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A Slacks-based Malmquist Productivity Index

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Any comments are welcomed.

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Abstract

We propose a new Malmquist productivity index using non-radial slacks-based measures of efficiency. It differs from the traditional one using the radial measure in the definition of the distance function employed. We utilize a weighted l_1-norm for this purpose that enables us to account non-radial slacks in the index. Also, we employ an "exclusive" policy for evaluating a decision making unit with respect to a group of evaluators, so that sharp identification of the catch-up effect, as well as the frontier shift can be obtained. We discuss the rationale of the new index by comparing it with the radial one.

Keywords: DEA; Malmquist productivity index; non-radial measure; slacks; non-parametric method

1 Introduction

We present a new Malmquist productivity index using non-radial slacks-based measures of efficiency. Most non-parametric Malmquist indices proposed so far utilize radial measures in implementation. The radial measures have a common shortcoming: their neglect of slacks. In an effort to overcome this shortcoming, the author has developed two measures: the slacks-based measure of efficiency (SBM) (Tone (2001a)) and the slacks-based measure of super-efficiency (Super SBM) (Tone (2001b)). The former corresponds to

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the CCR model (Charnes, Cooper and Rhodes (1978)) and the latter to the super-efficiency model by Andersen and Petersen (1993) in the radial measurement framework. Using the SBM and super SBM models, we formulate a new Malmquist index. As another feature, we employ an "exclusive" scheme for evaluating a decision making unit (DMU). This policy means that when we evaluate the efficiency of a DMU with respect to a group of DMUs, we always exclude the concerned DMU from the group. Hence, the examinee or target DMU is definitely separated from the evaluator group. In doing so, we can be free from the upper bound of efficiency scores which usually take the value 1 for efficient DMUs, and thus we can discriminate between efficient DMUs. The effect of this scheme appears in the evaluation of the Malmquist index, resulting in sharp identification of the catch-up effect and frontier shift.

The rest of the paper is organized as follows. Section 2 summarizes the SBM and Super SBM models. The new Malmquist index is introduced in Section 3. Then the rationale of the new index will be discussed in Section 4. Extensions to the output-orientation and to variable returns-to-scale environments will be presented in Section 5.

2 Slacks-based Measure of Efficiency

In this section, we will briefly survey the slacks-based measure of efficiency (Tone (2001a), Cooper, Seiford and Tone (2000)) and the slacks-based measure of super-efficiency (Tone (2001b)), putting emphasis on the input-oriented case.

2.1 Slacks-based Measure (SBM) and Super SBM

We consider a set of n Decision Making Units (DMUs) called "evaluator" which is described by the input and output matrices $X = (x_{ij}) \in R^{m \times n}$ and $Y = (y_{ij}) \in R^{s \times n}$, respectively. So, we suppose n DMUs with m inputs and s outputs. In addition, we have a "examinee" or "target" DMU denoted by $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ with $\boldsymbol{x}_o \in R^m$ and $\boldsymbol{y}_o \in R^s$. The target DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is not a member of the evaluator group. However, this does not exclude the possibility that some evaluators coincide with the target by chance. We assume that $X > O$ and $\boldsymbol{x}_o > \mathbf{0}$.

In order to evaluate the efficiency of the examinee $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, we formulate the following linear programming problem with variables $\boldsymbol{\lambda} \in R^n$, $\boldsymbol{s}^- \in R^m$ and $\boldsymbol{s}^+ \in R^s$:

$$
\begin{aligned}
\text{[SBM-I]} \qquad \rho_I^* &= \min \ \rho_I = 1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io} \qquad (1)\\
\text{subject to } \boldsymbol{x}_o &= X\boldsymbol{\lambda} + \boldsymbol{s}^- \\
\boldsymbol{y}_o &= Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\
\boldsymbol{\lambda} &\geq \mathbf{0}, \ \boldsymbol{s}^- \geq \mathbf{0}, \ \boldsymbol{s}^+ \geq \mathbf{0}.
\end{aligned}
$$

If [SBM-I] has a finite minimum ρ_I^*, then $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ belongs to the production possibility set P spanned by the evaluator group. From the assumption $X > O$ and $\boldsymbol{x}_o > \mathbf{0}$, it holds

$$0 < \rho_I^* \leq 1.$$

The score ρ_I^* is units invariant. If [SBM-I] has no feasible solution, the examinee $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is positioned outside P. In this case, we solve the following problem [SuperSBM-I]:

$$\text{[SuperSBM-I]} \qquad \delta_I^* = \min \delta_I = 1 + \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io} \qquad (2)$$

$$
\begin{aligned}
\text{subject to} \quad & \boldsymbol{x}_o \geq X\boldsymbol{\lambda} - \boldsymbol{s}^- \\
& \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} \\
& \boldsymbol{\lambda} \geq 0, \ \boldsymbol{s}^- \geq 0.
\end{aligned}
$$

[SuperSBM-I] always has a finite optimum δ_I^* (≥ 1).

