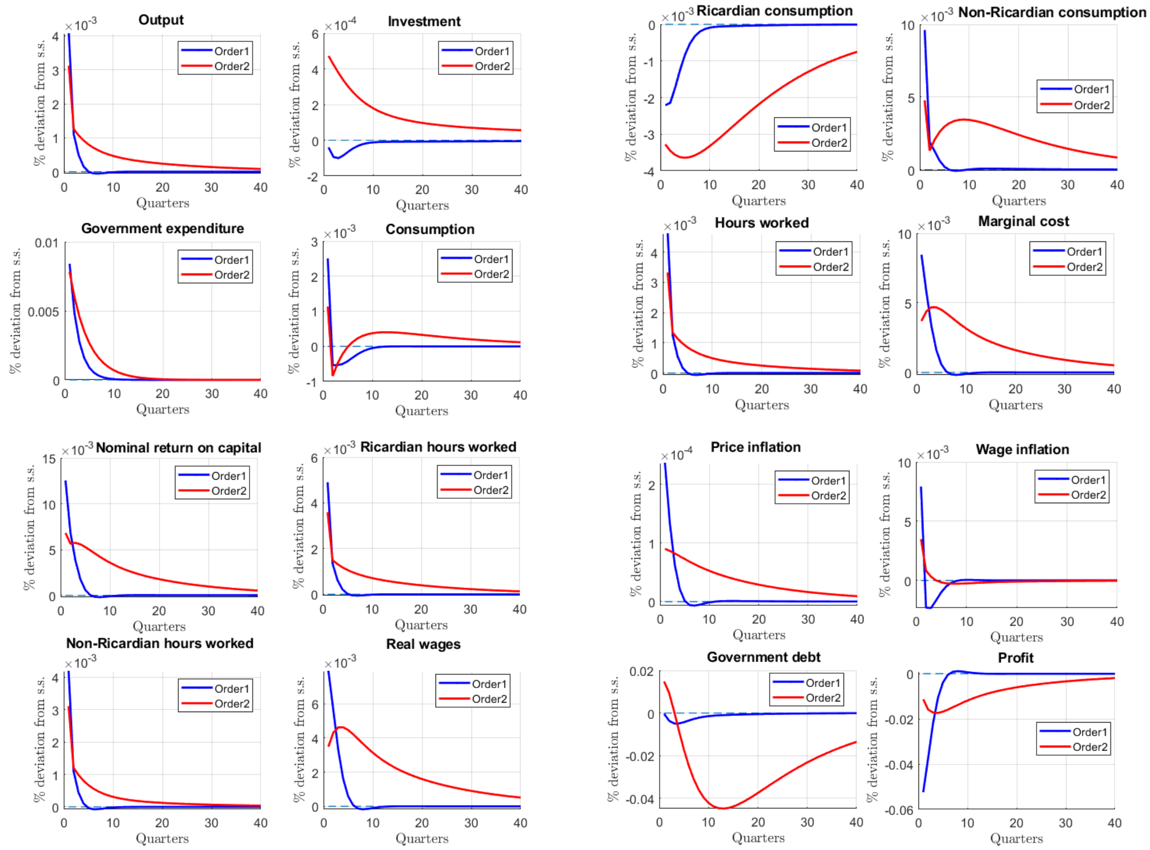

Figure 3.4: IRFs to expansionary government spending shock

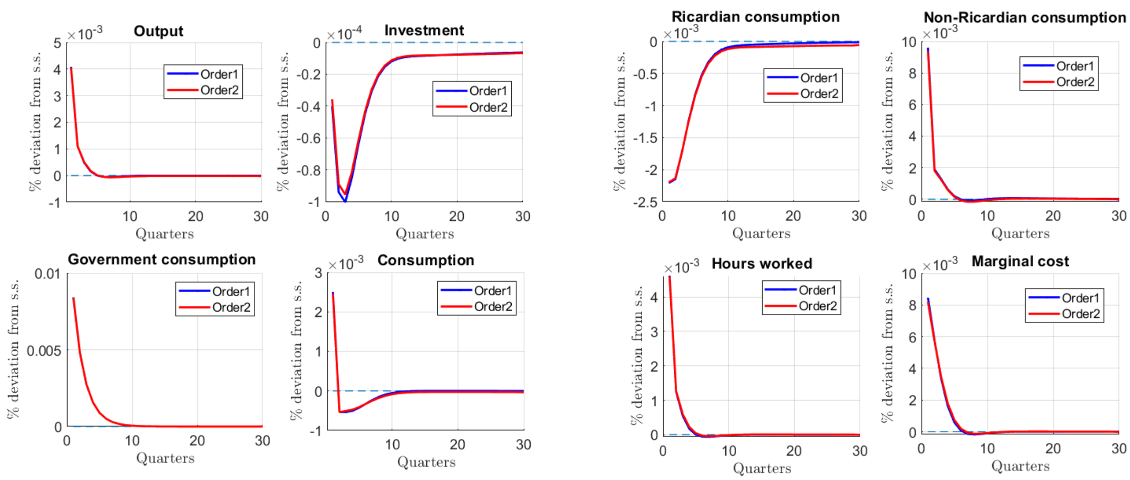

Figure 3.5: IRFs to expansionary government spending shock (common posterior)

