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An Adjusted Projection in Oriented DEA Models

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Abstract

One of the main concerns of Data Envelopment Analysis (DEA) is the projection of an inefficient Decision Making Unit (DMU) onto the referenced efficient frontier. This task needs care when organizations with several different characteristics are involved. This paper discusses such situations and proposes a simple and practical method for coping with target setting in such mixed business environments. A method of "adjusted projection" is proposed which synthesizes a global view with local conditions. A public library case is analyzed using this projection.

Recent management tends to aim at attaining "global optimum." However, globalization is not always possible since there exist many regulative or transcendental conditions which restrict this objective. We hope the proposed method will help decision-makers to reach a solution to this problem.

Keywords: Data Envelopment Analysis, projection, categorical data, environmental condition, weight restriction, target setting.

1 Introduction

Data Envelopment Analysis introduced by Charnes, Cooper and Rhodes ([6]) has been widely utilized for evaluating the performance of organizations such

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as hospitals, schools, financial institutions, hotels etc. In the CR-ROM attached to Cooper, Seiford and Tone [11], about 1500 papers, which employ DEA, are compiled with titles and abstracts. Even more applications are expanding rapidly in many fields of interest. DEA identifies the efficient frontiers spanned by efficient DMUs (decision making units). For an inefficient DMU, these frontiers operate as a benchmark, and inefficient DMU can be improved in efficiency by projecting onto the efficient frontiers. However, this operation needs care. Especially when we compare the performance of DMUs operating under different conditions, a straightforward comparison may cause problems of rationality and hence the induced projections will result in impractical proposals for improvement. As an example, we observe three categories of public libraries operating under different environments, i.e., those in business, commercial and residential districts. Usually, libraries in residential districts outperform others (in business and commercial districts). Hence, comparing public libraries from scratch might be unfair to libraries in the latter categories. Similar situations may arise in the case of comparisons of hotels, restaurants, supermarkets, hospitals etc., which are operating countrywide or worldwide.

As another example, we consider the comparisons of firms belonging to different types of businesses, e.g., chemical, machinery, electric, construction and food. Even if they are using some common resources as inputs, e.g., personnel, materials and capital to produce some common products as outputs, e.g., revenue and profit, comparing and proposing improvements to DMUs belonging to different types of business needs care.

Since the classical work by Banker and Morey ([2]) many authors have analyzed these situations. These papers can be classified into:

1. Papers that measure efficiencies of categorized DMUs and find difference between categories by using some statistical tests. (See Lewin and Morey [15], Chilingerian [7], Brockett and Golany [3], Ahn and Seiford [1], McMillan and Datta [16], Ozcan and Luke [17], Ozcan, Luke and Haksever [18], and Puig-Junoy [20], among others.)

2. Papers that assume a hierarchical structure among categories and evaluate the efficiency of DMUs in a certain category referring to DMUs in the same or less advantageous category classes. (See Banker and Morey [2], Kamakura [14], Cook and Kress [8], Cook and Kress [9],

Cook, Kress and Seiford [10], and Golany, Roll and Rybak [13], among others.)

Most of them put emphasis on the measurement of efficiency and relatively few addressed projection methodology under these situations. This may reflect the view that projection/improvement belongs to managerial decision and is not within the scope of DEA. This might be partially true but we need to develop some systematic way to deal with this problem as an aid for management.

Typical improvements by input (output) oriented DEA models consist of a radial reduction (enlargement) of inputs (outputs) plus deletion of input excesses (output shortfalls). Considering the above mentioned subjects, this paper concentrates on projection of inefficient DMUs under different business environments and proposes an "adjusted projection." The basic idea underlying this method is to think "globally" and to act "locally." First, we evaluate efficiency of each DMU with respect to all DMUs in the problem. Hence, big differences in efficiency may be found between DMUs in different categories. Then we select a "champion" DMU from each category. Using the efficiency of this champion DMU, we do an "adjusted projection" of DMUs in the same category. It will be demonstrated that the projected DMUs in a category exhibit the same level of efficiency when evaluated with respect to all DMUs in the problem. Furthermore, the projected DMUs have full efficiency when evaluated within the category. Thus, it may safely be said that we evaluate DMUs "globally" and project them "locally" still accounting and reflecting the global standard.

Recent management tends to aim at attaining a "global optimum." However, globalization is not always possible, since there exist many local constraints which restrict this movement. This paper may present a solution to this problem.

In Section 2, we introduce an adjusted projection and apply it to performance evaluation and improvement of public libraries in Tokyo in Section 3 as an elucidative example. Some concluding remarks follow in Section 4.

2 Methodology

We will deal with the input-oriented CCR (Charnes-Cooper-Rhodes [6]) model, although this methodology can be applied to the output-oriented case as well and to other returns-to-scale models, i.e., variable, increasing and decreasing ones.

2.1 Notation

We denote inputs and outputs of a DMU (decision making unit) by $\boldsymbol{x} = (x_1, \ldots, x_m)^T$ and $\boldsymbol{y} = (y_1, \ldots, y_s)^T$, respectively, where m and s are the numbers of inputs and outputs, and the symbol ^{T} designates transposition. We assume that the DMUs concerned are classified into several groups depending on their characteristics, e.g., environmental conditions, type of business etc. The total set of DMUs is presented by T and the production possibility set P spanned by T is defined by

$$P = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \geq \sum_{j \in T} \boldsymbol{x}_j \lambda_j,\ \boldsymbol{y} \leq \sum_{j \in T} \boldsymbol{y}_j \lambda_j \right\}, \tag{1}$$

where $\lambda_j \geq 0$ ($\forall j \in T$) and some other constraints will be imposed on $\boldsymbol{\lambda}$ occasionally. We assume that the data set is nonnegative, i.e., $X = (\boldsymbol{x}_j) \geq \mathbf{0}$ and $Y = (\boldsymbol{y}_j) \geq \mathbf{0}$.