Now we will observe how the slacks-based measure relates to the radial measure of efficiency and reduces the score by accounting the slacks. The typical radial measure evaluates the efficiency of DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ by solving the following problem:

$$
\begin{aligned}
[\text{CCR-I}] \quad \theta_I^* &= \min \ \theta_I \\
\text{subject to} \ \theta_I \boldsymbol{x}_o &= X\boldsymbol{\mu} + \boldsymbol{t}^- \\
\boldsymbol{y}_o &\leq Y\boldsymbol{\mu} - \boldsymbol{t}^+ \\
\boldsymbol{\mu} &\geq 0, \ \boldsymbol{t}^- \geq 0, \ \boldsymbol{t}^+ \geq 0.
\end{aligned} \tag{3}
$$

In this case we assume that DMU DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is included in the evaluator group (X, Y). Let an optimal solution of [CCR-I] be $(\theta_I^*, \boldsymbol{\mu}^*, \boldsymbol{t}^{-*}, \boldsymbol{t}^{+*})$. Then we have:

Theorem 1 (Tone (2001b)) *The* CCR-I *model returns the same efficiency score θ_I^* for any DMU represented by $(\boldsymbol{x}_o - \alpha \boldsymbol{t}^{-*}/\theta_I^*, \boldsymbol{y}_o)$ for the range $0 \leq \alpha \leq 1$.*

This contradicts our common understanding that a reduction of input values usually increases efficiency. Specifically, we have the following relationship between the optimal SBM-I and CCR-I scores.

Theorem 2 (Tone (2001a)) *The optimal* SBM-I ρ_I^* *is not greater than the optimal* CCR-I θ_I^* *and it holds*

$$
\rho_I^* \leq \theta_I^* - \frac{1}{m}\sum_{i=1}^{m} t_i^{-*}/x_{io}. \tag{4}
$$

The radial super-efficiency measure (Andersen and Petersen (1993)) can be obtained by solving [CCR-I] under the condition that $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-efficient and is excluded from the evaluator set (X, Y). We call this model [SuperCCR-I]. Then, we have the following inequality between the optimal values of [SuperSBM-I] and [SuperCCR-I].

Theorem 3 (Tone (2001b)) *The optimal value* δ_I^* *of* [SuperSBM-I] *is not greater than the optimal value* θ_I^* *of* [SuperCCR-I] *and it holds:*

$$\delta_I^* \leq \theta_I^* - \frac{\alpha^*}{m} \sum_{i=1}^{m} t_i^{-*}/x_{io}, \tag{5}$$

where α^* *is defined by*

$$\begin{aligned} \alpha^* &= \min \left\{ \frac{(\theta_I^* - 1)x_{io}}{t_i^{-*}} \mid t_i^{-*} > 0 \right\} \\ &= 0 \quad \text{if } s^{-*} = 0. \end{aligned} \tag{6}$$

The above theorems characterize the difference between the two approaches and demonstrate that the radial measures overestimate efficiency by neglecting slacks. Furthermore, we have

Theorem 4 (i) *The optimal* ρ_I^* *of* [SBM-I] *increases strictly in any decrease of the input* $\boldsymbol{x}_o$ *as long as* [SBM-I] *is feasible, and* (ii) *the optimal* δ_I^* *of* [SuperSBM-I] *increases in any decrease of the input* $\boldsymbol{x}_o$.

(See Appendix A for a proof.)

As to the output $\boldsymbol{y}_o$, since an increase of $\boldsymbol{y}_o$ reduces the feasible region of [SBM-I] as well as [SuperSBM-I], the following theorem holds.

