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A Hybrid Measure of Efficiency in DEA

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Abstract

This paper proposes a new hybrid measure of efficiency for DEA. It provides a unifying framework for radial and non-radial approaches.
Keywords: DEA, radial, non-radial, orientation, SBM

1 Introduction

There are two types of measures or approaches in DEA: *radial* and *non-radial*. Differences exist in the characterization of input or output items. Suppose that there are four inputs x_1, x_2, x_3 and x_4, in the concerned problem, where x_1 and x_2 are radial, and x_3 and x_4 are non-radial. That is, $(x_1,\ x_2)$ are subject to change proportionally, such as $(\alpha x_1, \alpha x_2)$ (with $\alpha > 0$), while x_3 and x_4 are subject to change non-radially. These differences should be reflected in the evaluation of efficiency. The radial input part $(x_1,\ x_2)$ satisfies the efficiency status if there is no proportionally reduced input $(\alpha x_1, \alpha x_2)$ (with $\alpha < 1$) that can produce the observed outputs. The non-radial input part x_3 (x_4) satisfies the efficiency status if there is no reduced x_3 (x_4) that can produce the observed outputs. Analogously, the output part can be divided into the radial and non-radial outputs.

The radial approach is represented by the CCR (Charnes, Cooper and Rhodes (1978)) and BCC (Banker, Charnes and Cooper (1984)) models. Its shortcoming is that it neglects the non-radial input/output slacks. The non-radial approach includes Russell (1985), Pastor et al. (1999) and Tone (2001).

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It deals with slacks directly, but it neglects the radial characteristics of inputs and/or outputs.

In this paper, we integrate these approaches in a unified framework and propose a hybrid measure of efficiency.

This paper unfolds as follows. Section 2 defines the hybrid measure. Section 3 discusses economic interpretations of the model. We extend the basic model in Section 4. Comparisons of the Hybrid model with the CCR and SBM are described in Section5. An illustrative example follows in Section 6 and we conclude the paper in Section 7.

2 A hybrid measure

Let the observed input and output data matrices be $X \in R_{+}^{m\times n}$ and $Y \in R_{+}^{s\times n}$, respectively, where n, m and s designate the numbers of DMU (decision making unit), inputs and outputs. We decompose the input matrix into the radial part, $X^{R} \in R_{+}^{m_1\times n}$, and non-radial part, $X^{NR} \in R_{+}^{m_2\times n}$, with $m = m_1 + m_2$, as follows:

$$X = \begin{pmatrix} X^{R} \\ X^{NR} \end{pmatrix}. \tag{1}$$

Analogously, we decompose the output matrix Y into the radial part, $Y^{R} \in R^{s_1\times n}$ and the non-radial part, $Y^{NR} \in R^{s_2\times n}$, with $s = s_1 + s_2$, as follows:

$$Y = \begin{pmatrix} Y^{R} \\ Y^{NR} \end{pmatrix}. \tag{2}$$

We assume that the data set is positive, i.e., $X > 0$ and $Y > 0$. The production possibility set P is defined by

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \boldsymbol{0}\}, \tag{3}$$

where $\boldsymbol{\lambda}$ is a nonnegative vector in R^n. (We can impose some constraints to $\boldsymbol{\lambda}$, such as $\sum_{j=1}^{n} \lambda_j = 1$, which is the variable returns to scale model. We will return to this subject in Section 4.)

We consider an expression for describing a certain DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o) = (\boldsymbol{x}_o^{R}, \boldsymbol{x}_o^{NR},$

$y_o^R, y_o^{NR}) \in P$ as

$$\begin{aligned} \theta \boldsymbol{x}_o^R &= X^R \boldsymbol{\lambda} + \boldsymbol{s}^{R-} \\ \boldsymbol{x}_o^{NR} &= X^{NR} \boldsymbol{\lambda} + \boldsymbol{s}^{NR-} \\ \phi \boldsymbol{y}_o^R &= Y^R \boldsymbol{\lambda} - \boldsymbol{s}^{R+} \\ \boldsymbol{y}_o^{NR} &= Y^{NR} \boldsymbol{\lambda} - \boldsymbol{s}^{NR+}, \end{aligned} \tag{4}$$

with $\theta \leq 1,\ \phi \geq 1,\ \boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{s}^{R-} \geq \mathbf{0},\ \boldsymbol{s}^{NR-} \geq \mathbf{0},\ \boldsymbol{s}^{R+} \geq \mathbf{0},\ \boldsymbol{s}^{NR+} \geq \mathbf{0}$. The vectors $\boldsymbol{s}^{R-} \in R^{m_1}$ and $\boldsymbol{s}^{NR-} \in R^{m_2}$ indicate the *excesses* for the radial and non-radial inputs, respectively, while $\boldsymbol{s}^{R+} \in R^{s_1}$ and $\boldsymbol{s}^{NR+} \in R^{s_2}$ indicate the *shortfalls* for the radial and non-radial outputs, respectively. They are called *slacks.*

As such, $\theta = 1,\ \phi = 1, \lambda_o = 1, \lambda_j = 0 (\forall j \neq o)$, with all slacks being zero is a feasible expression. Based on the expression (4), we define an index ρ as follows:

$$\rho = \frac{1 - \frac{m_1}{m}(1-\theta) - \frac{1}{m}\sum_{i=1}^{m_2} s_i^{NR-}/x_{io}^{NR}}{1 + \frac{s_1}{s}(\phi - 1) + \frac{1}{s}\sum_{r=1}^{s_2} s_r^{NR+}/y_{ro}^{NR}}. \tag{5}$$

This index ρ is designed so that it is decreasing with respect to decreases in θ and increases in ϕ, s_i^{NR-} $(\forall i)$ and s_r^{NR+} $(\forall r)$, but is not affected by $\boldsymbol{s}^{R-}$ and $\boldsymbol{s}^{R+}$ directly, reflecting free disposability of these radial slacks. This index is also units invariant, i.e., invariant with respect to the measurement units of the data.