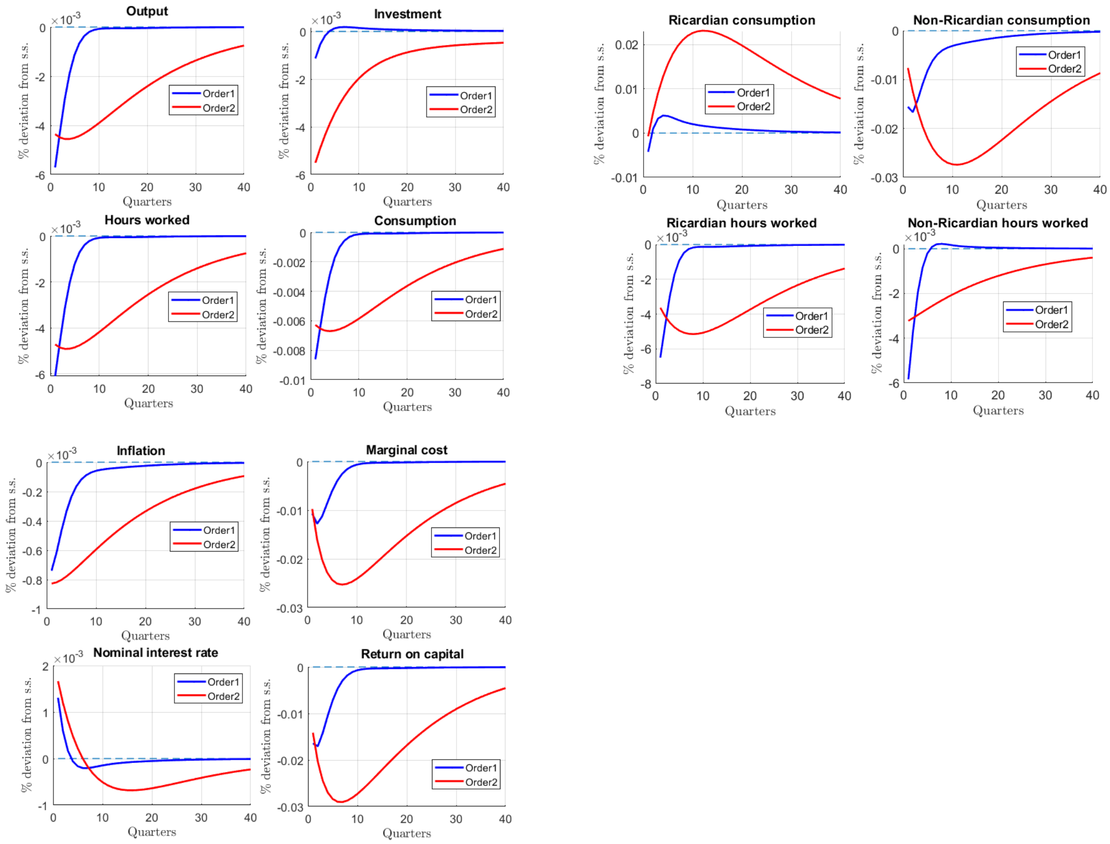

Figure 3.6: IRFs to contractionary monetary policy shock

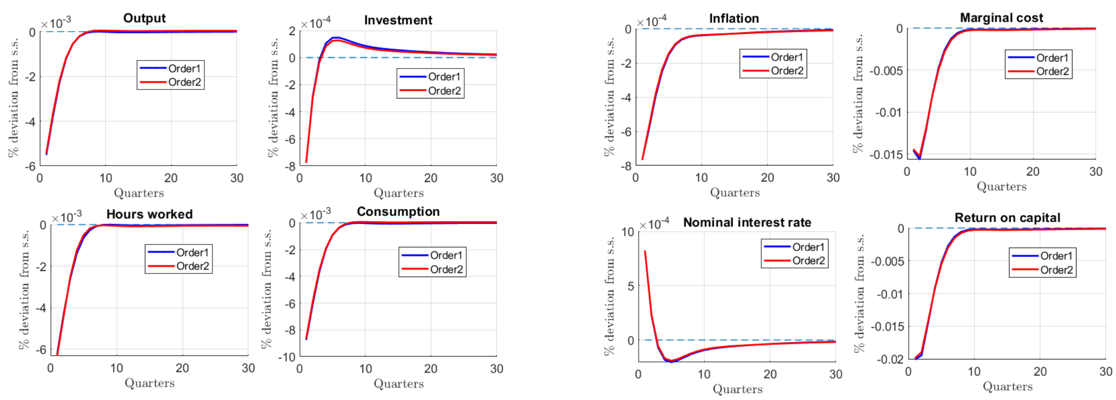

Figure 3.7: IRFs to contractionary monetary policy shock (common posterior)