Theorem 5 (i) *The optimal* ρ_I^* *of* [SBM-I] *increases in any increase of the output* $\boldsymbol{y}_o$ *as long as* [SBM-I] *is feasible, and* (ii) *the optimal* δ_I^* *of* [SuperSBM-I] *increases in any increase of the output* $\boldsymbol{y}_o$.

2.2 Dual Problems

We observe the dual problems of [SBM-I] and [SuperSBM-I], and present their economic meanings.

2.2.1 Dual SBM-I

The dual LP problem corresponding to the input-oriented [SBM-I] (1) is expressed, $\boldsymbol{v} \in R^m$ and $\boldsymbol{u} \in R^s$ as variables, as follows:

$$
\begin{aligned}
\text{[DSBM-I]} \quad \pi^* &= \max \pi = 1 + \boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o && (7)\\
\text{subject to} \quad -\boldsymbol{v}X + \boldsymbol{u}Y &\leq 0 && (8)\\
\boldsymbol{v} &\geq \frac{1}{m}[1/\boldsymbol{x}_o] && (9)\\
\boldsymbol{u} &\geq 0, && (10)
\end{aligned}
$$

where the notation $[1/\boldsymbol{x}_o]$ designates the row vector $(1/x_{1o}, \ldots, 1/x_{mo})$.

The dual variables $\boldsymbol{v}$ and $\boldsymbol{u}$ can be interpreted as the virtual costs and prices of input and output items, respectively. The dual problem aims to find the optimal virtual costs and prices for the examinee DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ so that it maximizes the profit $\boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o$ under the condition that the profit does not exceed zero for any evaluator DMU $(\boldsymbol{x}_j, \boldsymbol{y}_j)$. From condition (9), the optimal profit of the examinee is not greater than zero and hence the optimal objective value π^* is at best one. This interpretation of the dual problem gives a sound economic meaning for the SBM model. It contrasts to the ratio maximization operation of the CCR model (Charnes *et al.* (1978)), where the virtual cost $\boldsymbol{v}$ and price $\boldsymbol{u}$ are determined so that it maximizes the ratio $\boldsymbol{u}\boldsymbol{y}_o/\boldsymbol{v}\boldsymbol{x}_o$ under the condition that the ratio $\boldsymbol{u}\boldsymbol{y}_j/\boldsymbol{v}\boldsymbol{x}_j$ does not exceed one for every $(\boldsymbol{x}_j, \boldsymbol{y}_j)$ including the examinee $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. We are dealing with the virtual profit instead of the virtual ratio of the CCR model.

2.2.2 Dual SuperSBM-I

The dual LP to [SuperSBM-I] (2) can be described as:

$$
\begin{aligned}
\text{[DsuperSBM-I]}\ \pi^* &= \max \pi = 1 + \boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o && (11)\\
\text{subject to}\quad -\boldsymbol{v}X + \boldsymbol{u}Y &\leq 0 && (12)\\
\boldsymbol{v} &\leq \frac{1}{m}[1/\boldsymbol{x}_o] && (13)\\
\boldsymbol{u} &\geq 0 && (14)
\end{aligned}
$$

We can give a similar interpretation to the above expression as with the [DSBM-I] case. In this case we maximize the virtual profit of the examinee while keeping that of evaluators non-positive. So we seek the virtual costs and prices that discriminate the examinee against the evaluators at maximum. The condition (13) ensures $\boldsymbol{v}\boldsymbol{x}_o \leq 1$, and hence the optimal π^* is not less than one.

3 A New Malmquist Productivity Index

The Malmquist productivity index was introduced by Malmquist (1953), and has further been studied and developed in the non-parametric framework by several authors, e.g. Caves, Christensen and Diewert (1982), Färe and Grosskopf (1992), Färe, Grosskopf, Lindgren and Roos (1994), and Thrall (2000). Typically, it is expressed by a geometric mean of two indices representing the productivity change between two periods t and $t+1$. Each index indicates progress or regress in productivity as measured by the corresponding two period technologies. In order to develop a new Malmquist index using slacks-based measures, we employ the following notation:

$$\delta((X,Y)^{t_1}, (\boldsymbol{x}_o, \boldsymbol{y}_o)^{t_2}),\ (t_1 = t,\ t+1 \text{ and } t_2 = t,\ t+1). \qquad (15)$$