2.2 Adjusted Projection

Before going into group oriented projections, we will demonstrate several propositions which are valid in general situations and will be utilized later on.

The input oriented CCR model for evaluating the efficiency of a DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is described by the following linear programming (LP) problem:

$$\begin{aligned} & \min\ \theta & (2) \\ \text{subject to}\quad & \theta \boldsymbol{x}_o = \sum_{j \in T} \lambda_j \boldsymbol{x}_j + \boldsymbol{s}^- \\ & \boldsymbol{y}_o = \sum_{j \in T} \lambda_j \boldsymbol{y}_j - \boldsymbol{s}^+ \\ & \boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{s}^- \geq \mathbf{0},\ \boldsymbol{s}^+ \geq \mathbf{0}. \end{aligned}$$

Let the optimal solution of (2) be θ^*. Then, fixing θ at θ^*, we maximize the sum of the *input excesses* $\boldsymbol{s}^- \in R^m$ and the *output shortfalls* $\boldsymbol{s}^+ \in R^s$ by the following LP:

$$
\begin{aligned}
& \max \ \boldsymbol{e}\boldsymbol{s}^- + \boldsymbol{e}\boldsymbol{s}^+ \\
\text{subject to} \quad & \theta^* \boldsymbol{x}_o = \sum_{j \in T} \lambda_j \boldsymbol{x}_j + \boldsymbol{s}^- \\
& \boldsymbol{y}_o = \sum_{j \in T} \lambda_j \boldsymbol{y}_j - \boldsymbol{s}^+ \\
& \boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{s}^- \geq \mathbf{0},\ \boldsymbol{s}^+ \geq \mathbf{0},
\end{aligned}
\tag{3}
$$

where $\boldsymbol{e}$ is a row vector with all elements equal to one.

Let an optimal solution of this problem be $(\boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$.

The above procedure is known as the two phase process in DEA literature (see for example [11]), i.e., minimize θ in the first phase and then maximize the sum of slacks in the second phase.

Definition 1 (Radial efficient) *DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is radial efficient if $\theta^* = 1$ holds.*

Definition 2 (Slackless) *DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is slackless if $\boldsymbol{s}^{-*} = \mathbf{0}$ and $\boldsymbol{s}^{+*} = \mathbf{0}$ hold.*

Definition 3 (CCR-efficient) *DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-efficient, if it is radial efficient and slackless.*

The *CCR-projection* $(\bar{\boldsymbol{x}}_o, \bar{\boldsymbol{y}}_o)$ is defined by

$$
\begin{aligned}
\bar{\boldsymbol{x}}_o &= \theta^* \boldsymbol{x}_o - \boldsymbol{s}^{-*} \\
\bar{\boldsymbol{y}}_o &= \boldsymbol{y}_o + \boldsymbol{s}^{+*}.
\end{aligned}
\tag{4}
$$

It can be demonstrated that $(\bar{\boldsymbol{x}}_o, \bar{\boldsymbol{y}}_o)$ is CCR-efficient (see [11] for example).

Lemma 1 (Adjusted projection) *Let us define*

$$
\tilde{\boldsymbol{x}}_o = \bar{\boldsymbol{x}}_o / \alpha \text{ and } \tilde{\boldsymbol{y}}_o = \bar{\boldsymbol{y}}_o, \tag{5}
$$

where α is a positive scalar not greater than one $(0 < \alpha \leq 1)$. Then the radial efficiency of $(\tilde{\boldsymbol{x}}_o, \tilde{\boldsymbol{y}}_o)$ is α and it is slackless.

Proof: It can be seen that $(\tilde{\boldsymbol{x}}_o, \tilde{\boldsymbol{y}}_o) \in P$, since $(0 < \alpha \leq 1)$. The LP problem associated with $(\tilde{\boldsymbol{x}}_o, \tilde{\boldsymbol{y}}_o)$ is described as:

$$\begin{aligned} & \min\ \tilde{\theta} \\ \text{subject to} \quad & \tilde{\theta}\tilde{\boldsymbol{x}}_o = \sum_{j \in T} \lambda_j \boldsymbol{x}_j + \tilde{\boldsymbol{s}}^- \\ & \tilde{\boldsymbol{y}}_o = \sum_{j \in T} \lambda_j \boldsymbol{y}_j - \tilde{\boldsymbol{s}}^+ \\ & \boldsymbol{\lambda} \geq \mathbf{0},\ \tilde{\boldsymbol{s}}^- \geq \mathbf{0},\ \tilde{\boldsymbol{s}}^+ \geq \mathbf{0}. \end{aligned} \tag{6}$$

Using the definition of $(\tilde{\boldsymbol{x}}_o, \tilde{\boldsymbol{y}}_o)$ and (4), this LP can be transformed into:

$$\begin{aligned} & \min\ \tilde{\theta} \\ \text{subject to} \quad & \frac{\tilde{\theta}\theta^*}{\alpha}\boldsymbol{x}_o = \sum_{j \in T} \lambda_j \boldsymbol{x}_j + \tilde{\boldsymbol{s}}^- + \frac{\tilde{\theta}}{\alpha}\boldsymbol{s}^{-*} \\ & \boldsymbol{y}_o = \sum_{j \in T} \lambda_j \boldsymbol{y}_j - \tilde{\boldsymbol{s}}^+ - \boldsymbol{s}^{+*} \\ & \boldsymbol{\lambda} \geq \mathbf{0},\ \tilde{\boldsymbol{s}}^- \geq \mathbf{0},\ \tilde{\boldsymbol{s}}^+ \geq \mathbf{0}. \end{aligned} \tag{7}$$