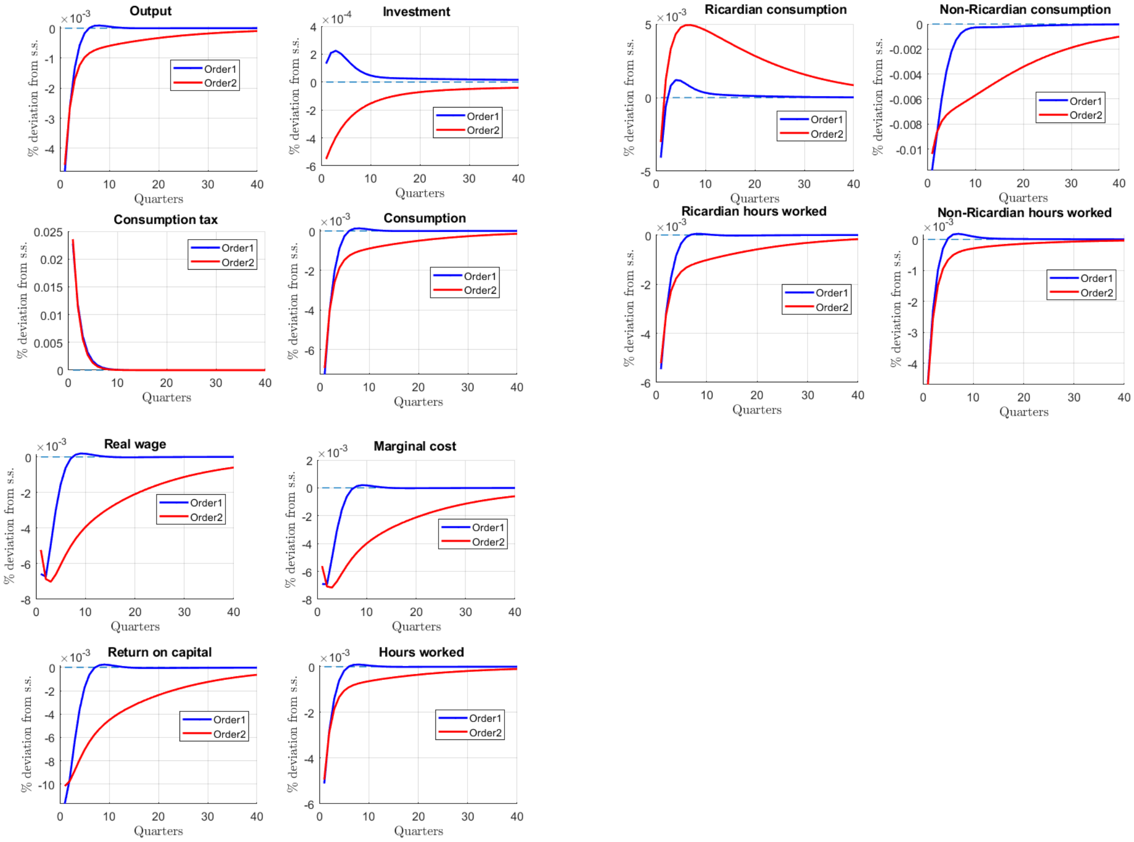

Figure 3.8: IRFs to positive consumption tax shock

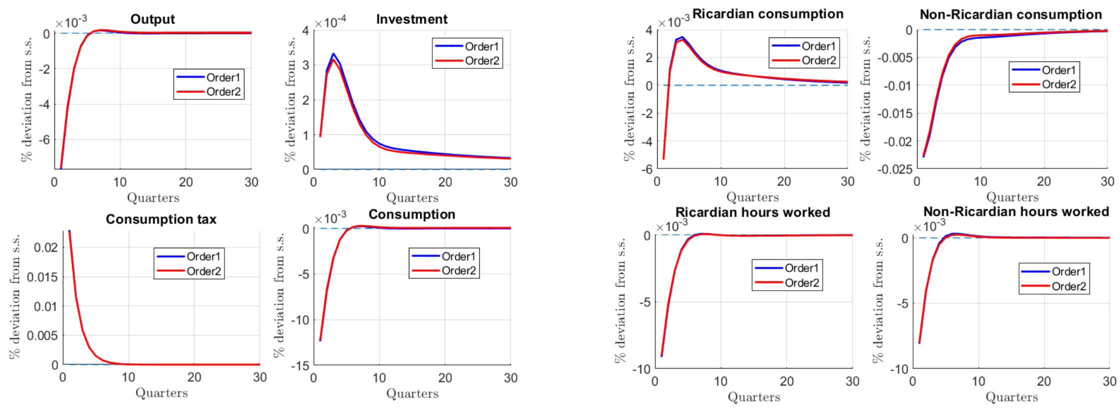

Figure 3.9: IRFs to positive consumption tax shock (common posterior)