This implies the distance between the production (X, Y) at the period t_1 and the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ at the period t_2. The distance is measured as a slacks-based l_1-norm as follows. First we measure the distance between $(\boldsymbol{x}_o, \boldsymbol{y}_o)^{t_2}$ and $(X, Y)^{t_1}$ by using the [SBM-I] (1). In the case $t_1 = t_2$, we exclude the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ from the evaluator group (X, Y). If the distance is finite, we define this as $\delta((X, Y)^{t_1}, (\boldsymbol{x}_o, \boldsymbol{y}_o)^{t_2})$. Otherwise if [SBM-I] is infeasible, i.e. $(\boldsymbol{x}_o, \boldsymbol{y}_o)^{t_2}$ is outside $(X, Y)^{t_1}$, then the distance is evaluated by the [Super-SBM-I] model (2). Using this notation, the Malmquist index for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is defined by:

$$M(\boldsymbol{x}_o, \boldsymbol{y}_o) = \left[\frac{\delta((X, Y)^t, (\boldsymbol{x}_o, \boldsymbol{y}_o)^{t+1})}{\delta((X, Y)^t, (\boldsymbol{x}_o, \boldsymbol{y}_o)^t)} \times \frac{\delta((X, Y)^{t+1}, (\boldsymbol{x}_o, \boldsymbol{y}_o)^{t+1})}{\delta((X, Y)^{t+1}, (\boldsymbol{x}_o, \boldsymbol{y}_o)^t)} \right]^{1/2}. \quad (16)$$

In the similar way as Färe *et al.* (1994), we decompose $M(\boldsymbol{x}_o, \boldsymbol{y}_o)$ into the catch-up term and the frontier shift term as follows:

$$\begin{aligned} M(\boldsymbol{x}_o, \boldsymbol{y}_o) &= C \times F, & (17) \\ \text{where } C &= \frac{\delta((X, Y)^{t+1}, (\boldsymbol{x}_o, \boldsymbol{y}_o)^{t+1})}{\delta((X, Y)^t, (\boldsymbol{x}_o, \boldsymbol{y}_o)^t)} \quad \text{and} & (18) \\ F &= \left[\frac{\delta((X, Y)^t, (\boldsymbol{x}_o, \boldsymbol{y}_o)^t)}{\delta((X, Y)^{t+1}, (\boldsymbol{x}_o, \boldsymbol{y}_o)^t)} \times \frac{\delta((X, Y)^t, (\boldsymbol{x}_o, \boldsymbol{y}_o)^{t+1})}{\delta((X, Y)^{t+1}, (\boldsymbol{x}_o, \boldsymbol{y}_o)^{t+1})} \right]^{1/2} & (19) \end{aligned}$$

The term C indicates the *catch-up effect* of the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, while the term F the *frontier shift* for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ between the two periods.

4 Rationale of the New Index

We will discuss rationale of the proposed new index in this section.

4.1 Radial or Non-radial?

We utilize the non-radial approach, although the majority of DEA literatures, especially those related with the Malmquist index, are radial-oriented. The exceptions are Grifell-Tatje, Lovell and Pastor (1998), and Thrall (2000). Reasons for this approach are as follows.

1. As is well known, the radial measures in DEA neglect the existence of slacks. If slacks are free disposal in economy, they should be duly appreciated. However, in many situations, this is not the case: large slacks indicate inefficiency. On the other hand, non-radial approaches, e.g. Tone (2001a, 2001b), account slacks, and efficiency scores drop from those of radial ones by accounting slacks.

2. One of the reasons why non-radial approaches have not had justifiable position is that they have no economic interpretation (Forsund (1998, page 31)). However, as we have pointed out in Section 2.2, their dual problems play the role of maximizing virtual profits. This corresponds to the maximizing operation of the ratio scale (virtual output *vs.* virtual input) in the radial cases, and justifies their economic interpretation.

3. As is seen from Theorem 4, the optimal values of the input-oriented SBM and Super SBM are monotone decreasing with respect to input increase. In the radial approaches, these important features cannot always be expected. (cf. Theorem 1.)