Since this LP can be seen as one for evaluating $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, we have

$$\frac{\tilde{\theta}^*\theta^*}{\alpha} = \theta^*,\ \tilde{\boldsymbol{s}}^{-*} = \mathbf{0} \text{ and } \tilde{\boldsymbol{s}}^{+*} = \mathbf{0}.$$

Thus, $(\tilde{\boldsymbol{x}}_o, \tilde{\boldsymbol{y}}_o)$ has $\tilde{\theta}^* = \alpha$ and is slackless. □

2.3 Evaluation and Projection of DMUs in a Group Category

We tend to make evaluations and projections of DMUs in a group category while reflecting evaluation under the total production possibility set P. Let the group of interest be denoted by A. We will follow the steps below:

Step 1 We first evaluate the efficiency of DMUs in Group A with respect to the total production possibility set P. Let the optimal solution for

$A_j = (\boldsymbol{x}_j, \boldsymbol{y}_j)$ $(j \in A)$ be $(\theta_j^*, \boldsymbol{\lambda}_j^*, \boldsymbol{s}_j^{-*}, \boldsymbol{s}_j^{+*})$. We project A_j onto the efficient frontier of P and denote it by $\bar{A}_j = (\bar{\boldsymbol{x}}_j, \bar{\boldsymbol{y}}_j)$. Thus,

$$\begin{aligned} \bar{\boldsymbol{x}}_j &= \theta_j^* \boldsymbol{x}_j - \boldsymbol{s}_j^{-*} \\ \bar{\boldsymbol{y}}_j &= \boldsymbol{y}_j + \boldsymbol{s}_j^{+*}. \end{aligned} \tag{8}$$

Step 2 Find the maximum θ_{max}^* of θ_j^* $(j \in A)$, i.e.,

$$\theta_{max}^* = \max\{\theta_j^* | j \in A\}. \tag{9}$$

Step 3 Define the projection $\widetilde{A}_j = (\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ of A_j within Group A by the following formula:

$$\widetilde{\boldsymbol{x}}_j = \bar{\boldsymbol{x}}_j / \theta_{max}^*, \quad \widetilde{\boldsymbol{y}}_j = \bar{\boldsymbol{y}}_j. \tag{10}$$

The thus obtained projection $(\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ has the following characteristics:

Theorem 1 $(\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ $(j \in A)$ *has the radial efficiency equal to* θ_{max}^* *and is slackless.*

Proof: Putting $\alpha = \theta_{max}^*$ in Lemma 1 demonstrates that the radial efficiency of $(\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ is θ_{max}^* and it is slackless. □

Theorem 2 *If we evaluate the efficiency of* $\widetilde{A}_j = (\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ *within Group* $\widetilde{A}$*, then* $\widetilde{A}_j$ *is CCR-efficient.*

Proof: Let an optimal solution of the LP for evaluating $\widetilde{A}_j$ within Group $\widetilde{A}$ be $(\widetilde{\theta}^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$. Then it holds,

$$\begin{aligned} \widetilde{\theta}^* \widetilde{\boldsymbol{x}}_j &= \sum_{k \in A} \lambda_k^* \widetilde{\boldsymbol{x}}_k + \boldsymbol{s}^{-*} \\ \widetilde{\boldsymbol{y}}_j &= \sum_{k \in A} \lambda_k^* \widetilde{\boldsymbol{y}}_k - \boldsymbol{s}^{+*}. \end{aligned} \tag{11}$$

Returning to $(\bar{\boldsymbol{x}}_j, \bar{\boldsymbol{y}}_j)$ via (10), we have

$$\begin{aligned} \widetilde{\theta}^* \bar{\boldsymbol{x}}_j &= \sum_{k \in A} \lambda_k^* \bar{\boldsymbol{x}}_k + \boldsymbol{s}^{-*} \theta_{max}^* \\ \bar{\boldsymbol{y}}_j &= \sum_{k \in A} \lambda_k^* \bar{\boldsymbol{y}}_k - \boldsymbol{s}^{+*}. \end{aligned} \tag{12}$$

Since $(\bar{\boldsymbol{x}}_j, \bar{\boldsymbol{y}}_j)$ is CCR-efficient with respect to the production possibility set P, it must hold that $\widetilde{\theta}^* = 1, \boldsymbol{s}^{-*} = \mathbf{0}$ and $\boldsymbol{s}^{+*} = \mathbf{0}$. □

So far, we have discussed efficiency issues mainly with respect to the original production possibility set P define by (1). Another production possibility set can be defined based on the projected DMUs $(\widetilde{\boldsymbol{x}}_j, \widetilde{\boldsymbol{y}}_j)$ $(j \in T)$ as

$$\widetilde{P} = \left\{ (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) \mid \widetilde{\boldsymbol{x}} \geq \sum_{j \in T} \widetilde{\boldsymbol{x}}_j \lambda_j, \ \widetilde{\boldsymbol{y}} \leq \sum_{j \in T} \widetilde{\boldsymbol{y}}_j \lambda_j \right\}. \tag{13}$$

Between P and $\widetilde{P}$ we have a relationship:

Theorem 3 $P = \widetilde{P}$.