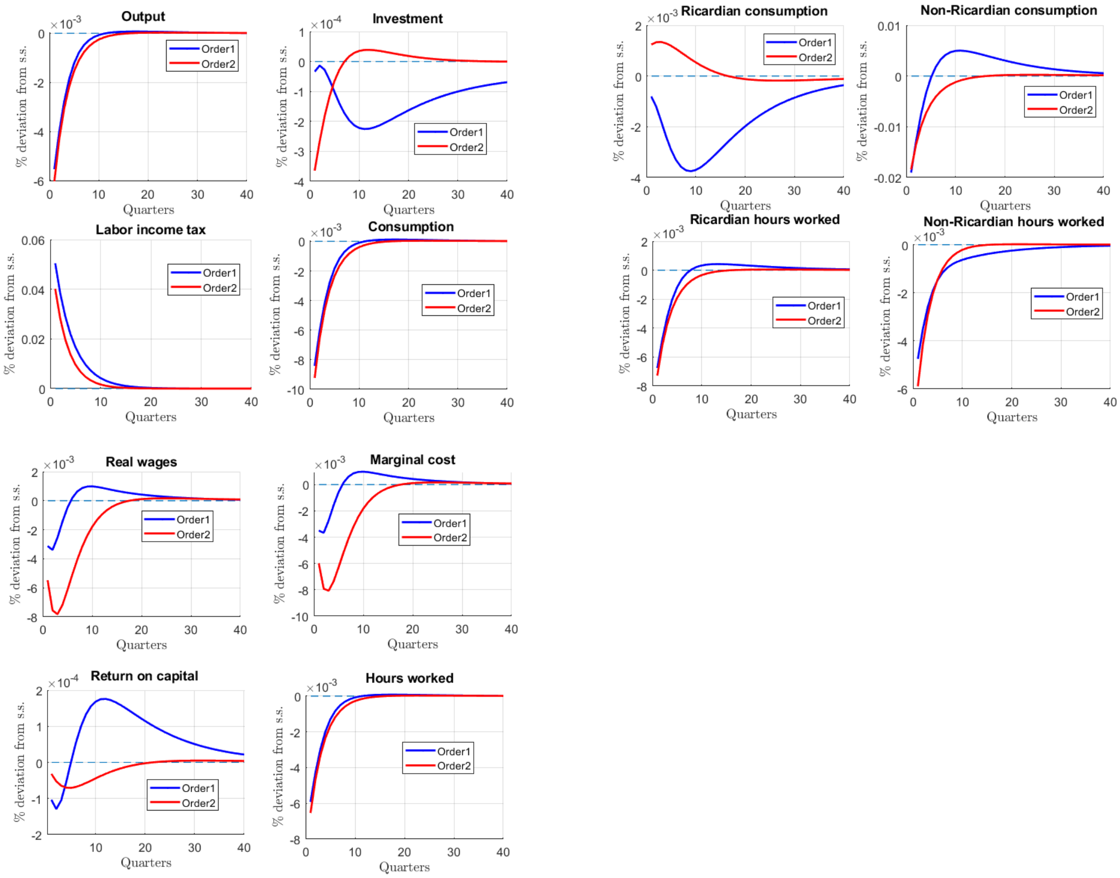

Figure 3.10: IRFs to positive labor income tax shock

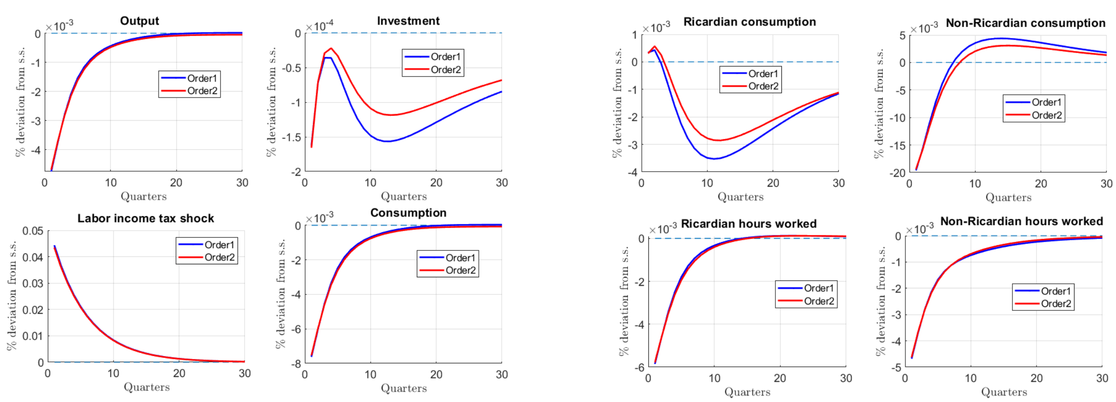

Figure 3.11: IRFs to positive consumption labor income shock (common posterior)

Chapter 4 Conclusions

4.1 Introduction

This chapter presents a summary of the key findings and their implications, highlighting the main contributions of the implemented research. Section 4.2 provides a summary of the main objectives, research methodologies, and findings, while section 4.3 offers policy recommendations based on the research outcomes. Section 4.4 explores the limitation of the studies and proposes promising avenues for future research.

4.2 Conclusions

The estimation method using nonlinear DSGE models is quite restrictive, which motivates us to design a straightforward, faster approach. Thus, this dissertation investigates the nonlinear estimation of DSGE models using the likelihood-based approach for estimating the models resulting from the second-order perturbation technique. Notably, in Chapter 2, we leverage the fact that the shocks in some DSGE models can be recovered uniquely from some of the control variables, implying that the policy function is invertible with respect to the shocks. Therefore, we implicitly used an invertible approximation of the policy function and a Lagrange inversion formula to approximate an inverse of the policy function, enabling us to derive a higher-order likelihood approximation. In the evaluation of the likelihood function of the DSGE model, the current method does not require the introduction of measurement errors or resorting to particle filter theory to compute the likelihood function. This distinguishes our approach from those in the literature on the nonlinear estimation of DSGE models.

To demonstrate the efficacy of the technique, we apply it to the well-known neoclassical growth model of Fernández-Villaverde (2010), using US data for 1959 Q1–2019 Q4. In addition, to estimate the models resulting from first and second-order approximation, we modify the growth model by introducing heteroscedasticity through GARCH processes (including restricted and unrestricted versions of the GARCH model) to capture uncertainty in shocks.

The main findings of Chapter 2 suggest that our nonlinear model with restricted GARCH processes delivers a better fit of the model (as measured by marginal and predictive likelihoods) to the data than the other three models. The model resulting from second-order approximation with homoskedastic shocks shows better results than the linearized model obtained from the first-order perturbation method. The posterior analyses indicate that the monetary policy shock in a nonlinear heteroscedastic model is the main driving force of uncertainty in the economy.

In Chapter 3 we develop a TANK DSGE model with two types of households, constrained and unconstrained, with constant shares in population. The choice of this model is primarily driven by its tractability relative to the recently developed HANK model and its ability to provide an approximation of the impact of the HANK model impact on aggregate variables. In comparison with the RANK model, where all agents are assumed to be identical and have the same preferences, the TANK model is more realistic, as it allows for some degree of heterogeneity in the population. As existing studies show, the TANK models give a better explanation than the RANK models of the effects of aggregate fluctuations on the economy. Our TANK model incorporates price and wage rigidity dynamics, capital adjustment costs, and various structural shocks related to technology, monetary policy, government spending, labor income, and consumption tax. We also model real and nominal frictions nonlinearly, with investment costs as a component of the former and price and wage frictions as part of the latter. Regarding the estimation, estimations in existing studies with two-agent structure are mainly linear. Since second-order approximation allows us to capture some nonlinearities featured in macroeconomic data (which could be missing in the linearized model) we estimate the model by means of the proposed likelihood-based approach using the perturbation method of order one and two for the Korean economy with quarterly aggregate data for 1999 Q4–2021 Q4.