4. The radial measure uses an input distance function $d(\boldsymbol{x}, \boldsymbol{y})$ defined in such a way as:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \min\{\theta \mid \theta \boldsymbol{x} \in L(\boldsymbol{y})\},$$

   where $L(\boldsymbol{y})$ represents the set of all input $\boldsymbol{x}$ which can produce the output $\boldsymbol{y}$, i.e. $L(\boldsymbol{y}) = \{\boldsymbol{x} \mid \boldsymbol{x} \text{ can produce } \boldsymbol{y}\}$. Hence proportional changes in inputs play a central role. The non-radial approach, instead, deals with non-proportional change in inputs using a slack vector s^- (≥ 0), such that

$$x - s^- \in L(y).$$

It is evident that the latter includes the former, and it can be applied to analyze actual production correspondences with flexibility.

4.2 Inclusive or Exclusive?

There are typically two stances in evaluating the performance of DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. One is to evaluate it among the DMUs including it and the other excluding it. Our approach belongs to the latter. The reasons are:

1. When we compare a DMU in the period t $(t+1)$ with the set of DMUs in the period $t+1$ (t), we apply the latter approach, i.e. exclusive. If the DMU in time t lies in the relative interior of the production possibility set spanned by the set of DMUs in time $t+1$, the score of the DMU is less than one. Otherwise the score takes the value greater than or equal to one. Thus, no upper bound is set to the score *a priori.* Similarly, there is no reason why we cannot employ the same policy when evaluating a DMU with respect to the set of DMUs in the same period. If we apply the "inclusive" policy, the upper bound is one, and we cannot discriminate superiority between the efficient DMUs. The "exclusive" model enables this discrimination. The dual problem (11) gives support to this approach, too.

2. To be more specific, we explain this situation using the $M(\boldsymbol{x}_o, \boldsymbol{y}_o)$ in (17). The catch-up term C as expressed by (18) is the ratio of two measures $\delta((X,Y)^{t+1}, (\boldsymbol{x}_o, \boldsymbol{y}_o)^{t+1})$ *vs.* $\delta((X,Y)^t, (\boldsymbol{x}_o, \boldsymbol{y}_o)^t)$. If both $(\boldsymbol{x}_o, \boldsymbol{y}_o)^t$

and $(x_o, y_o)^{t+1}$ are efficient with respect to t and $t+1$ technology, respectively, then the traditional measure assigns one to their efficiency and hence C, their ratio, turns out to be 1.

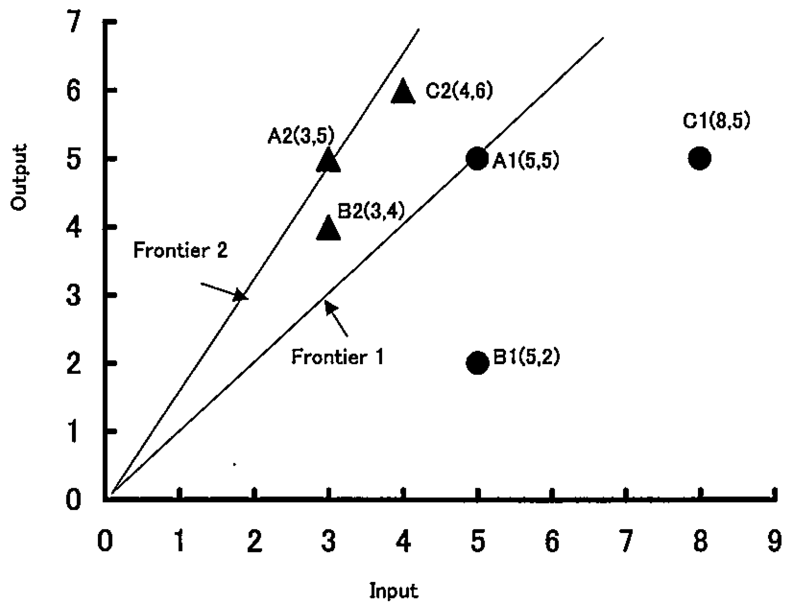

Figure 1: Three DMUs in Two Periods

However, consider the case depicted in Figure 1, where we have three DMUs, A, B and C with a single input and a single output in two periods 1 and 2. For example, A2 indicates DMU A in the period 2. From the figure we can see that, under the constant returns-to-scale assumption, DMU A has advantage against their competitors B and C in both periods. However, the degree of superiority decreased in the period 2 compared with that in the period 1. In short, DMU A is caught by its competitors. The traditional radial Malmquist index returns 1 to A as its catch-up term. Let us use the notation $C_{old}(A) = 1$. However, we should take account of the catch-up (or caught-up) effect by its competitors. This becomes possible by using the Super-SBM model. A1's superiority against B1 and C1 is evaluated as 1.6 (=1+3/5), while A2 has 1.1111 (=1+0.3333/3) against B2 and C2. Apparently, A reduces its superiority in the period 2. Hence, A's catch-up effect is given by 1.1111/1.6= 0.6944 which is less than 1. Thus, we have $C_{new}(A) = 0.6944$.

