Univariate and Multivariate Inverse Gamma Stochastic Volatility Models

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Introduction

It is difficult, for most non-linear or non-Gaussian state space models, to obtain the likelihood function in closed form . This prevents the use of Maximum Likelihood Estimation (MLE) to estimate the parameters and unobserved states for these state space model. Without the likelihood expression, estimation of non linear non gaussian state space models generally involves bayesian methods such as Markov Chain Monte Carlo. Generalized Autoregressive Conditional Heteroscedasticity (GARCH) models are simpler to estimate than Stochastic Volatility (SV) models, because the likelihood for a GARCH model can be easily calculated in closed form. However, SV models have often been found to outperform GARCH models in empirical studies for both macroeconomic and financial data (e.g.Kim et al. (1998)). In addition, unlike GARCH models, SV models provide not only filtered estimates but also smoothed estimates of the volatility. Although in linear Gaussian state space models the likelihood is available in closed form and can easily be calculated with the Kalman Filter algorithm, few studies have attempted to obtain a closed form expression for the likelihood in nonlinear non-Gaussian state space models.

To extend these univariate SV approaches to estimating large vector autoregressions, Carriero et al. (2016) observed that the volatility estimate patterns across macroeconomic variables exhibit similarities across the variables in support of their common drifting volatility model. They employ a common factor model whose volatility is log normal. These, Common Stochastic Volatility (CSV) models capture the commonality that is often observed in volatility patterns. A number of papers either apply or build on this specification of the variance to model this feature of macroeconomic variables. Inclusion of this common volatility feature in the error structure consistently increased forecasting accuracy in the above literature. However, CSV models assume that all the time variation in volatility is driven by a single multiplicative factor. Given the empirical evidence on fat tailed distributions, and the commonality that is observed in volatility patterns it is worthwhile to improve the efficiency of CSV models, by adding these properties to the model specifications. However, these specifications imply significant complexity of estimating these CSV models.

Methodology

To address these challenges, I proposes a novel method to explicitly calculate the likelihood for a stationary inverse gamma Stochastic Volatility model, which is conventionally approximated using sampling methods. The derived likelihood is expressed by infinite series of functions and its calculation is implemented by truncating higher order terms. Moreover, the likelihood function proposed in this dissertation, can be calculated efficiently using a simple recursion. The calculations can be accelerated by implementing computations in parallel, as well as by applying Euler or other acceleration techniques to the Gauss hypergeometric functions in the likelihood. I show that by marginalising out the volatilities, the model that is obtained has the resemblance of a GARCH in the sense that the formulas are similar, which simplifies computations significantly. This expression of the likelihood allows the estimation of the parameters and unobserved states for this model class by MLE. In addition to obtaining the exact likelihood, I obtain analytically the expressions for the smoothed and filtered estimates of the volatilities as mixture of gammas. The computer code to perform the analysis has been publicly made available through a R package that is freely available from the Comprehensive R Archive Network (CRAN).

To extend the univariate approach to the multivariate analysis, the study combines both properties of the heavy tailed distributions observed in most macroeconomic and financial applications, and a common heteroscedastic latent factor. The volatility for this novel CSV model follows an inverse gamma process, which implies student-t type tails for the observed data. First, I obtain an analytical expression for the likelihood for the variance of the error structure, where the time varying volatility is inverse gamma. This specification allows for heavy tails, which are an important feature of economic and financial data empirically. I show that by marginalising out the volatilities, the analytical expression of the likelihood that I obtain is simple to estimate and computationally efficient. Using this likelihood, I then obtain numerically, the marginal likelihood and the one step ahead out of sample forecasts that are the basis of comparison.

Results and Discussion

The first paper evaluates the performance of the proposed method using several macroeconomic and financial data applications. The macroeconomic application uses quarterly inflation data observations of data ranging from 1960Q2 to 2022Q1 for the US, UK and Japan while Brazil data consists of observations for the period 1981Q1 to 2021Q4. To check the accuracy of the computations proposed in our model, I used the particle filter log likelihood comparison with the model computation. To test efficiency, I check the computation time needed to obtain the log likelihood using the proposed approach. Results show that the proposed method achieved accurate calculation of the likelihood with low computational cost. For instance, using a truncation point of 350, the computation time for one evaluation of the likelihood in seconds for the UK inflation dataset ($T = 244$) is 0.24, 0.39, 0.72 and 2.60 when using 18, 8, 4, or just one computing thread, respectively. For the UK exchange rate dataset ($T = 999$) that I use in Section 4.2 a truncation point of 350 was also adequate, and the computation times for the same increase to 0.82, 1.42, 2.72, 10.07, respectively. The coding was done in C++, linked to the R software and executed in a Ryzen threadripper 3970x processor.