The estimation result demonstrates that, unlike a growth model in Chapter 2, the linear representation of the TANK model generates a better fit of the model to the data (as measured by log-likelihood at the posterior mode). However, the nonlinear model is preferable in terms of log marginal and predictive likelihoods than the linearized one. Nevertheless, these results support the idea of including nonlinearities when analyzing the behavior of the TANK model.

The posterior analyses show that the posterior distribution for some parameters diverts from the parameters estimated using the first-order perturbation method. Standard deviations of the parameters are estimated to be much higher for the second-order approximation model than for the first-order, indicating that the posterior distribution of the nonlinear model is more dispersed.

Bayesian impulse analyses show that expansionary spending shock positively impacts aggregate output and consumption. It positively affects the consumption of constrained agents in nonlinear and linear models. It should be noted that this effect is pronounced for nonlinear models since the share of constrained agents is estimated to be higher than for the linear model. However, increased public spending has a negative effect on the consumption of unconstrained agents for both models due to a negative wealth effect. In addition, we find an overall negative impact on aggregate output, consumption, and hours worked from increased labor income and consumption tax shocks. This negative effect is also pronounced for the nonlinear model.

Bayesian impulse analyses for positive monetary policy shocks have a contractionary effect on the overall economy, which is in line with the findings of existing studies.

4.3 Policy implication

The findings of Chapter 2 argue in favor of the nonlinear estimation of DSGE models to assess the role of monetary and fiscal policy rules on macroeconomic outcomes. Linear DSGE models cannot capture important features such as consumer risk aversion, time-varying variances, asset pricing, and welfare evaluations. Although nonlinear DSGE models help address these issues, they present challenges for estimation. The new method presented here facilitates the estimation of nonlinear DSGE models and makes it possible to evaluate some nonlinear DSGE models that cannot be estimated with existing methods. As a practical policy issue, for example, the design of optimal monetary or fiscal policies to improve the welfare of individuals in an economy. As mentioned above, there are already methods to do this in the literature; the method proposed in this dissertation widens the tools for empirical analysis of these issues.

Moreover, our results show that the nonlinear model with changes in uncertainty over time performs better empirically than all other estimated models. The findings of this study tend to support to the rising interest shown by central banks in the estimation of nonlinearly solved DSGE models, particularly in modeling and estimating DSGE models with time-varying variances. The study also contributes to the understanding the sources of uncertainty in the economy and the role of different shocks in driving fluctuations in economic activity. This relevance has become even more significant in the context of the recent situation surrounding the Covid-19 pandemic and ongoing global conflicts.

In addition, the results of this study provide evidence that nonlinear models can deliver a more accurate representation of the economy, allowing for nonlinear relationships between variables. These features can affect the transmission of monetary policy and the response of macroeconomic variables to aggregate shocks, which are important considerations for central banks in their policy-making decisions. The implications of the findings can be useful for central banks in their efforts to stabilize the economy and achieve their policy objectives.

The findings from Chapter 3 on the expansionary effect of public spending support the argument that despite wage rigidities, fiscal stimulus in the form of public spending can have an expansionary effect on the economy and support economic recovery. Meanwhile, the positive shocks to income and consumption tax have a hindering impact on the entire economy. However, it is important to note that the effectiveness of fiscal stimuli depends on various factors, including the magnitude of the stimulus, the timing of its implementation, and the specific characteristics of the economy in question.

4.4 Study limitations and further research

The proposed estimation technique is a relatively new approach, and we recognize several limitations to our study. First, the method is not applicable in all cases, as it requires the policy function to be invertible concerning shocks. Furthermore, the analysis is restricted to second-order approximation and to the case in which the number of shocks equals the number of observed variables. Since these two assumptions are rather restrictive, future studies could usefully address these challenges by further exploring the estimation of nonlinear DSGE models. Therefore, subsequent research analysis can investigate this issue further in order to work out how to relax these two assumptions.

As recent articles suggest, uncertainty shock can support comprehension of the principal features of business cycle fluctuations. Time-varying uncertainty became the object of a booming line of research during the last decade. In modeling these uncertainty shocks, the macroeconomic literature has pointed out three main alternative approaches to specifying changes uncertainty over time: stochastic volatility (SV), GARCH processes, and Markov regime switching (Fernández-Villaverde and A.Guerrón-Quintana (2020)).

In this dissertation, we analyze and estimate nonlinear models with heteroscedastic structural shocks employing GARCH processes. However, some studies show that regarding SV and GARCH models, the latter employs one shock, driving the dynamics of the level of volatility of random variables of interest. Thus, separating a volatility shock from a level shock is impossible: higher volatilities are triggered only by a large past-level innovation. This assumption is partly restrictive in analyses of the DSGE model. Furthermore, some studies show that SV models tend to fit the data better (Nakajima (2012)). Therefore, modeling and analysis of SV in the context of a nonlinear DSGE model would also be an intriguing direction for future research.

From the computation side, since DSGE models with uncertainty shocks are highly dimensional, as in this work, future exploration cannot rely on linearizing the equilibrium conditions of the model. The limitation of linearized models motivates us to develop a new approach for estimating models resulting from a second-order approximation of the solution of the DSGE model. Nevertheless, the literature shows that second-order approximation involves uncertainty shock comprising only cross-products of level and volatility shock. Thus, exploring the third-order perturbation method could extend the insights gained in this dissertation.

From another perspective, the quantitative aspect of the macroeconomic model could be fertile for future exploration. There are disputes arising regarding the utilization of, for example, New Keynesian DSGE models, particularly to obtain quantitative conclusions about the overall economic impact of a policy. While we acknowledge that the estimated TANK model is simpler than other heterogeneous agent models, we recognize that more complex models may be necessary to explain complex subjects such as the monetary and fiscal mechanisms of the entire economy. Nonetheless, the TANK model can be expanded in various ways while maintaining its relative tractability. For instance, one possible extension is the inclusion of idiosyncratic income shocks, as suggested by Debortoli (2018). However, estimating the parameters and evaluating the likelihood of such an extended model may require a different approach.