As to the frontier shift of DMU A, it is given by

$$\left[\frac{\delta((X,Y)^1, A1)}{\delta((X,Y)^2, A1)} \times \frac{\delta((X,Y)^1, A2)}{\delta((X,Y)^2, A2)}\right]^{1/2}.$$

The traditional method evaluates each element as follows:

$$\begin{array}{ll} \delta((X,Y)^1, A1) = 1 & \delta((X,Y)^1, A2) = 5/3 \\ \delta((X,Y)^2, A1) = 3/5 & \delta((X,Y)^2, A2) = 1 \end{array}$$

Hence we have $F_{old}(A) = 5/3 = 1.6667$. However, our Super-SBM evaluates them as follows:

$$\begin{array}{ll} \delta((X,Y)^1, A1) = 8/5 & \delta((X,Y)^1, A2) = 5/3 \\ \delta((X,Y)^2, A1) = 3/5 & \delta((X,Y)^2, A2) = 3.3333/3 \end{array}$$

Thus, it turns out $F_{new}(A) = 2$. The difference is caused by the "exclusive" evaluation policy employed, in that we always measure the distance from the "exclusive" frontiers. This results in the differences in the terms $\delta((X,Y)^1, A1)$ and $\delta((X,Y)^2, A2)$. In total we have the Malmquist index of A:

$$\begin{array}{lllllll} M_{old}(A) & = & C_{old} \times F_{old}(A) & = & 1 \times 1.6667 & = & 1.6667 \\ M_{new}(A) & = & C_{new} \times F_{new}(A) & = & 0.6944 \times 2 & = & 1.3889 \end{array}$$

As can be seen, A lost superiority over B and C, while its frontier still holds a high score due to its performance in period 1. The new scheme devalues A more than the old one does.

4.3 Continuos-smooth or Discrete?

There is another misunderstanding; that the slacks phenomenon may reflect basic mis-specification of the production function and should therefore at best be ignored. (Refer to discussions in Forsund (1998) for example.) This assertion stems from the continuous-smooth production frontier assumption in which case there occur no slacks in the radial measurements. However, as we pointed out in Section 4.1, slacks have their own right, even in the case of smooth and continuos frontiers. Moreover, most actual data sets are discrete and hence we are obliged to deal with slacks.

5 Extensions

In this section we extend our results to the output-oriented case and to other types of returns to scale, i.e., variable, increasing and decreasing.

5.1 Output-oriented Malmquist Index

The output-oriented SBM and super SBM models are represented by the following programs. In this case we assume $Y > O$ and $\boldsymbol{y}_o > 0$.

$$
\begin{array}{rrcl}
\text{[SBM-O]} & \rho_O^* & = & \min \;\; \rho_O = \dfrac{1}{1 + \frac{1}{s}\sum_{i=1}^{s} s_i^+/y_{io}} \\
\text{subject to} & \boldsymbol{x}_o & = & X\boldsymbol{\lambda} + \boldsymbol{s}^- \\
 & \boldsymbol{y}_o & = & Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\
 & \boldsymbol{\lambda} & \geq & 0, \; \boldsymbol{s}^- \geq 0, \; \boldsymbol{s}^+ \geq 0.
\end{array}
\tag{20}
$$

$$
\text{[SuperSBM-O]} \qquad \delta_O^* = \min \delta_O = \frac{1}{1 - \frac{1}{s}\sum_{i=1}^{s} s_i^+/y_{io}} \tag{21}
$$

$$\begin{aligned} \text{subject to} \quad & \boldsymbol{x}_o \geq X\boldsymbol{\lambda} \\ & \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} + \boldsymbol{s}^+ \\ & \boldsymbol{\lambda} \geq \boldsymbol{0},\ \boldsymbol{s}^+ \geq \boldsymbol{0}. \end{aligned}$$

The distance $\delta((X,Y)^{t_1}, (\boldsymbol{x}_o, \boldsymbol{y}_o)^{t_2})$ between $(\boldsymbol{x}_o, \boldsymbol{y}_o)^{t_2}$ and $\delta(X,Y)^{t_1}$ is determined in the same way as in the input-oriented case by replacing the roles of [SBM-I] and [SuperSBM-I] by [SBM-O] and [SuperSBM-O], respectively. Utilizing these distances we define the output-oriented Malmquist index by the formula (16).