Proof : Let the set of efficient DMUs in P be E.

$$E = \left\{ j \middle| \ (\boldsymbol{x}_j, \boldsymbol{y}_j) \ (j \in T) \text{ is CCR-efficient.} \right\}. \tag{14}$$

E is the generator of the set P.

For $(\boldsymbol{x}_j, \boldsymbol{y}_j)$ $(j \in E)$ we have

$$\widetilde{\boldsymbol{x}}_j = \boldsymbol{x}_j \ \text{ and } \widetilde{\boldsymbol{y}}_j = \boldsymbol{y}_j. \tag{15}$$

Furthermore, for every $(\widetilde{\boldsymbol{x}}_o, \widetilde{\boldsymbol{y}}_o) \in \widetilde{P}$, it holds

$$\widetilde{\boldsymbol{x}}_o = \bar{\boldsymbol{x}}_o / \theta^*_{max} \geq \bar{\boldsymbol{x}}_o = \sum_{j \in E} \lambda^*_j \boldsymbol{x}_j \tag{16}$$

$$\widetilde{\boldsymbol{y}}_o = \bar{\boldsymbol{y}}_o = \sum_{j \in E} \lambda^*_j \boldsymbol{y}_j. \tag{17}$$

This implies that E is also the generator of $\widetilde{P}$. □

Corollary 1 *The efficiency evaluation with respect to the data set (X, Y) is equivalent to that by the data set $(\widetilde{X}, \widetilde{Y})$.*

2.4 Adjusted Projection under Assurance Region Constarints

The CCR model described in (2) has its dual LP expressed by

$$\max \ \boldsymbol{u}\boldsymbol{y}_o \tag{18}$$

$$\text{subject to} \quad \boldsymbol{v}\boldsymbol{x}_o = 1 \tag{19}$$

$$-\boldsymbol{v}X + \boldsymbol{u}Y \le \mathbf{0} \tag{20}$$

$$\boldsymbol{v} \ge \mathbf{0}, \ \boldsymbol{u} \ge \mathbf{0}, \tag{21}$$

where $\boldsymbol{v} \in R^m$ and $\boldsymbol{u} \in R^s$ are dual variables (weights) corresponding to inputs and outputs, respectively. This LP is equivalent to the following fractional program (refer to Charnes and Cooper [4] for this transformation):

$$\max \ \frac{\boldsymbol{u}\boldsymbol{y}_o}{\boldsymbol{v}\boldsymbol{x}_o} \tag{22}$$

$$\text{subject to} \quad -\boldsymbol{v}X + \boldsymbol{u}Y \le \mathbf{0} \tag{23}$$

$$\boldsymbol{v} \ge \mathbf{0}, \ \boldsymbol{u} \ge \mathbf{0}. \tag{24}$$

This program means that we are seeking optimal weights $\boldsymbol{v}^*$ and $\boldsymbol{u}^*$ for maximizing the ratio (22) of the virtual output $\boldsymbol{u}\boldsymbol{y}_o$ vs. the virtual input $\boldsymbol{v}\boldsymbol{x}_o$, subject to the constraint (23). The optimality condition of both LPs demands the complementary slackness among the input (output) slacks $\boldsymbol{s}^{-*}$ ($\boldsymbol{s}^{+*}$) and the weight $\boldsymbol{v}^*$ ($\boldsymbol{u}^*$):

$$\boldsymbol{v}^*\boldsymbol{s}^{-*} = 0 \text{ and } \boldsymbol{u}^*\boldsymbol{s}^{+*} = 0. \tag{25}$$

This asserts that if s_i^{-*} (s_j^{+*}) > 0, then the corresponding v_i^* (u_j^*) must be zero. In the dual problem, this means that the input (output) i (j) has no contribution to the efficiency evaluation of DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ if it has a positive slack in input (output) i (j) at its optimal solution.

In order to prevent such inconvenience, Thompson *et al.* [21] introduced the Assurance Region Model, which employs, for example, constraints like:

$$l_{1,2} \le \frac{v_2}{v_1} \le u_{1,2}, \tag{26}$$

where $l_{1,2}$ and $u_{1,2}$ are lower and upper bounds that the ratio v_2/v_1 may assume. (See also Dyson and Thanassoulis [12].)

More generally, we can constrain all of the values of the input (output) weights in the following manner.

$$v_1\ l_{1,i} \le v_i \le v_1\ u_{1,i} \quad (i = 2, \ldots, m) \tag{27}$$

$$u_1\ L_{1,r} \le u_r \le u_1\ U_{1,r}. \quad (r = 2, \ldots, s) \tag{28}$$

If some are not required, we can delete them from the constraints. Thus, the CCR-AR model is described as:

$$\begin{aligned}
& \max\ \boldsymbol{u}\boldsymbol{y}_o && (29)\\
\text{subject to}\quad & \boldsymbol{v}\boldsymbol{x}_o = 1 && (30)\\
& -\boldsymbol{v}X + \boldsymbol{u}Y \le \mathbf{0} && (31)\\
& \boldsymbol{v}P \le \mathbf{0} && (32)\\
& \boldsymbol{u}Q \le \mathbf{0} && (33)\\
& \boldsymbol{v} \ge \mathbf{0},\ \boldsymbol{u} \ge \mathbf{0}, && (34)
\end{aligned}$$

where

$$P = \begin{pmatrix} l_{12} & -u_{12} & l_{13} & -u_{13} & \ldots & \ldots & \ldots & \ldots \\ -1 & 1 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & -1 & 1 & \ldots & \ldots & \ldots & \ldots \\ \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{pmatrix}$$

and

$$Q = \begin{pmatrix} L_{12} & -U_{12} & L_{13} & -U_{13} & \ldots & \ldots & \ldots & \ldots \\ -1 & 1 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & -1 & 1 & \ldots & \ldots & \ldots & \ldots \\ \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{pmatrix}.$$