Appendix

$$
\begin{aligned}
W_t^* &= \frac{\varepsilon}{\varepsilon-1} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left[\frac{(L_{t+s}^{d,R})^{\varphi}}{\zeta_{t+s}^{R}(1-\tau_{t+s}^{l})} \right] = \\
&= \frac{\varepsilon_w}{\varepsilon_w-1} \frac{(L_t^{d,R})^{\varphi}}{\zeta_t^{R}(1-\tau_t^{l})} + \frac{\varepsilon_w}{\varepsilon_w-1} E_t \sum_{s=1}^{\infty} (\theta_w \beta)^s \left[\frac{(L_{t+s}^{d,R})^{\varphi}}{\zeta_{t+s}^{R}(1-\tau_{t+s}^{l})} \right] = \\
&\frac{\varepsilon_w}{\varepsilon_w-1} \frac{(L_t^{d,R})^{\varphi}}{\zeta_t^{R}(1-\tau_t^{l})} + (\theta_w \beta) \frac{\varepsilon_w}{\varepsilon_w-1} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left[\frac{(L_{t+1+s}^{d,R})^{\varphi}}{\zeta_{t+1+s}^{R}(1-\tau_{t+1+s}^{l})} \right] = \\
&\frac{\varepsilon_w}{\varepsilon_w-1} \frac{(L_t^{d,R})^{\varphi}}{\zeta_t^{R}(1-\tau_t^{l})} + (\theta_w \beta) W_{t+1}^*
\end{aligned}
\tag{1}
$$

Similarly, for Non-Ricardian households:

$$
\begin{aligned}
W_t^* &= \frac{\varepsilon}{\varepsilon-1} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left[\frac{(L_{t+s}^{d,NR})^{\varphi}}{\zeta_{t+s}^{NR}(1-\tau_{t+s}^{l})} \right] = \\
&= \frac{\varepsilon_w}{\varepsilon_w-1} \frac{(L_t^{d,NR})^{\varphi}}{\zeta_t^{NR}(1-\tau_t^{l})} + \frac{\varepsilon_w}{\varepsilon_w-1} E_t \sum_{s=1}^{\infty} (\theta_w \beta)^s \left[\frac{(L_{t+s}^{d,NR})^{\varphi}}{\zeta_{t+s}^{NR}(1-\tau_{t+s}^{l})} \right] = \\
&\frac{\varepsilon_w}{\varepsilon_w-1} \frac{(L_t^{d,NR})^{\varphi}}{\zeta_t^{NR}(1-\tau_t^{l})} + (\theta_w \beta) \frac{\varepsilon_w}{\varepsilon_w-1} E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left[\frac{(L_{t+1+s}^{d,NR})^{\varphi}}{\zeta_{t+1+s}^{NR}(1-\tau_{t+1+s}^{l})} \right] = \\
&\frac{\varepsilon_w}{\varepsilon_w-1} \frac{(L_t^{d,NR})^{\varphi}}{\zeta_t^{NR}(1-\tau_t^{l})} + (\theta_w \beta) W_{t+1}^*
\end{aligned}
\tag{2}
$$

$$g_t^2 = \sum_{i=0}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{1-\psi} \Pi_t^* Y_{t+i} \right] =$$

$$= (C_t^R)^{-\sigma} Y_t \Pi_t^* + \sum_{i=1}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{1-\psi} \Pi_t^* Y_{t+i} \right] =$$

$$= (C_t^R)^{-\sigma} Y_t \Pi_t^* + E_t \left(\frac{P_t}{P_{t+1}} \right)^{1-\psi} \frac{\Pi_t^*}{\Pi_{t+1}^*} \sum_{i=1}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_{t+1}}{P_{t+i}} \right)^{1-\psi} \Pi_{t+1}^* Y_{t+i} \right] = \quad (3)$$

$$= (C_t^R)^{-\sigma} Y_t \Pi_t^* + \beta\theta E_t \left(\frac{P_t}{P_{t+1}} \right)^{1-\psi} \frac{\Pi_t^*}{\Pi_{t+1}^*} g_{t+1}^2 =$$

$$= (C_t^R)^{-\sigma} Y_t \Pi_t^* + \beta\theta E_t \Pi_{t+1}^{\psi-1} \frac{\Pi_t^*}{\Pi_{t+1}^*} g_{t+1}^2$$

$$g_t^1 = \sum_{i=0}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{-\psi} Y_{t+i} MC_{t+i} \right] =$$

$$= (C_t^R)^{-\sigma} Y_t MC_t + \sum_{i=1}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_t}{P_{t+i}} \right)^{-\psi} Y_{t+i} MC_{t+i} \right] =$$

$$(4)$$

$$= (C_t^R)^{-\sigma} Y_t MC_t + E_t \left(\frac{P_t}{P_{t+1}} \right)^{-\psi} \sum_{i=1}^{\infty} \theta^i E_t \left[\beta^i (C_{t+i}^R)^{-\sigma} \left(\frac{P_{t+1}}{P_{t+i}} \right)^{-\psi} Y_{t+i} MC_{t+i} \right] =$$

$$= (C_t^R)^{-\sigma} Y_t MC_t + \beta\theta E_t \Pi_{t+1}^{\psi} g_{t+1}^1$$

Steady State model, derivation

At the steady state we have: $A_{ss}=1$, $S^p_{ss}=1$, $S^w_{ss}=1$, $\Pi^w_{ss}=1$, $\Pi^{w*}_{ss}=1$