The hybrid efficiency status of the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o) = (\boldsymbol{x}_o^R, \boldsymbol{x}_o^{NR}, \boldsymbol{y}_o^R, \boldsymbol{y}_o^{NR}) \in P$ is defined as follows:

Definition 1 (Hybrid efficient status) *The DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is hybrid efficient if and only if $\rho = 1$ for every feasible expression of (4), i.e., $\theta = 1,\ \phi = 1,\ \boldsymbol{s}^{NR-} = \mathbf{0},\ \boldsymbol{s}^{NR+} = \mathbf{0}$.*

This status can be identified by solving the following program with the variables θ, ϕ, $\boldsymbol{\lambda}$, $\boldsymbol{s}^{NR-}$, $\boldsymbol{s}^{NR+}$.

$$\begin{aligned}
&[\text{Hybrid}] \quad && \rho^* = \min \frac{1 - \frac{m_1}{m}(1-\theta) - \frac{1}{m}\sum_{i=1}^{m_2} s_i^{NR-}/x_{io}^{NR}}{1 + \frac{s_1}{s}(\phi - 1) + \frac{1}{s}\sum_{r=1}^{s_2} s_r^{NR+}/y_{ro}^{NR}} \qquad (6)\\
&\text{subject to} && \theta \boldsymbol{x}_o^R \geq X^R \boldsymbol{\lambda}\\
&&& \boldsymbol{x}_o^{NR} = X^{NR}\boldsymbol{\lambda} + \boldsymbol{s}^{NR-}\\
&&& \phi \boldsymbol{y}_o^R \leq Y^R \boldsymbol{\lambda}\\
&&& \boldsymbol{y}_o^{NR} = Y^{NR}\boldsymbol{\lambda} - \boldsymbol{s}^{NR+}\\
&&& \theta \leq 1,\ \phi \geq 1,\ \boldsymbol{\lambda} \geq \boldsymbol{0},\ \boldsymbol{s}^{NR-} \geq \boldsymbol{0},\ \boldsymbol{s}^{NR+} \geq \boldsymbol{0}.
\end{aligned}$$

Let an optimal solution for this program be $(\theta^*, \phi^*, \ \boldsymbol{\lambda}^*, \ \boldsymbol{s}^{NR-*}, \ \boldsymbol{s}^{NR+*})$. Then we have a theorem:

Theorem 1 *The DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is hybrid-efficient if and only if $\rho^* = 1$, i.e., $\theta^* = 1, \phi^* = 1, \ \boldsymbol{s}^{NR-*} = \boldsymbol{0}, \ \boldsymbol{s}^{NR+*} = \boldsymbol{0}$.*

The [Hybrid] can be transformed into a linear program using the Charnes-Cooper transformation (Charnes and Cooper (1962)) as follows:

$$\begin{aligned}
&[\text{LP}] \quad && \tau^* = \min\ t - \frac{m_1}{m}(t - \Theta) - \frac{1}{m}\sum_{i=1}^{m_2} S_i^{NR-}/x_{io}^{NR} \qquad (7)\\
&\text{subject to} && t + \frac{s_1}{s}(\Phi - t) + \frac{1}{s}\sum_{r=1}^{s_2} S_r^{NR+}/y_{ro}^{NR} = 1\\
&&& \Theta \boldsymbol{x}_o^R \geq X^R \Lambda\\
&&& t\boldsymbol{x}_o^{NR} = X^{NR}\Lambda + \boldsymbol{S}^{NR-}\\
&&& \Phi \boldsymbol{y}_o^R \leq Y^R \Lambda\\
&&& t\boldsymbol{y}_o^{NR} = Y^{NR}\Lambda - \boldsymbol{S}^{NR+},\\
&&& \Theta \leq t,\ \Phi \geq t,\ \Lambda \geq 0,\ \boldsymbol{S}^{NR-} \geq 0,\ \boldsymbol{S}^{NR+} \geq 0.
\end{aligned}$$

(See Tone (2001) for additional details.)

Let an optimal solution of [LP] be $(t^*, \ \Theta^*, \ \Phi^*, \ \Lambda^*, \ \boldsymbol{S}^{NR-*}, \ \boldsymbol{S}^{NR+*})$. Then we have an optimal solution of [Hybrid] as defined by:

$$\begin{aligned}
&\rho^* = \tau*,\ \theta^* = \Theta^*/t^*,\ \phi^* = \Phi^*/t^*,\\
&\boldsymbol{\lambda}^* = \Lambda^*/t^*,\ \boldsymbol{s}^{NR-*} = \boldsymbol{S}^{NR-*}/t^*,\ \boldsymbol{s}^{NR+*} = \boldsymbol{S}^{NR+*}/t^*. \qquad (8)
\end{aligned}$$

For a hybrid-inefficient DMU, i.e., $\rho^* < 1$, the hybrid-*projection* is given by:

$$\bar{\boldsymbol{x}}_o^R \leftarrow \theta^* \boldsymbol{x}_o^R \tag{9}$$
$$\bar{\boldsymbol{x}}_o^{NR} \leftarrow \boldsymbol{x}_o^{NR} - \boldsymbol{s}^{NR-*} \tag{10}$$
$$\bar{\boldsymbol{y}}_o^R \leftarrow \phi^* \boldsymbol{y}_o^R \tag{11}$$
$$\bar{\boldsymbol{y}}_o^{NR} \leftarrow \boldsymbol{y}_o^{NR} + \boldsymbol{s}^{NR+*}. \tag{12}$$

We notice that the radial slacks $\boldsymbol{s}^{R-*}$ and $\boldsymbol{s}^{R+*}$, if they exist, are not accounted for in the above projection, since they are assumed to be freely disposable and have no effect on efficiency evaluation.