The dual side (the envelopment model) is expressed as:

$$\begin{aligned}
(DAR_o)\quad & \min\ \theta && (35)\\
\text{subject to}\quad & \theta\boldsymbol{x}_o - X\boldsymbol{\lambda} + P\boldsymbol{\pi} \ge \mathbf{0} && (36)\\
& Y\boldsymbol{\lambda} + Q\boldsymbol{\tau} \ge \boldsymbol{y}_o && (37)\\
& \boldsymbol{\lambda} \ge \mathbf{0},\ \boldsymbol{\pi} \ge \mathbf{0},\ \boldsymbol{\tau} \ge\ \mathbf{0}. && (38)
\end{aligned}$$

Let an optimal two phase solution of (DAR_o) be $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{\pi}^*, \boldsymbol{\tau}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$, where slacks $\boldsymbol{s}^{-*}$ and $\boldsymbol{s}^{+*}$ are defined by:

$$\boldsymbol{s}^{-*} = \theta^* \boldsymbol{x}_o - X\boldsymbol{\lambda}^* + P\boldsymbol{\pi}^* \tag{39}$$

$$\boldsymbol{s}^{+*} = -\boldsymbol{y}_o + Y\boldsymbol{\lambda}^* + Q\boldsymbol{\tau}^*. \tag{40}$$

Based on the solution, we define "AR-efficiency" as:

Definition 4 (AR-Efficiency) *The DMU associated with* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ *is* AR-efficient, *if and only if it satisfies*

$$\theta^* = 1, \quad \boldsymbol{s}^{-*} = \mathbf{0} \quad \textit{and} \quad \boldsymbol{s}^{+*} = \mathbf{0}.$$

An improvement (*AR-projection*) of an AR-inefficient $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is represented by:

$$\bar{\boldsymbol{x}}_o = \theta^* \boldsymbol{x}_o - \boldsymbol{s}^{-*} + P\boldsymbol{\pi}^* \quad (= X\boldsymbol{\lambda}^*) \tag{41}$$

$$\bar{\boldsymbol{y}}_o = \boldsymbol{y}_o + \boldsymbol{s}^{+*} - Q\boldsymbol{\tau}^* \quad (= Y\boldsymbol{\lambda}^*). \tag{42}$$

Theorem 4 *The activity* $(\bar{\boldsymbol{x}}_o, \bar{\boldsymbol{y}}_o)$ *is AR-efficient.*

(See [11] pp. 154-155 for a proof.)

Lemma 2 (Adjusted AR-projection) *Let us define*

$$\tilde{\boldsymbol{x}}_o = \bar{\boldsymbol{x}}_o/\alpha \textit{ and } \tilde{\boldsymbol{y}}_o = \bar{\boldsymbol{y}}_o, \tag{43}$$

where $\bar{\boldsymbol{x}}_o$ *and* $\bar{\boldsymbol{y}}_o$ *are defined by (41) and (42), respectively, and* α *is a positive scalar not greater than one* $(0 < \alpha \leq 1)$. *Then the radial AR-efficiency of* $(\tilde{\boldsymbol{x}}_o, \tilde{\boldsymbol{y}}_o)$ *is* α *and it is slackless.*

Proof: We can prove this lemma in the similar way as in the case of Lemma 1. □

Using this lemma, we can develop an adjusted AR-projection in the same way as the CCR-model. So we will not repeat it here.

3 An Application to Public Library Evaluation

We apply the above analysis to the efficiency evaluation/improvement of public libraries in Tokyo, Japan. Table 1 shows the data for public libraries in the 23 Wards of the Tokyo Metropolitan Area in 1986. As the measurement items of efficiency we use the number of books (unit=1000) and staff as inputs and the number of registered residents (unit=1000) and borrowed books (unit=1000) as outputs. We classify these 23 Wards into three categories: business (Group A), commercial (Group B) and residential (Group C) districts. In the table, libraries L1A to L6A belong to Group A, L7B to L16B to Group B and L17C to L23C to Group C, respectively. However, it should be noted that this classification is not rigid but temporary.

Table 1

We first tried bilateral comparisons between each combination of paired groups using non-parametric tests based on the rank-sum statistics by Wilcoxon-Man-Whiteney (see Brockett and Golany [3] and Cooper, Seiford and Tone [11] pp. 200-203). For this purpose, we used the input-oriented CCR model for testing the null hypothesis: the two groups have the same distribution of efficiency score. The results were as follows:

Between Group A and Group B The null hypothesis is rejected at the significance level 3.93%. Group B outperforms Group A.

Between Group A and Group C The null hypothesis is rejected at the significance level 0.27%. Group C outperforms Group A.

Between Group B and Group C The null hypothesis is rejected at the significance level 0.18%. Group C outperforms Group B.

As can be seen, significant gaps exist in the efficiency scores among the three groups, especially between A and C, and B and C. Hence, it seems that we need to add a certain handicap to A and B when comparing them with C. We followed the processes proposed in the preceding section.

3.1 Adjusted CCR-Projection

From the data we obtained the input-oriented CCR efficiency (θ^*) and the CCR-projection ($\bar{X}, \bar{Y}$) of each library as exhibited in Table 2.