Similarly to the input-oriented case, we have the following corollaries.

Corollary 1 (i) ρ_O^* *increases in any decrease in* $\boldsymbol{x}_o$ *as long as* [SBM-O] *is feasible, and* (ii) δ_O^* *increases in any decrease in* $\boldsymbol{x}_o$.

Corollary 2 (i) ρ_O^* *increases in any increase in* $\boldsymbol{y}_o$ *as long as* [SBM-O] *is feasible, and* (ii) δ_O^* *increases in any increase in* $\boldsymbol{y}_o$.

5.2 Extensions to Variable Returns to Scale

So far we have observed the Malmquist index under the constant returns-to-scale assumption. We can extend our results to other returns-to-scale environments as follows.

5.2.1 Variable Returns to Scale

We add the constraint below to the corresponding programs.

$$\sum_j \lambda_j = 1. \tag{22}$$

In this case, it may occur that both [SBM] and [SuperSBM] have no feasible solution in both input and output orientations. One solution for avoiding this difficulty might be to assign 1 to the distance, since we have no means to evaluate the DMU within the evaluator.

5.2.2 Increasing Returns to Scale

The additional constraint is

$$\sum_j \lambda_j \geq 1. \tag{23}$$

It is to be noted that, in this case, [SuperSBM-I] is always feasible, while [SuperSBM-O] may have no solution.

5.2.3 Decreasing Returns to Scale

The additional constraint is

$$\sum_j \lambda_j \leq 1. \tag{24}$$

In this case, [SuperSBM-O] is always feasible, while [SuperSBM-I] may have no solution.

6 Concluding Remarks

This paper has proposed a new Malmquist productivity index using the slacks-based and non-radial measure of efficiency. Although we focused our discussion mainly on the motivation of our study and the methodology employed, further theoretical analysis on the properties of this new index will be required. Also, we need empirical case studies using real world data. These are future research subjects.

Appendix A: Proof of Theorem 4

Proof of (i): Let us perturb $\boldsymbol{x}_o$ to $\boldsymbol{x}_o - \Delta\boldsymbol{x}$ ($\geq \mathbf{0}$) with $\Delta\boldsymbol{x} \geq \mathbf{0}$. The SBM-I for $(\boldsymbol{x}_o - \Delta\boldsymbol{x}, \boldsymbol{y}_o)$ is described as:

$$\begin{aligned}
[\text{SBM-I}(\Delta\boldsymbol{x})] \quad \rho_I^*(\Delta\boldsymbol{x}) &= \min \quad \rho_I(\Delta\boldsymbol{x}) = 1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / (x_{io} - \Delta x_i) \\
\text{subject to } \boldsymbol{x}_o - \Delta\boldsymbol{x} &= X\boldsymbol{\lambda} + \boldsymbol{s}^- \\
\boldsymbol{y}_o &= Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\
\boldsymbol{\lambda} &\geq 0, \ \boldsymbol{s}^- \geq 0, \ \boldsymbol{s}^+ \geq 0.
\end{aligned}$$

Let an optimal solution of [SBM-I($\Delta\boldsymbol{x}$)] be $(\boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$. Then $(\boldsymbol{\lambda}^*, \boldsymbol{s}^{-*} + \Delta\boldsymbol{x}, \boldsymbol{s}^{+*})$ is feasible for [SBM-I]. Hence we have

$$\rho_I^* \leq 1 - \frac{1}{m}\sum_{i=1}^{m}(s_i^{-*} + \Delta x_i)/x_{io} \leq 1 - \frac{1}{m}\sum_{i=1}^{m} s_i^{-*}/(x_{io} - \Delta x_i) = \rho_I^*(\Delta\boldsymbol{x}).$$

The last inequality holds, since we have

$$\frac{s_i^{-*} + \Delta x_i)}{x_{io}} - \frac{s_i^{-*}}{(x_{io} - \Delta x_i)} = \frac{\Delta x_i (X\boldsymbol{\lambda})_i}{x_{io}(x_{io} - \Delta x_i)} \geq 0,$$

and strict inequality holds if $\Delta x_i > 0$ for some i.