Theorem 2 *The projected DMU* $(\bar{\boldsymbol{x}}_o^R,\ \bar{\boldsymbol{x}}_o^{NR},\ \bar{\boldsymbol{y}}_o^R,\ \bar{\boldsymbol{y}}_o^{NR})$ *is hybrid-efficient.*

See Appendix A for a proof.

Using the optimal solution $(\theta^*, \phi^*, \boldsymbol{s}^{NR-*}, \boldsymbol{s}^{NR+*})$, we can decompose the hybrid efficiency indicator ρ^* into four factors as follows:

$$\text{Radial input inefficiency:} \quad \alpha_1 = \frac{m_1}{m}(1-\theta^*) \tag{13}$$
$$\text{Non-radial input inefficiency:} \quad \alpha_2 = \frac{1}{m}\sum_{i=1}^{m_2} s_i^{NR-*}/x_{io}^{NR} \tag{14}$$
$$\text{Radial output inefficiency:} \quad \beta_1 = \frac{s_1}{s}(\phi^*-1) \tag{15}$$
$$\text{Non-radial output inefficiency:} \quad \beta_2 = \frac{1}{s}\sum_{r=1}^{s_2} s_r^{NR+*}/y_{ro}^{NR}. \tag{16}$$

Also we define input and output inefficiencies as:

$$\text{Input inefficiency:} \quad \alpha = \alpha_1 + \alpha_2 \tag{17}$$
$$\text{Output inefficiency:} \quad \beta = \beta_1 + \beta_2. \tag{18}$$

Thus, ρ^* can be expressed as:

$$\rho^* = \frac{1-\alpha}{1+\beta} = \frac{1-\alpha_1-\alpha_2}{1+\beta_1+\beta_2}. \tag{19}$$

This expression is useful for finding the sources of inefficiency and the magnitude of their influence on the efficiency score ρ^*.

Based on $\boldsymbol{\lambda}^*$, we define the reference set to DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ as follows:

Definition 2 (Reference set) *The set of indices corresponding to the positive λ_j^*s is called the reference set to $(\boldsymbol{x}_o, \boldsymbol{y}_o)$.*

The reference set R_o is described as

$$R_o = \{j|\ \lambda_j^* > 0\} \quad (j \in \{1, \cdots, n\}). \tag{20}$$

3 Economic interpretations

The dual program corresponding to the linear program [LP] can be described in terms of dual variables $\boldsymbol{v}^R \in R^{m_1}$, $\boldsymbol{v}^{NR} \in R^{m_2}$, $\boldsymbol{u}^R \in R^{s_1}$, $\boldsymbol{u}^{NR} \in R^{s_2}$, $w \in R$ as follows:

$$\begin{array}{lll}
\text{[Dual]} & w^* = \max\ w & (21)\\
\text{subject to} & w = 1 - \boldsymbol{v}^R\boldsymbol{x}_o^R - \boldsymbol{v}^{NR}\boldsymbol{x}_o^{NR} + \boldsymbol{u}^R\boldsymbol{y}_o^R + \boldsymbol{u}^{NR}\boldsymbol{u}_o^{NR} & (22)\\
& -\boldsymbol{v}^R X^R - \boldsymbol{v}^{NR} X^{NR} + \boldsymbol{u}^R Y^R + \boldsymbol{u}^{NR} Y^{NR} \leq 0 & (23)\\
& \boldsymbol{v}^{NR} \geq \frac{1}{m}[1/\boldsymbol{x}_o^{NR}] \ (\text{or } v_i^{NR} x_{io}^{NR} \geq \frac{1}{m},\ i = 1, \ldots, m_2) & (24)\\
& \boldsymbol{u}^{NR} \geq \frac{w}{s}[1/\boldsymbol{y}_o^{NR}] \ (\text{or } u_r^{NR} y_{ro}^{NR} \geq \frac{w}{s},\ r = 1, \ldots, s_2) & (25)\\
& \boldsymbol{v}^R\boldsymbol{x}_o^R \geq \frac{m_1}{m} & (26)\\
& \boldsymbol{u}^R\boldsymbol{y}_o^R \geq \frac{s_1}{s} w & (27)\\
& \boldsymbol{v}^R \geq 0,\ \boldsymbol{u}^R \geq 0, & (28)
\end{array}$$

where the notation $[1/\boldsymbol{x}_o^{NR}]$ designates the row vector $(1/x_{1o}^{NR}, \cdots, 1/x_{m_2o}^{NR})$.

The dual variables $\boldsymbol{v}^R \in R^{m_1}$, $\boldsymbol{v}^{NR} \in R^{m_2}$, $\boldsymbol{u}^R \in R^{s_1}$, and $\boldsymbol{u}^{NR} \in R^{s_2}$ can be interpreted as the virtual unit-costs and unit-prices of the corresponding input and output items, respectively. The dual program aims at finding the optimal virtual unit-costs and unit-prices for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ so that the profit $\boldsymbol{u}^R\boldsymbol{y}_j^R + \boldsymbol{u}^{NR}\boldsymbol{y}_j^{NR} - \boldsymbol{v}^R\boldsymbol{x}_j^R - \boldsymbol{v}^{NR}\boldsymbol{x}_j^{NR}$ does not exceed zero for any DMU (including $(\boldsymbol{x}_o, \boldsymbol{y}_o)$), and maximizes the profit $\boldsymbol{u}^R\boldsymbol{y}_o^R + \boldsymbol{u}^{NR}\boldsymbol{y}_o^{NR} - \boldsymbol{v}^R\boldsymbol{x}_o^R - \boldsymbol{v}^{NR}\boldsymbol{x}_o^{NR}$ for the DMU_o concerned. Apparently, the optimal profit is at best zero and hence $w^* = 1$ for the hybrid efficient DMU.