Table 2

Then we try an adjusted projection by choosing a champion from each group. Following Step 2 in Section 2.3, L5A ($\theta^* = 0.911$) is the champion for Group A, L9B ($\theta^* = 0.897$) for B and L17C (L19C, L23C) ($\theta^* = 1$) for Group C, respectively.

Using these value as θ^*_{max} in (10), we adjust the projection and obtained the results as depicted in Table 3. The last column of this table shows the global efficiency ($\tilde{\theta}^*$) of the adjusted projection ($\widetilde{X}, \widetilde{Y}$). This value ($\theta^*_{max}$) is the same within the same group as demonstrated by Theorem 1. Furthermore, the adjusted projection is locally CCR-efficient by Theorem 2.

Table 3

3.2 Adjusted AR-Projection

Table 4 exhibits the optimal weights ($\boldsymbol{v}^*, \boldsymbol{u}^*$) for the CCR-model. As can be observed, there are zeros in the optimal weights for inefficient DMUs. These zeros correspond to the positive slacks in inputs or outputs by the complementary slackness condition as noted by (25) in the previous section.

Table 4

In an effort to prevent such zero weights, we imposed the following weight restrictions on inputs and outputs. In this case, we assume that the two inputs, i.e., the numbers of books and staff, have equal weights in evaluating the library efficiency. This means that, for example, Library L1A has 163.523 ($\times$ 1000) books and 26 staff and hence the weight of 163.523 books is equal to that of 26 staff. Thus, for L1A, we have

$$\frac{v_1}{v_2} = \frac{163.523}{26} = 6.289,$$

where v_1 and v_2 are weights for 1000 books and for one member of staff, respectively.

After accounting for the same ratios regarding all 23 libraries, we set the upper and lower bounds to v_1/v_2 as:

$$4 \leq \frac{v_1}{v_2} \leq 10. \tag{44}$$

In the same vein, for outputs, we imposed the ratio of weights for a registered residence and for a borrowed book as below:

$$0.03 \leq \frac{u_1}{u_2} \leq 0.08. \tag{45}$$

The results of this CCR-AR model is presented in Table 5, where the AR-score is exhibited along with the optimal input/output weights.

Table 5

These optimal weights satisfy the bounds expressed by (44) and (45). It should be noted that the optimal scores dropped from these in Table 4 by dint of the weight restrictions, e.g., L5A (0.911 $\rightarrow$ 0.587), L9B (0.897 $\rightarrow$ 0.859), L17C (1 $\rightarrow$ 0.843), and L19C (1 $\rightarrow$ 0.848). The scores from the AR model have a lower average and a larger standard deviation than those from the CCR model. Thus, the former is more discriminatory than the latter with respect to the efficiency evaluation. Not only the magnitudes, but also the rank of scores changed considerably. The correlation coefficient of the two inputs is 0.941 and that of two outputs 0.932, showing that the values of the two inputs (outputs) are almost linearly related. Even using such highly correlated input/output data, the optimal scores change significantly depending on whether the weight restrictions are employed or not. This fact demonstrates an instance of the importance of employing weight restrictions in DEA. See Pedraja-Chaparro *et al.* [19] for similar discussions.

Table 6

3.3 A Summary

Table 7 summarizes the average rate of reduction for inputs in each group.

Table 7

The "CCR" rows indicate the average reduction rates of inputs by the original input-oriented CCR model, i.e., the target level being one for all groups. The "Adjusted CCR" and "Adjusted AR" rows exhibit those by the adjusted projection using the target level of the group as denoted by the efficiency score of the benchmark within the corresponding group. The values of "CCR" and "Adjusted CCR" show similarity whereas those of "Adjusted AR" have a little different tendencies. This is caused by difference between models. The values of "Adjusted AR" seem to be more balanced between groups.

Since the employed models were input-oriented, we saw no significant differences in output projections between models.

4 Concluding Remarks

We have proposed a projection/improvement method using DEA. This method reflects the global evaluation of performance and then utilizes results thus obtained in order to set a local level improvement target. As the library example shows, this local level is determined by the efficiency value of a champion DMU in the group. Although, in the library example, we chose the DMU with θ^*_{max} for this purpose, we can utilize other values as standard, e.g., the θ^*-value of a typical DMU in the group, the mean or median of efficiency values across the group, unless this setting affects input and output values of any efficient DMUs in P. For example, if we use L4A ($\theta^* = 0.593$) instead of L5A ($\theta^* = 0.911$) as the representative of Group A, then the adjusted CCR projection and the efficiency values become as those exhibited in Table 8. As can be seen, a big change occurs in the projected input values of Group A and the efficiency level of Group A drops to that of L4A. We should note that this library study is only for explanatory purposes and is still preliminary. Further research in cooperation with library managers is still required.

Table 8

This paper has also pointed out the importance of weight restrictions in DEA and proposes the adjusted projection under weight restrictions. This will enhance usability of the method, if the restrictions reflect the real world weights to inputs/outputs.

The setting of local targets may be a matter for managers or decision-makers. However, once the local standard is set, the proposed method will contribute to determining how much improvement in inputs and outputs is necessary so that the projected DMUs result in the same level of global efficiency designated by the standard, and the full efficiency within the group.

Future research includes extensions of this method to non-oriented DEA models such as the Additive models (Charnes *et al.* [5]) and the SBM (slack-based measure of efficiency models [11]).