Proof of (ii): For the same perturbation as (i), the SuperSBM-I for $(\boldsymbol{x}_o - \Delta\boldsymbol{x})$ is described as:

$$\begin{aligned}
[\text{SuperSBM-I}(\Delta\boldsymbol{x})] \qquad & \delta_I^*(\Delta\boldsymbol{x}) = \min \delta_I(\Delta\boldsymbol{x}) = 1 + \frac{1}{m}\sum_{i=1}^{m} s_i^- / (x_{io} - \Delta x_i) \\
\text{subject to} \qquad & \boldsymbol{x}_o - \Delta\boldsymbol{x} \geq X\boldsymbol{\lambda} - \boldsymbol{s}^- \\
& \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} \\
& \boldsymbol{\lambda} \geq 0, \ \boldsymbol{s}^- \geq 0.
\end{aligned}$$

Let an optimal solution of [SuperSBM-I($\Delta\boldsymbol{x}$)] be $(\boldsymbol{\lambda}^*, \boldsymbol{s}^{-*})$. Then, $(\boldsymbol{\lambda}^*, \boldsymbol{s}^{-*})$ is also feasible for [SuperSBM-I] and hence it holds

$$\delta_I^* \leq 1 + \frac{1}{m}\sum_{i=1}^{m} s_i^{-*}/x_{io} \leq 1 + \frac{1}{m}\sum_{i=1}^{m} s_i^{-*}/(x_{io} - \Delta x_i) = \delta_I^*(\Delta\boldsymbol{x}).$$

If both $s_i^{-*} > 0$ and $\Delta x_i > 0$ hold for some i, we have

$$\delta_I^* < \delta_I^*(\Delta x).$$

This concludes the proof of Theorem 4. $\square$

References

[1] Andersen, P. and N.C. Petersen (1993) "A Procedure for Ranking Efficient Units in Data Envelopment Analysis," *Management Science*, 39, 1261-1264.

[2] Caves, D.W., L.R. Christensen and W.E. Diewert (1982) "The Economic Theory of Index Numbers and the Measurement of Input, Output, and Productivity," *Econometrica,* 50, 1393-1414.

[3] Charnes, A., W.W. Cooper and E. Rhodes (1978) "Measuring the Efficiency of Decision Making Units," *European Journal of Operations Research,* 2, 429-444.

[4] Cooper, W.W., L.M. Seiford and K. Tone (2000) *Data Envelopment Analysis - A Comprehensive text with Models, Applications, References and DEA-Solver Software*, Kluwer Academic Publishers.

[5] Färe and R.,S. Grosskopf (1992) "Malmquist Indexes and Fisher Ideal Indexes," *The Economic Journal*, 102, 158-160.

[6] Färe, R.,S. Grosskopf, B. Lindgren and P. Roos (1994) "Productivity Change in Swedish Hospitals: A Malmquist Output Index Approach," In A. Charnes, W.W. Cooper, A.Y. Lewin and L.M. Seiford (eds.) *Data Envelopemnt Analysis: Theory, Methodology and Applications.* Boston: Kluwer Academic Publishers.

[7] Forsund, F.R. (1998) "The Rise and Fall of Slacks: Comments on Quasi-Malmquist Productivity Indices," *Journal of Productivity Analysis*, 10, 21-34.

[8] Grifell-Tatje, E., C.A.K. Lovell and J.P. Pastor (1998) "A Quasi-Malmquist Productivity Index," *Journal of Productivity Analysis*, 10, 7-20.

[9] Malmquist, S. (1953) "Index Numbers and Indifference Surfaces," *Trabajos de Estadistica* 4, 209-242.

[10] Thrall, R.M. (2000) "Measures in DEA with an Application to the Malmquist Index," *Journal of Productivity Analysis*, 13, 125-137.

[11] Tone, K. (2001a) "A Slacks-based Measure of Efficiency in Data Envelopment Analysis," *European Journal of Operational Research*, 130, 498-509.

[12] Tone, K. (2001b) "A Slacks-based Measure of Super-efficiency in Data Envelopment Analysis," *European Journal of Operational Research*, forthcoming.