Constraints (24) and (25) restrict the feasible virtual unit-cost and virtual unit-price $\boldsymbol{v}^{NR}$ and $\boldsymbol{u}^{NR}$ of the non-radial inputs and outputs, respectively, to the positive orthant, while constraints (26) and (27) set the lower bound for the radial cost $\boldsymbol{v}^R\boldsymbol{x}_o^R$ and the radial price $\boldsymbol{u}^R\boldsymbol{y}_o^R$, respectively.

4 Extensions

In this section, we extend the basic [Hybrid] model to the variable returns-to-scale environment and to oriented-models.

4.1 Variable returns-to-scale model

We can evaluate the hybrid efficiency under the variable returns to scale (VRS) condition by imposing the following constraint to [Hybrid]:

$$\sum_{j=1}^{n} \lambda_j = 1. \tag{29}$$

This effects introduction of a free variable $w_o \in R$ to the [Dual] as follows:

$$
\begin{aligned}
&[\text{Dual-VRS}] \quad w^* = \max\ w && (30)\\
&\text{subject to} &&\\
&w = 1 - \boldsymbol{v}^R \boldsymbol{x}_o^R - \boldsymbol{v}^{NR} \boldsymbol{x}_o^{NR} + \boldsymbol{u}^R \boldsymbol{y}_o^R + \boldsymbol{u}^{NR} \boldsymbol{u}_o^{NR} + w_0 && (31)\\
&-\boldsymbol{v}^R X^R - \boldsymbol{v}^{NR} X^{NR} + \boldsymbol{u}^R Y^R + \boldsymbol{u}^{NR} Y^{NR} + w_0 \boldsymbol{e} \leq \boldsymbol{0} && (32)\\
&\boldsymbol{v}^{NR} \geq \frac{1}{m}[1/\boldsymbol{x}_o^{NR}] && (33)\\
&\boldsymbol{u}^{NR} \geq \frac{w}{s}[1/\boldsymbol{y}_o^{NR}] && (34)\\
&\boldsymbol{v}^R \boldsymbol{x}_o^R \geq \frac{m_1}{m} && (35)\\
&\boldsymbol{u}^R \boldsymbol{y}_o^R \geq \frac{s_1}{s} w && (36)\\
&\boldsymbol{v}^R \geq \boldsymbol{0},\ \boldsymbol{u}^R \geq \boldsymbol{0}, && (37)
\end{aligned}
$$

where $\boldsymbol{e}$ is a row vector with all elements equal to 1. The free variable w_0 plays the role of determining the returns-to-scale characteristics of efficient DMUs, i.e., *increasing, constant, decreasing*, in a manner similar to the procedure developed by Banker and Thrall (1992).

4.2 Oriented model

The input (output)-oriented Hybrid model can be defined by neglecting the denominator (numerator) of the objective function (6) of [Hybrid]. Thus, the efficiency values ρ_I^* and ρ_O^* can be obtained as follows:

$$\text{[Hybrid-I]} \quad \rho_I^* = \min \quad 1 - \frac{m_1}{m}(1-\theta) - \frac{1}{m}\sum_{i=1}^{m_2} s_i^{NR-}/x_{io}^{NR} \tag{38}$$

$$\begin{aligned} \text{subject to} \quad & \theta \boldsymbol{x}_o^R \geq X^R \boldsymbol{\lambda} \\ & \boldsymbol{x}_o^{NR} = X^{NR}\boldsymbol{\lambda} + \boldsymbol{s}^{NR-} \\ & \boldsymbol{y}_o^R \leq Y^R \boldsymbol{\lambda} \\ & \boldsymbol{y}_o^{NR} \leq Y^{NR}\boldsymbol{\lambda} \\ & \theta \leq 1,\ \boldsymbol{\lambda} \geq \boldsymbol{0},\ \boldsymbol{s}^{NR-} \geq \boldsymbol{0}. \end{aligned}$$

$$\text{[Hybrid-O]} \quad \rho_O^* = \min \frac{1}{1 + \frac{s_1}{s}(\phi - 1) + \frac{1}{s}\sum_{r=1}^{s_2} s_r^{NR+}/y_{ro}^{NR}} \tag{39}$$

$$\begin{aligned} \text{subject to} \quad & \boldsymbol{x}_o^R \geq X^R \boldsymbol{\lambda} \\ & \boldsymbol{x}_o^{NR} \geq X^{NR}\boldsymbol{\lambda} \\ & \phi \boldsymbol{y}_o^R \leq Y^R \boldsymbol{\lambda} \\ & \boldsymbol{y}_o^{NR} = Y^{NR}\boldsymbol{\lambda} - \boldsymbol{s}^{NR+} \\ & \phi \geq 1,\ \boldsymbol{\lambda} \geq \boldsymbol{0},\ \boldsymbol{s}^{NR+} \geq \boldsymbol{0}. \end{aligned}$$

This leads to:

Theorem 3

$$\rho_I^* \geq \rho^* \quad \textit{and} \quad \rho_O^* \geq \rho^*. \tag{40}$$

The decomposition (19) is valid in these models too, i.e., $\beta = \beta_1 = \beta_2 = 0$ in [Hybrid-I] and $\alpha = \alpha_1 = \alpha_2 = 0$ in [Hybrid-O].