$$W^*_{ss}=\frac{\varepsilon_w}{\varepsilon_w-1}\frac{(L^{d,R}_{ss})^{\varphi}}{\zeta^R_{ss}(1-\tau^l_{ss})}+(\theta_w\beta)W^*_{ss} \tag{1}$$

and

$$W^*_{ss}=\frac{\varepsilon_w}{\varepsilon_w-1}\frac{(L^{d,NR}_{ss})^{\varphi}}{\zeta^{NR}_{ss}(1-\tau^l_{ss})}+(\theta_w\beta)W^*_{ss} \tag{2}$$

and

$$\mu^w=\frac{\varepsilon_w}{(\varepsilon_w-1)}$$

We assume at steady-state:

$$C^R_{ss}=C^{NR}_{ss}=C_{ss}$$

$$L^d_{ss}=(L^R_{ss})^d=(L^{NR}_{ss})^d$$

and $C_{ss}=\gamma_c Y_{ss}$.

We have:

$$\zeta^R_{ss}=\frac{(C^R_{ss})^{-\sigma}}{(1+\tau^c_{ss})}=\frac{(\gamma_c Y_{ss})^{-\sigma}}{(1+\tau^c_{ss})} \tag{3}$$

Substituting (3) in (1) we have:

$$W^*_{ss}(1-\theta_w\beta)=\frac{\mu^w(L^d_{ss})^{\varphi}(1+\tau^c_{ss})(\gamma_c Y_{ss})^{\sigma}}{(1-\tau^l_{ss})} \tag{4}$$

and noting: $W^*_{ss}=W_{ss}$

So, we have first steady-state equation for $Y_s s$

$$(Y_{ss})^{\sigma}=\frac{W_{ss}(1-\theta_w\beta)(1-\tau^l_{ss}}{\mu^w(L^d_{ss})^{\varphi}(1+\tau^c_{ss})\gamma^{\sigma}_c} \tag{5}$$

From (3.24)

$$Y_{ss}=\frac{A_{ss}K^{\alpha}_{ss}(L^d_{ss})^{1-\alpha}}{S^p_{ss}}$$

$$Y_{ss}=K^{\alpha}_{ss}(L^d_{ss})^{1-\alpha}$$

From (3.13)

$$\frac{K_{ss}}{L^d_{ss}} = \frac{\alpha}{(1-\alpha)}\frac{W_{ss}}{R_{ss}}$$

$$K_{ss} = \frac{L^d_{ss}\alpha W_{ss}}{(1-\alpha)R_{ss}}$$

$$Y_{ss} = \left(\frac{L^d_{ss}\alpha W_{ss}}{(1-\alpha)R_{ss}}\right)^{\alpha}(L^d_{ss})^{1-\alpha}$$

Or

$$Y_{ss} = \left(\frac{\alpha W_{ss}}{(1-\alpha)R_{ss}}\right)^{\alpha} L^d_{ss}$$

and we raise Y_{ss} to power σ:

$$(Y_{ss})^{\sigma} = \left[\left(\frac{\alpha W_{ss}}{(1-\alpha)R_{ss}}\right)^{\alpha} L^d_{ss}\right]^{\sigma}$$

Since we have previously received from (5):

$$(Y_{ss})^{\sigma} = \frac{W_{ss}(1-\theta_w\beta)(1-\tau^l_{ss})}{\mu^w(L^d_{ss})^{\varphi}(1+\tau^c_{ss})\gamma^{\sigma}_c}$$

we can equate two equations for Y_{ss} to obtain the steady-state for L^d_{ss}:

$$\frac{W_{ss}(1-\theta_w\beta)(1-\tau^l_{ss})}{\mu^w(L^d_{ss})^{\varphi}(1+\tau^c_{ss})\gamma^{\sigma}_c} = \left[\left(\frac{\alpha W_{ss}}{(1-\alpha)R_{ss}}\right)^{\alpha} L^d_{ss}\right]^{\sigma}$$

$$\frac{W_{ss}(1-\theta_w\beta)(1-\tau^l_{ss})}{\mu^w(L^d_{ss})^{\varphi}(1+\tau^c_{ss})\gamma^{\sigma}_c} = (L^d_{ss})^{\sigma}\left[\left(\frac{\alpha W_{ss}}{(1-\alpha)R_{ss}}\right)^{\alpha}\right]^{\sigma}$$

We fraction out L^d_{ss}:

$$(L^d_{ss})^{\sigma+\varphi} = \left[\frac{W_{ss}(1-\theta_w\beta)(1-\tau^l_{ss})}{\mu^w(L^d_{ss})^{\varphi}(1+\tau^c_{ss})\gamma^{\sigma}_c}\left(\frac{(1-\alpha)R_{ss}}{\alpha W_{ss}}\right)^{\alpha\sigma}\right]$$

$$L^d_{ss} = \left[\frac{W_{ss}(1-\tau^l_{ss})(1-\theta_w\beta)}{\mu^w\gamma^{\sigma}_c(1+\tau^c_{ss})}\left(\frac{(1-\alpha)R_{ss}}{\alpha W_{ss}}\right)^{\alpha\sigma}\right]^{\frac{1}{\sigma+\varphi}} \tag{6}$$