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Table 1: Data for Public Libraries in Tokyo

	INPUT(X)		OUTPUT(Y)	
Lib.	Book	Employee	Registered	Borrowed
L1A	163.523	26	5.561	105.321
L2A	338.671	30	18.106	314.682
L3A	281.655	51	16.498	542.349
L4A	400.993	78	30.810	847.872
L5A	363.116	69	57.279	758.704
L6A	541.658	114	66.137	1438.746
L7B	508.141	61	35.295	839.597
L8B	338.804	74	33.188	540.821
L9B	511.467	84	65.391	1562.274
L10B	393.815	68	41.197	978.117
L11B	509.682	96	47.032	930.437
L12B	527.457	92	56.064	1345.185
L13B	601.594	127	69.536	1164.801
L14B	528.799	96	37.467	1348.588
L15B	394.158	77	57.727	1100.779
L16B	515.624	101	46.160	1070.488
L17C	566.708	118	102.967	1707.645
L18C	467.617	74	47.236	1223.026
L19C	768.484	103	84.510	2299.694
L20C	669.996	107	69.576	1901.465
L21C	844.949	120	89.401	1909.698
L22C	1258.981	242	97.941	3055.193
L23C	1148.863	202	191.166	4096.300
Average	549.772	96	59.402	1351.382
S.D.	250.536	47	37.530	870.917

Table 2: Efficiency and Projection by the CCR Model

		X		Y	
Lib	θ^*	Book	Employee	Registered	Borrowed
L1A	0.226	33.42	5.88	5.56	119.16
L2A	0.638	108.81	19.13	18.11	387.97
L3A	0.540	152.11	26.74	25.31	542.35
L4A	0.593	237.80	41.81	39.57	847.87
L5A	0.911	330.90	62.88	57.28	1099.76
L6A	0.745	403.52	70.95	67.14	1438.75
L7B	0.650	256.52	39.63	35.30	839.60
L8B	0.539	182.66	38.03	33.19	550.40
L9B	0.897	458.60	75.32	69.13	1562.27
L10B	0.705	277.70	47.95	45.02	978.12
L11B	0.539	274.57	51.12	47.03	930.44
L12B	0.719	379.32	66.16	62.40	1345.19
L13B	0.638	383.92	79.47	69.54	1164.80
L14B	0.715	378.23	66.50	62.94	1348.59
L15B	0.844	332.70	63.51	57.73	1100.78
L16B	0.582	300.23	52.79	49.96	1070.49
L17C	1	566.71	118.00	102.97	1707.65
L18C	0.787	367.88	58.22	52.48	1223.03
L19C	1	768.48	103.00	84.51	2299.69
L20C	0.849	568.54	90.80	82.23	1901.47
L21C	0.787	537.28	94.47	89.40	1915.68
L22C	0.681	856.87	150.66	142.58	3055.19
L23C	1	1148.86	202.00	191.17	4096.30

Table 3: Adjusted Projection and Efficiency by the CCR Model

	$\tilde{X}$		$\tilde{Y}$		
Lib	Book	Employee	Registered	Borrowed	θ^*
L1A	36.674	6.448	5.561	119.161	0.911
L2A	119.406	20.995	18.106	387.975	0.911
L3A	166.917	29.348	25.310	542.349	0.911
L4A	260.947	45.881	39.568	847.872	0.911
L5A	363.116	69.000	57.279	1099.758	0.911
L6A	442.799	77.856	67.143	1438.746	0.911
L7B	286.092	44.198	35.295	839.597	0.897
L8B	203.715	42.418	33.188	550.403	0.897
L9B	511.467	84.000	69.132	1562.274	0.897
L10B	309.707	53.477	45.024	978.117	0.897
L11B	306.220	57.017	47.032	930.437	0.897
L12B	423.049	73.789	62.399	1345.185	0.897
L13B	428.177	88.636	69.536	1164.801	0.897
L14B	421.829	74.169	62.936	1348.588	0.897
L15B	371.049	70.831	57.727	1100.779	0.897
L16B	334.841	58.874	49.958	1070.488	0.897
L17C	566.708	118.000	102.967	1707.645	1
L18C	367.877	58.216	52.484	1223.026	1
L19C	768.484	103.000	84.510	2299.694	1
L20C	568.540	90.797	82.227	1901.465	1
L21C	537.279	94.468	89.401	1915.682	1
L22C	856.870	150.660	142.580	3055.193	1
L23C	1148.863	202.000	191.166	4096.300	1

Table 4: Optimal Weight for the CCR Model

DMU	Score	v_1^*	v_2^*	u_1^*	u_2^*
L1A	0.226	0	3.85E-02	4.06E-02	0
L2A	0.638	0	3.33E-02	0.035222442	0
L3A	0.540	3.55E-03	0	0	9.96E-04
L4A	0.593	2.49E-03	0	0	6.99E-04
L5A	0.911	1.33E-03	7.51E-03	1.59E-02	0
L6A	0.745	1.85E-03	0	0	5.18E-04
L7B	0.650	0	1.64E-02	7.48E-03	4.59E-04
L8B	0.5399	2.95E-03	0	1.62E-02	0
L9B	0.897	6.63E-04	7.87E-03	0	5.74E-04
L10B	0.705	8.33E-04	9.88E-03	0	7.21E-04
L11B	0.539	1.96E-03	0	7.40E-03	2.05E-04
L12B	0.719	6.17E-04	7.33E-03	0	5.35E-04
L13B	0.638	1.66E-03	0	6.27E-03	1.74E-04
L14B	0.715	1.89E-03	0	0	5.30E-04
L15B	0.844	2.54E-03	0	9.57E-03	2.65E-04
L16B	0.582	1.94E-03	0	0	5.44E-04
L17C	1	1.38E-03	1.86E-03	7.89E-03	1.10E-04
L18C	0.787	7.43E-04	8.82E-03	0	6.43E-04
L19C	1	2.87E-04	7.56E-03	1.89E-03	3.65E-04
L20C	0.849	5.15E-04	6.12E-03	0	4.46E-04
L21C	0.787	0	8.33E-03	8.81E-03	0
L22C	0.681	7.94E-04	0	0	2.23E-04
L23C	1	7.04E-04	9.48E-04	4.03E-03	5.60E-05
Average	0.721				
S.D.	0.180				