4.3 Weighted Hybrid model

We can impose the relative importance of radial vs. non-radial inputs (outputs) as well as the relative importance among the non-radial inputs and outputs as follows:

$$\begin{aligned} \rho^* = \quad & \min \frac{1 - \frac{w_I^R m_1}{m}(1-\theta) - \frac{w_I^{NR}}{m}\sum_{i=1}^{m_2}\frac{w_i^{NR-}s_i^{NR-}}{x_{io}^{NR}}}{1 + \frac{w_O^R s_1}{s}(\phi - 1) + \frac{w_O^{NR}}{s}\sum_{r=1}^{s_2}\frac{w_r^{NR+}s_r^{NR+}}{y_{ro}^{NR}}} \\ \text{with} \quad & w_I^R + w_I^{NR} \leq 2,\ w_I^R,\ w_I^{NR} \geq 0 \\ & w_O^R + w_O^{NR} \leq 2,\ w_O^R,\ w_O^{NR} \geq 0 \\ & \sum_{i=1}^{m_2} w_i^{NR-} = m_2,\ \sum_{r=1}^{s_2} w_r^{NR+} = s_2,\ w_i^{NR-} \geq 0\ (\forall i)\ w_r^{NR+} \geq 0\ (\forall r). \end{aligned} \tag{41}$$

This will contribute to discriminate inputs and outputs according to their importance in evaluation.

5 Comparisons with the CCR and SBM models

We define the CCR and SBM models as special cases of the Hybrid model and compare efficiency values among them. Furthermore, we observe a single radial input (output) case and show that this reduces to a non-radial model.

5.1 The CCR and SBM models

The Hybrid model can be transformed into the CCR or SBM model by setting all inputs and outputs as *radial* or *non-radial*, respectively. Thus, we can define:

$$\begin{aligned} \text{[CCR]} \quad & \rho^*_{CCR} = \min \frac{\theta}{\phi} \\ \text{subject to} \quad & \theta \boldsymbol{x}_o \geq X\boldsymbol{\lambda} \\ & \phi \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} \\ & \theta \leq 1,\ \phi \geq 1,\ \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned} \tag{42}$$

$$
\text{[SBM]} \qquad \rho^*_{SBM} = \min \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r/y_{ro}} \tag{43}
$$

$$
\begin{aligned}
\text{subject to} \quad & \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^- \\
& \boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\
& \boldsymbol{\lambda} \geq 0,\ \boldsymbol{s}^- \geq 0,\ \boldsymbol{s}^+ \geq 0.
\end{aligned}
$$

In a similar vein as described in Section 4.2, we can define the oriented CCR or SBM models, i.e., CCR-I, CCR-O, or SBM-I, SBM-O, by neglecting output or input side efficiency. Between the CCR, CCR-I and CCR-O models we have:

Lemma 1

$$
\rho^*_{CCR} = \rho^*_{CCR-I} = \rho^*_{CCR-O}. \tag{44}
$$

See Appendix B for a proof.

However, this lemma is no longer true when we deal with the BCC model, i.e., the radial and VRS model. Usually, ρ^*_{BCC}, ρ^*_{BCC-I}, and ρ^*_{BCC-O} are different each other.

Among these indices, we have the following relationship.

Theorem 4

$$
\begin{aligned}
& \rho^*_{CCR} \geq \rho^* \geq \rho^*_{SBM} \\
& \rho^*_{CCR-I} \geq \rho^*_I \geq \rho^*_{SBM-I} \\
& \rho^*_{CCR-O} \geq \rho^*_O \geq \rho^*_{SBM-O} \\
& \rho^*_{SBM-I} \geq \rho^*_{SBM}, \quad \rho^*_{SBM-O} \geq \rho^*_{SBM} \\
& \rho^*_{BCC-I} \geq \rho^*_{BCC}, \quad \rho^*_{BCC-O} \geq \rho^*_{BCC}.
\end{aligned} \tag{45}
$$

See Appendix C for a proof.

5.2 The case having a single radial input (output) item

We will demonstrate that a single radial input (output) case can be reduced to a non-radial input (output) model.

Let us assume that the number of the radial inputs (outputs) is one, i.e., $m_1 = 1$ ($s_1 = 1$). Then the first constraint of [Hybrid] becomes:

$$\theta x_o^R \geq X^R \boldsymbol{\lambda}.$$

We can transform this single (not vector) inequality into:

$$x_o^R \geq X^R \boldsymbol{\lambda} + (1-\theta) x_o^R. \tag{46}$$

Let us define $s^{R-} = (1-\theta)x_o^R$ (≥ 0). Then, the θ-related term in the objective function of [Hybrid] becomes:

$$-\frac{1}{m} s^{R-} / x_o^R.$$

Thus, the model is reduced to the non-radial case. The inequality in (46) is binding at the optimal solution.

Similarly, the single radial output case can be reduced to a non-radial case.

However, the single non-radial input (output) case cannot be reduced to a radial model.

6 An illustrative example

Table 1 exhibits a simple example consisting of six DMUs, each having two radial inputs (x_1, x_2), a single non-radial input (x_3), two radial outputs (y_1, y_2), and a single non-radial output (y_3).

Insert Table 1 about here.

We measured the efficiency score using the [Hybrid], [CCR] and [SBM] models and obtained the results as shown in the last three columns of Table 1, where [CCR] and [SBM] assume all variables are Radial and Non-radial, respectively.

Observing the data set, we can see that the DMU A is the only superstar in the sense that other DMUs have input excesses or output shortfalls against A. The non-radial [SBM] captures these drawbacks effectively and gives the full score unity only to A, whereas others are judged to be inefficient. On the other hand, the radial [CCR] model neglects these slacks and assigns the full efficiency score unity to all DMUs except F. The [Hybrid] model gives the score between [CCR] and [SBM]. The optimal slacks are common to all models except DMU F as exhibited in Table 2.