As for the consumption:

$$C_{ss} = Y_{ss} - I_{ss} - G_{ss}$$

$$C_{ss} = Y_{ss} - \delta K_{ss} - \gamma_g Y_{ss}$$

$$C_{ss} = Y_{ss}\left(1 - \frac{\delta K_{ss}}{Y_{ss}} - \gamma_g\right)$$

Since

$$K_{ss} = \alpha MC_{ss}\frac{Y_{ss}}{R_{ss}}$$

or

$$K_{ss} = \alpha\frac{1}{\mu_p}\frac{Y_{ss}}{R_{ss}}$$

and

$$C_{ss} = Y_{ss}\left(1 - \frac{\delta\alpha}{\alpha\frac{Y_{ss}}{K_{ss}}} - \gamma_g\right)$$

we can rearrange C_{ss}:

$$C_{ss} = Y_{ss}\left(1 - \frac{\delta\alpha}{\alpha\frac{Y_{ss}}{\frac{\alpha Y_{ss}}{\mu_p R_{ss}}}} - \gamma_g\right)$$

$$C_{ss} = Y_{ss}\left(1 - \frac{\delta\alpha}{\alpha\frac{Y_{ss}}{\frac{\alpha Y_{ss}}{\mu_p R_{ss}}}} - \gamma_g\right)$$

$$C_{ss} = Y_{ss}\left(1 - \frac{\delta\alpha}{\mu_p R_{ss}} - \gamma_g\right)$$

Since:

$$R_{ss} = \frac{1}{\beta} - (1-\delta)$$

$$C_{ss} = Y_{ss}\left(1 - \frac{\delta\alpha}{\mu_p\left(\frac{1}{\beta} - (1-\delta)\right)} - \gamma_g\right)$$

we define:

$$\rho = \frac{1}{\beta} - 1$$

and we obtain following:

$$C_{ss} = Y_{ss}\left(1 - \frac{\delta\alpha}{\mu_p(\rho+\delta)} - \gamma_g\right)$$

Steady State model, aggregation

Steady state equations:

$$A_{ss} = 1 \tag{1}$$

$$S^p_{ss} = 1 \tag{2}$$

$$\Pi_{ss} = 1 \tag{3}$$

$$\Pi^*_{ss} = 1 \tag{4}$$

$$\Pi^w_{ss} = 1 \tag{5}$$

$$\Pi^{w*}_{ss} = 1 \tag{6}$$

$$R^B_{ss} = \frac{1}{\beta} \tag{7}$$

$$MC_{ss} = \frac{\psi - 1}{\psi} \tag{8}$$

$$\rho = \frac{1}{\beta} - 1 \tag{9}$$

$$\gamma_c = 1 - \gamma_g - \frac{\delta\alpha}{(\rho + \delta)\mu_p} \tag{10}$$

$$R_{ss} = \frac{1}{\beta} - (1 - \delta) \tag{11}$$

$$Q_{ss} = 1 \tag{12}$$

$$W_{ss} = (1 - \alpha) MC_{ss}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{R_{ss}}\right)^{\frac{\alpha}{1-\alpha}} \tag{13}$$

$$W^*_{ss} = W_{ss}\Pi^{w*}_{ss} \tag{14}$$

$$L^d_{ss} = \left[\frac{W_{ss}(1 - \tau^l_{ss})(1 - \theta_w\beta)}{\mu^w\gamma_c^\sigma(1 + \tau^c_{ss})}\left(\frac{(1 - \alpha)R_{ss}}{\alpha W_{ss}}\right)^{\alpha\sigma}\right]^{\frac{1}{\sigma+\varphi}} \tag{15}$$

$$L^d_{ss} = (L^R_{ss})^d = (L^{NR}_{ss})^d \tag{16}$$

$$K_{ss} = \frac{L^d_{ss}\alpha W_{ss}}{(1 - \alpha)R_{ss}} \tag{17}$$

$$K^R_{ss} = \frac{K_{ss}}{(1 - \lambda)} \tag{18}$$

$$Y_{ss} = {K_{ss}}^{\alpha} (L^d_{ss})^{(1-\alpha)} \tag{19}$$

$$I_{ss} = \delta K_{ss} \tag{20}$$

$$I^R_{ss} = \frac{I_{ss}}{(1-\lambda)} \tag{21}$$

$$G_{ss} = \gamma_g Y_{ss} \tag{22}$$

$$C_{ss} = Y_{ss} \left(1 - \frac{\delta\alpha}{\mu_p(\rho+\delta)} - \gamma_g \right) \tag{23}$$

$$C_{ss} = C^R_{ss} = C^{NR}_{ss} \tag{24}$$

$$T_{ss} = \gamma_T Y_{ss} \tag{25}$$

$$T^{NR}_{ss} = W_{ss} (L^d_{ss})^{NR} (1-\tau^l_{ss}) - C^{NR}_{ss} (1+\tau^c_{ss}) \tag{26}$$

$$T^R_{ss} = \frac{T_{ss} - \lambda T^{NR}_{ss}}{1-\lambda} \tag{27}$$

$$TAX_{ss} = \tau^c_{ss} C_{ss} + \tau^l_{ss} W_{ss} L^d_{ss} + T_{ss} \tag{28}$$

$$B_{ss} = \gamma_B Y_{ss} \tag{29}$$

$$B^R_{ss} = \frac{B_{ss}}{(1-\lambda)} \tag{30}$$

$$\zeta^R_{ss} = \frac{(C^R_{ss})^{-\sigma}}{(1+\tau^c_{ss})} \tag{31}$$

$$\zeta^{NR}_{ss} = \frac{(C^{NR}_{ss})^{-\sigma}}{(1+\tau^c_{ss})} \tag{32}$$

$$D_{ss} = Y_{ss} - W_{ss} L^d_{ss} - R_{ss} K_{ss} \tag{33}$$

$$D^R_{ss} = \frac{D_{ss}}{(1-\lambda)} \tag{34}$$

$$g^1_{ss} = \frac{(C^R_{ss})^{-\sigma} Y_{ss} MC_{ss}}{1-\theta\beta} \tag{35}$$

$$g^2_{ss} = \frac{(C^R_{ss})^{-\sigma} Y_{ss} \Pi^*_{ss}}{1-\theta\beta} \tag{36}$$

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