Table 5: Results of the AR Model

DMU	Score	v_1^*	v_2^*	u_1^*	u_2^*
L1A	0.181	5.88E-03	1.47E-03	1.37E-04	1.72E-03
L2A	0.266	2.89E-03	7.22E-04	6.74E-05	8.43E-04
L3A	0.54	3.49E-03	3.49E-04	2.98E-05	9.94E-04
L4A	0.592	2.45E-03	2.45E-04	2.09E-05	6.97E-04
L5A	0.587	2.70E-03	2.70E-04	6.15E-05	7.68E-04
L6A	0.742	1.81E-03	1.81E-04	1.55E-05	5.15E-04
L7B	0.47	1.91E-03	4.78E-04	1.68E-05	5.59E-04
L8B	0.446	2.89E-03	2.89E-04	6.57E-05	8.21E-04
L9B	0.859	1.88E-03	4.70E-04	1.65E-05	5.49E-04
L10B	0.697	2.43E-03	6.09E-04	2.14E-05	7.12E-04
L11B	0.512	1.93E-03	1.93E-04	4.38E-05	5.48E-04
L12B	0.715	1.82E-03	4.54E-04	1.59E-05	5.31E-04
L13B	0.542	1.63E-03	1.63E-04	3.70E-05	4.63E-04
L14B	0.714	1.86E-03	1.86E-04	1.59E-05	5.29E-04
L15B	0.782	2.49E-03	2.49E-04	5.66E-05	7.08E-04
L16B	0.581	1.90E-03	1.90E-04	1.63E-05	5.42E-04
L17C	0.843	1.73E-03	1.73E-04	3.93E-05	4.91E-04
L18C	0.736	2.06E-03	5.14E-04	1.80E-05	6.01E-04
L19C	0.848	1.26E-03	3.15E-04	1.10E-05	3.68E-04
L20C	0.799	1.44E-03	3.59E-04	1.26E-05	4.20E-04
L21C	0.639	1.14E-03	2.86E-04	2.67E-05	3.33E-04
L22C	0.679	7.79E-04	7.79E-05	6.66E-06	2.22E-04
L23C	1	8.55E-04	8.55E-05	1.95E-05	2.43E-04
Average	0.642				
S.D.	0.187				

Table 6: Adjusted Projection and Efficiency under the AR-Model

	$\tilde{X}$		$\tilde{Y}$		
Lib	Book	Employee	Registered	Borrowed	θ^*
L1A	39.806	6.999	4.918	105.372	0.742
L2A	118.980	20.920	14.698	314.955	0.742
L3A	204.783	36.006	25.298	542.085	0.742
L4A	320.200	56.300	39.556	847.610	0.742
L5A	287.273	50.510	35.489	760.447	0.742
L6A	543.502	95.562	67.142	1438.716	0.742
L7B	274.111	48.196	39.177	839.481	0.859
L8B	176.798	31.086	25.269	541.455	0.859
L9B	510.047	89.680	72.898	1562.049	0.859
L10B	319.336	56.148	45.641	977.984	0.859
L11B	303.904	53.434	43.435	930.725	0.859
L12B	439.170	77.218	62.768	1344.984	0.859
L13B	380.731	66.942	54.415	1166.011	0.859
L14B	440.098	77.381	62.900	1347.825	0.859
L15B	359.597	63.226	51.395	1101.286	0.859
L16B	349.503	61.452	49.952	1070.374	0.859
L17C	479.453	84.300	79.779	1709.500	1
L18C	342.932	60.296	57.062	1222.731	1
L19C	644.789	113.371	107.290	2299.011	1
L20C	533.131	93.738	88.711	1900.891	1
L21C	535.607	94.174	89.123	1909.720	1
L22C	856.495	150.594	142.517	3053.856	1
L23C	1148.863	202.000	191.166	4096.300	1

Table 7: Average Reduction Rates of Inputs for Each Group

	Model	Book	Employee	Target Level
	CCR	39%	38%	1
Group A	Adjusted CCR	33%	32%	0.911
	Adjusted AR	28%	28%	0.742
	CCR	33%	34%	1
Group B	Adjusted CCR	26%	26%	0.897
	Adjusted AR	26%	29%	0.859
	CCR	16%	15%	1
Group C	Adjusted CCR	16%	15%	1
	Adjusted AR	21%	17%	1

Table 8: Adjusted CCR Projection by L4A and Efficiency

	X		Y		
Lib	Book	Employee	Registered	Borrowed	θ^*
L1A	56.356	9.909	5.561	119.161	0.593
L2A	183.489	32.262	18.106	387.975	0.593
L3A	256.499	45.099	25.310	542.349	0.593
L4A	400.993	70.505	39.568	847.872	0.593
L5A	557.994	106.031	57.279	1099.758	0.593
L6A	680.441	119.639	67.143	1438.746	0.593
The values for Group B are the same with those in Table 3.					
The values for Group C are the same with those in Table 3.					