Insert Table 2 about here.

The differences among the three models can be explained more clearly by observing the dual side solution. An optimal dual solution and its corresponding weighted data for [CCR] are reported in Tables 3 and 4, respectively. In this model, we can conclude by means of Lemma 1 that the dual problem becomes:

$$w^* = \max \boldsymbol{u}^R y_o^R \tag{47}$$

$$\text{subject to} \quad -\boldsymbol{v}X^R + \boldsymbol{u}Y^R \leq 0 \tag{48}$$

$$\boldsymbol{v}^R \boldsymbol{x}_o^R = 1 \tag{49}$$

$$\boldsymbol{v}^R \geq 0, \; \boldsymbol{u}^R \geq 0. \tag{50}$$

At optimality, the following complementary slackness conditions should hold: $s_i^- v_i = 0$ for $i = 1, 2, 3$, and $s_r^+ u_r = 0$ for $r = 1, 2, 3$. Hence, in Table 3, the underlined parts that correspond to non-zero slacks in Table 2 must be zero. The dual values in Table 3 satisfy the above dual constraints for each DMU and provide a dual objective value that is equal to the objective value of the primal problem. Hence they are optimal, although they are not necessarily unique. The [CCR] model takes no account of the remaining slacks in evaluating efficiency. Table 4 reports the weighted inputs and outputs for each DMU. They satisfy the relationship $\sum_{i=1}^{3} v_i x_i = 1$ and $\sum_{r=1}^{3} u_r y_r = w^*$ for each DMU.

Insert Table 3 about here.

Insert Table 4 about here.

We now turn to [SBM]. In this model, we have, from (24) and (25), $v_i x_{io} \geq 1/3$ $(i = 1, 2, 3)$ and $u_r y_{ro} \geq w/3$ $(r = 1, 2, 3)$. (We omitted the notation NR, since all variables are non-radial in [SBM].) Thus, the optimal $\boldsymbol{v}$ must be positive, and $\boldsymbol{u}$ must also be positive if $w^* > 0$. This reduces the feasible region of the dual program, and the optimal w^* is not greater than that of [CCR]. Tables 5 and 6 exhibit the results. All entries are positive, indicating that all inputs and outputs are accounted for in the efficiency evaluation.

Insert Table 5 about here.

Insert Table 6 about here.

As we pointed out in Theorem 4, [Hybrid] is positioned between [CCR] and [SBM]. Tables 7 and 8 show the optimal dual variables and weighted inputs/outputs. Since x_1 and x_2 are radial, their dual variables are constrained to be non-negative, while x_3 is non-radial and hence $v_3 \geq 1/(3x_3)$ is required.

Insert Table 7 about here.

Insert Table 8 about here.

Table 9 exhibits the decomposition of the efficiency score by means of the expression (19). From this table, we can see the sources of inefficiency and their magnitudes. For example, DMU C's inefficiency is caused by the input inefficiency ($\alpha = 0.167$), which stems from the non-radial input inefficiency ($\alpha_2 = 0.167$). DMU F's inefficiency can be attributed to the output inefficiency ($\beta = 5.67$), which stems from the radial output inefficiency ($\beta_1 = 4.67$) and the non-radial output inefficiency ($\beta_2 = 1$). This tallies with the value $\phi_F = 8$ in the table. $\theta = 1$ and $\phi = 1$ indicate the absence of radial inefficiency in inputs and outputs, respectively.

Insert Table 9 about here.

7 Concluding remarks

In this paper we have proposed a hybrid measure of efficiency that combines the radial and non-radial measures of efficiency in a unified manner. This model is useful for measuring the efficiency of DMUs when radial and non-radial inputs (outputs) are mixed in the problem. We have also presented a formula for decomposing the efficiency score into radial input (output) and non-radial inputs (outputs) inefficiencies. Using this decomposition we can see the sources of inefficiency and the magnitude of their influence on the score. The efficiency score of [Hybrid] ranks between [CCR] (the easiest) and [SBM] (the hardest), reflecting the partial incorporation of slacks. We recommend that, if slacks for an input (output) are considered important in measuring efficiency, the input (output) should be handled as *non-radial.* Meanwhile, if the slacks are freely-disposable, the item can be classified as *radial.*

The proposed model can be extended and utilized in several dimensions, e.g., super-efficiency measurements (Tone (2002)) and application to the multi-stage DEA (Fried et al. (2002)). These are future research subjects.

Appendix A: Proof of Theorem 2

Let an optimal solution of the [Hybrid] for $(\bar{\boldsymbol{x}}_o, \bar{\boldsymbol{y}}_o)$ be $(\theta^{**}, \phi^{**}, \boldsymbol{\lambda}^{**}, \boldsymbol{s}^{NR-**}, \boldsymbol{s}^{NR+**})$. It holds that

$$\theta^{**}\bar{\boldsymbol{x}}_o^R = \theta^{**}\theta^*\boldsymbol{x}_o^R \geq X^R\boldsymbol{\lambda}^{**} \tag{51}$$
$$\bar{\boldsymbol{x}}_o^{NR} = X^{NR}\boldsymbol{\lambda}^{**} + \boldsymbol{s}^{NR-**} \tag{52}$$
$$\phi^{**}\bar{\boldsymbol{y}}_o^R = \phi^{**}\phi^*\boldsymbol{y}_o^R \geq Y^R\boldsymbol{\lambda}^{**} \tag{53}$$
$$\bar{\boldsymbol{y}}_o^{NR} = Y^{NR}\boldsymbol{\lambda}^{**} - \boldsymbol{s}^{NR+**} \tag{54}$$
$$\theta^{**} \leq 1, \quad \phi^{**} \geq 1. \tag{55}$$