In addition, using the same inflation data, a range of other models are also estimated to evaluate the empirical performance of the proposed model. The models used in the comparison were the homoscedastic model, local scale model (Shephard, (1994)), Garch with normal and student t errors models, log normal SV model (e.g. Kim et al. (1998)) and the Gamma SV model. Furthermore, the study uses 1000 daily exchange rate observations for 7 currencies (GBP, EUR, JPY, CND, AUD, BRL, ZAR) to the USD. Using the inflation data, the proposed model performs better than all other models in 50% of the applications in terms of the Bayesian Information Criterion (BIC), with very large gains for the Brazil dataset. The second application of 7 currencies finds that the empirical fit of the proposed model is overall better than alternative models in 4 of the 7 currencies in terms of the BIC, being much superior in the case of currencies with turbulent episodes, such as the Brazilian Real.

Lastly, using daily returns for 8 stocks on the Tokyo Stock Price Index (Topix) and the Standards and Poors 500 ($S\&P500$), the performance of the model is compared to the stochastic volatility model with leverage. The proposed model performs better 62.5% than the asymmetric SV model in terms of the Bayesian Information Criterion (BIC).

The second paper is estimated using 4 macroeconomic applications that use 20 vintage macroeconomic variables each for Japan, Brazil, US, and the UK. A second application uses daily exchange rate returns for a small VAR of 4 currencies and a larger VAR of 8 currencies all with 1000 observations each. The comparison method is based on marginal likelihoods and the one step ahead out of sample predictive likelihoods.

The proposed model is compared to other CSV models proposed in the literature. Using two evaluation periods of 24 and 50 data points and two priors, the empirical fit of the common factor inverse gamma SV model has better forecasting accuracy compared to alternative CSV models for the macroeconomic application in 13 of the 16 instances. The BVAR-CSV model is best for the Japan data over a longer evaluation period. Using the alternative prior, over the shorter period, the BVAR-CSV-MA-t is best for the UK while the BVAR-CSV-t model is best for Brazil. In the financial application, the proposed model performs better than alternative models when using the first prior, while the BVAR-CSV-t is best using the alternative prior. The marginal likelihoods all have a small standard error less than 1 unit, therefore all results are significant.

Conclusion

The study obtained an analytic expression for the likelihood of an inverse gamma SV model. As a result, it is possible to obtain the Maximum Likelihood estimator. The exact value of the likelihood is also useful for Bayesian estimation and model comparison. Within the literature of nonlinear or non Gaussian state space models this novel approach is one of the very few methods that allow MLE because I am able to obtain the likelihood exactly. I provide the explicit formulas for this likelihood as well as the code to calculate it. Furthermore, I obtained the filtering and smoothing distributions for the inverse volatilities as mixture of gammas, allowing exact sampling from these distributions. Inverse gamma SV models can account for fat tails, which are observed in most macroeconomic and financial data. The approach that I use to obtain the likelihood expression is a result of integrating out the volatilities in the model. The results show that this approach is computationally efficient, simple and accurate. The computer code to perform the analysis has been publicly made available through a R package that is freely available from CRAN.

Extending this approach to the multivariate case by using a factor model framework such as proposed in Kim et al. (1998), the study obtained an analytical expression for the likelihood in an inverse gamma CSV model that allows this study to obtain a better approximation of the marginal likelihood. My approach is a result of marginalising out the volatilities and the latent common factor in the model. The results in this case tend to favour the proposed common factor inverse gamma SV model in terms of forecasting accuracy.

While the models, derivations and analysis proposed in this dissertation are robust, they are not without limitations. Considering the univariate stochastic volatility model for instance, one such limitation is that for some datasets characterised with sporadic marginal jumps in volatility such as the Zimbabwean Dollar exchange rate, the analysis will fail to converge to a global maximum at least without further manipulations to the algorithm. However, this limitation is not peculiar to the inverse gamma model alone but all models that were used in the comparison exercise in this chapter. In future research, it would be useful to extend the proposed approaches to allow for the estimation of more complex data such as the data that has jumps in volatility which will especially be useful in forecasting exercises for developing economies.

References

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