From (52) and (54) we have:

$$\boldsymbol{x}_o^{NR} = X^{NR}\boldsymbol{\lambda}^{**} + (\boldsymbol{s}^{NR-**} + \boldsymbol{s}^{NR-*}) \tag{56}$$
$$\boldsymbol{y}_o^{NR} = Y^{NR}\boldsymbol{\lambda}^{**} - (\boldsymbol{s}^{NR+**} + \boldsymbol{s}^{NR+*}). \tag{57}$$

Thus, $(\theta^{**}\theta^*, \phi^{**}\phi^*, \boldsymbol{\lambda}^{**}, \boldsymbol{s}^{NR-**}+\boldsymbol{s}^{NR-*}, \boldsymbol{s}^{NR+**}+\boldsymbol{s}^{NR+*})$ is a feasible solution of the [Hybrid] for $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. From the optimality of $(\theta^*, \phi^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{NR-*}, \boldsymbol{s}^{NR+*})$, we have the inequality:

$$\frac{1 - \frac{m_1}{m}(1-\theta^{**}\theta^*) - \frac{1}{m}\sum_{i=1}^{m_2}(s_i^{NR-**} + s_i^{NR-*})/x_{io}^{NR}}{1 + \frac{s_1}{s}(\phi^{**}\phi^* - 1) + \frac{1}{s}\sum_{r=1}^{s_2}(s_r^{NR+**} + s_r^{NR+*})/y_{ro}^{NR}}$$
$$\geq \frac{1 - \frac{m_1}{m}(1-\theta^*) - \frac{1}{m}\sum_{i=1}^{m_2} s_i^{NR-*}/x_{io}^{NR}}{1 + \frac{s_1}{s}(\phi^* - 1) + \frac{1}{s}\sum_{r=1}^{s_2} s_r^{NR+*}/y_{ro}^{NR}}. \tag{58}$$

Noting that $\theta^{**} \leq 1,\ \phi^{**} \geq 1,\ s^{NR-**} \geq 0,\ s^{NR+**} \geq 0$, this inequality holds if and only if we have:

$$\theta^{**} = 1,\ \phi^{**} = 1,\ s^{NR-**} = 0,\ s^{NR+**} = 0. \tag{59}$$

Thus, the projected DMU $(\bar{x}_o, \bar{y}_o)$ is hybrid-efficient.

Appendix B: Proof of Lemma 1

In the [CCR] model, let us define

$$\pi = \theta/\phi. \tag{60}$$

Then [CCR] becomes:

$$\begin{array}{ll} \text{[CCR]} & \rho^*_{CCR} = \min\ \pi \\ \text{subject to} & \pi x_o \geq X(\lambda/\phi) \\ & y_o \leq Y(\lambda/\phi) \\ & \phi \geq 1,\ \lambda \geq 0. \end{array} \tag{61}$$

Essentially this is equivalent to [CCR-I]. Similarly, we can demonstrate that $\rho^*_{CCR} = \rho^*_{CCR-O}$.

Appendix C: Proof of Theorem 4

We will demonstrate the inequality (45), since others can be derived similarly. Let an optimal solution of [CCR] be $(\theta^*, \phi^*, \lambda^*)$. Then, after partition into radial and non-radial parts, we have:

$$\begin{array}{l} \theta^* x_o^R \geq X^R \lambda^* \\ \theta^* x_o^{NR} \geq X^{NR} \lambda^* \\ \phi^* y_o^R \leq Y^R \lambda^* \\ \phi^* y_o^{NR} \leq Y^{NR} \lambda^*. \end{array}$$

Let us define

$$s^{NR-} = (1 - \theta^*) x_o^{NR}\ (\geq 0),\quad s^{NR+} = (\phi^* - 1) y_o^{NR}\ (\geq 0).$$

Then, $(\theta^*, \phi^*, \boldsymbol{\lambda}^*, s^{NR-}, s^{NR+})$ is a feasible solution for [Hybrid]. Thus, we have $\rho^*_{CCR} \geq \rho^*$ showing the first inequality of (45).

Let an optimal solution of [Hybrid] be $(\theta^*, \phi^*,\ \boldsymbol{\lambda}^*,\ s^{NR-*},\ s^{NR+*})$ and define s^{R-*} and s^{R+*} as:

$$s^{R-*} = (1 - \theta^*)x_o^R\ (\geq 0), \quad s^{R+*} = (\phi^* - 1)y_o^R\ (\geq 0).$$

Then we have:

$$x_o^R = X^R\boldsymbol{\lambda}^* + s^{R-*}, \quad y_o^R = Y^R\boldsymbol{\lambda}^* - s^{R+*}.$$

Hence, after changing the radial part into non-radial and defining $s^{-*} = (s^{R-*}, s^{NR-*})$, $s^{+*} = (s^{R+*}, s^{NR+*})$, the obtained $(\boldsymbol{\lambda}^*, s^{-*}, s^{+*})$ is a feasible solution for [SBM]. Thus we have $\rho^* \geq \rho^*_{SBM}$.

References

[1] Banker, R. D., Thrall, R. M., 1992. Estimating most productive scale size using data envelopment analysis, European Journal of Operational Research 62, 74-84.

[2] Charnes, A., Cooper, W.W., 1962. Programming with linear fractional functionals. Naval Research Logistics Quarterly 15, 333-334.

[3] Charnes, A., Cooper, W.W., Rhodes, E., 1978. Measuring the efficiency of decision making units. European Journal of Operational Research 2, 429-444.

[4] Fried, H.O., Lovell, C.A.K., Schmidt, S.S., Yaisawamg, S., 2002. Accounting for environmental effects and statistical noise in data envelopment analysis, Journal of Productivity Analysis 17, 157-174.

[5] Pastor, J.T., Ruiz, J.L. and Sirvent, I., 1999. An enhanced DEA Russell-graph efficiency measure, European Journal of Operational Research 115, 596-607.

[6] Russell, R.R., 1985. Measures of technical efficiency. Journal of Economic Theory 35, 109-126.

[7] Tone, K., 2001. A slacks-based measure of efficiency in data envelopment analysis. European Journal of Operational Research 130, 498-509.

[8] Tone, K., 2002. A slacks-based measure of super-efficiency in data envelopment analysis. European Journal of Operational Research 143, 32-41.

Table 1: A simple example

	Input			Output			Efficiency score		
	Radial		Non-radial	Radial		Non-radial			
DMU	x_1	x_2	x_3	y_1	y_2	y_3	[Hybrid]	[CCR]	[SBM]
A	1	1	1	2	2	2	1	1	1
B	1	2	1	2	2	2	1	1	0.833
C	1	1	2	1	2	2	0.833	1	0.625
D	1	1	1	1	2	1	0.75	1	0.6
E	1	1	1	2	1	0.5	0.5	1	0.429
F	2	2	2	0.5	0.5	1	0.15	0.25	0.15

Table 2: Slacks

	Input			Output		
DMU	s_1^-	s_2^-	s_3^-	s_1^+	s_2^+	s_3^+
A	0	0	0	0	0	0
B	0	1	0	0	0	0
C	0	0	1	1	0	0
D	0	0	0	1	0	1
E	0	0	0	0	1	1.5
F(Hybrid)	0	0	0	0	0	3
F(CCR)	0	0	0	0.5	0.5	0
F(SBM)	0	0	0	3.5	3.5	3

Table 3: Dual variables: [CCR]

		Input Radial			Output Radial		
DMU	Score	v_1	v_2	v_3	u_1	u_2	u_3
A	1	0.667	0.167	0.167	0.167	0.167	0.167
B	1	1	0	0	0.5	0	0
C	1	1	0	0	0	0.5	0
D	1	1	0	0	0	0.5	0
E	1	1	0	0	0.5	0	0
F	0.25	0.5	0	0	0	0	0.25

Table 4: Weighted inputs and outputs: [CCR]

		Input Radial			Output Radial		
DMU	Score	v_1x_1	v_2x_2	v_3x_3	u_1y_1	u_2y_1	u_3y_3
A	1	0.667	0.167	0.167	0.333	0.333	0.333
B	1	1	0	0	1	0	0
C	1	1	0	0	0	1	0
D	1	1	0	0	0	1	0
E	1	1	0	0	1	0	0
F	0.25	1	0	0	0	0	0.25

Table 5: Dual variables: [SBM]

		Input Non-radial			Output Non-radial		
DMU	Score	v_1	v_2	v_3	u_1	u_2	u_3
A	1	0.333	0.333	0.333	0.167	0.167	0.167
B	0.833	0.333	0.167	0.333	0.139	0.139	0.139
C	0.625	0.333	0.333	0.167	0.208	0.104	0.104
D	0.6	0.333	0.333	0.333	0.2	0.1	0.2
E	0.429	0.333	0.333	0.333	0.071	0.143	0.286
F	0.15	0.167	0.167	0.167	0.1	0.1	0.05

Table 6: Weighted inputs and outputs: [SBM]

		Input			Output		
		Non-radial			Non-radial		
DMU	Score	v_1x_1	v_2x_2	v_3x_3	u_1y_1	u_2y_1	u_3y_3
A	1	0.333	0.333	0.333	0.333	0.333	0.333
B	0.833	0.333	0.333	0.333	0.278	0.278	0.278
C	0.625	0.333	0.333	0.333	0.208	0.208	0.208
D	0.6	0.333	0.333	0.333	0.2	0.2	0.2
E	0.429	0.333	0.333	0.333	0.143	0.143	0.143
F	0.15	0.333	0.333	0.333	0.05	0.05	0.05

Table 7: Dual variables: [Hybrid]

		Input			Output		
		Radial		Non-radial	Radial		Non-radial
DMU	Score	v_1	v_2	v_3	u_1	u_2	u_3
A	1	0.667	0	0.333	0.333	0	0.167
B	1	0.667	0	0.333	0.333	0	0.167
C	0.833	0.667	0	0.167	0	0.278	0.139
D	0.75	0.667	0	0.333	0	0.25	0.25
E	0.5	0.667	0	0.333	0.167	0	0.333
F	0.15	0.333	0	0.167	0.2	0	0.05

Table 8: Weighted inputs and outputs: [Hybrid]

		Input			Output		
		Radial		Non-radial	Radial		Non-radial
DMU	Score	v_1x_1	v_2x_2	v_3x_3	u_1y_1	u_2y_1	u_3y_3
A	1	0.667	0	0.333	0.667	0	0.333
B	1	0.667	0	0.333	0.667	0	0.333
C	0.833	0.667	0	0.333	0	0.556	0.278
D	0.75	0.667	0	0.333	0	0.5	0.25
E	0.5	0.667	0	0.333	0.333	0	0.167
F	0.15	0.667	0	0.333	0.1	0	0.05

Table 9: Measures of inefficiency: [Hybrid]

		Input			Output			Input	Output
DMU	Score	α	α_1	α_2	β	β_1	β_2	θ	ϕ
A	1	0	0	0	0	0	0	1	1
B	1	0	0	0	0	0	0	1	1
C	0.833	0.167	0	0.167	0	0	0	1	1
D	0.75	0	0	0	0.333	0	0.333	1	1
E	0.5	0	0	0	1	0	1	1	1
F	0.15	0	0	0	5.67	4.67	1	1	8