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Egoist's Dilemma : A DEA Game

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Ken Nakabayashi and Kaoru Tone

National Graduate Institute for Policy Studies
2-2 Wakamatsu-cho, Shinjuku-ku,
Tokyo 162-8677, Japan.

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Egoist's Dilemma: A DEA Game

Ken Nakabayashi* **and Kaoru Tone**†
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Abstract

This paper deals with problems of consensus-making among individuals or organizations with multiple criteria for evaluating their performance when the players are supposed to be egoistic, in the sense that each player sticks to his superiority regarding the criteria. We analyze this situation within the framework or concept developed in data envelopment analysis (DEA). This leads to a dilemma called the 'egoist's dilemma.' We examine this dilemma using cooperative game theory and propose a solution. The scheme developed in this paper can also be applied to attaining fair cost allocations as well as benefit-cost distributions.

Keywords
Game theory, cooperative game, DEA, variable weight, Shapley value, nucleolus, assurance region method, cost allocation

1. Introduction

Let us suppose n players each have m criteria for evaluating their competency or ability which is represented by a positive score for each criterion. Similar to usual classroom examinations, the higher the score for a criterion is, the better the player is judged to perform as regard to the criterion. For example, let the players be three students A, B and C, with three criteria, mathematics, literature and gymnastics. The scores are their records for the three subjects, measured by positive cardinal numbers. Now, we want to allocate a certain amount of fellowship grant to the three students in accordance with their scores in the three criteria. All players are supposed to be selfish or egoistic in the sense that they insist on their own advantage on the scores. However, they must reach a consensus in order to get the fellowship. Similar situations exist in many societal problems as discussed later. This paper proposes a new scheme for allocating or imputing the given benefit to the players under the framework of game theory and data envelopment analysis (DEA). This scheme can also be applied for attaining fair

* ken@stu.grips.ac.jp
† tone@grips.ac.jp

expense (cost) allocations, as well as benefit-cost distributions.

The rest of this paper unfolds as follows. Section 2 describes the basic model of the DEA game and its properties. Then in Section 3, we observe and propose several methods for imputation, including Shapley value and nucleolus. Also, we discuss the relationship of our DEA game with the linear production game by Owen (1975). Extensions of the basic model are discussed in Section 4. Section 5 presents several potential applications of this model. Finally, some concluding remarks follow in Section 6.

2. Basic models of the game

We introduce the basic models and structures of the game and uncover its mathematical properties.

2.1 Selfish behavior and egoist's dilemma

Let $X=(x_{ij})\in R_+^{m\times n}$ be the score matrix, consisting of the record x_{ij} of player j to the criterion i. It is assumed that the higher the score for a criterion is, the better the player is judged to perform as regard to the criterion. Each player k has a right to choose a set of nonnegative weights $w^k=(w_1^k,\cdots,w_m^k)$ to the criteria that are most preferable to the player. Using the weight w^k, we define the relative score of player k to the total score as follows:

$$\frac{\sum_{i=1}^{m} w_i^k x_{ik}}{\sum_{i=1}^{m} w_i^k (\sum_{j=1}^{n} x_{ij})}. \tag{1}$$

The denominator represents a total score of all players as measured by player k's weight selection, while the numerator indicates player k's self evaluation by the same weight selection. Hence, the expression (1) demonstrates player k's relative importance (share) under the weight (or value) selection w^k. We assume that the weighted scores are transferable. Player k wishes to maximize this ratio by selecting the most preferable weight, thus resulting in the following fractional program.

$$\max_{w^k} \frac{\sum_{i=1}^{m} w_i^k x_{ik}}{\sum_{i=1}^{m} w_i^k (\sum_{j=1}^{n} x_{ij})} \tag{2}$$

$$\text{subject to } w_i^k \geq 0 \; (\forall i).$$

The motivation behind this program is that player k aims to maximize his relative value as measured by the ratio: the weighted sum of his records vs. the weighted sum of all players' records. This arbitrary weight selection is the fundamental concept underlying data envelopment analysis (DEA) initiated by Charnes, Cooper and Rhodes (1978). DEA terms this as 'variable' weight that is contrasted to *a priori* 'fixed' one. Refer to Cooper *et al.* (1999) pp.12-13 for this issue.

Before going further, we reformulate the problem as follows, without losing generality. We normalize the data set X so that it is row-wise normalized, i.e.,

$$\sum_{j=1}^{n} x_{ij} = 1. \quad (\forall i)$$

For this purpose, we divide the row $(x_{i1}, \cdots, x_{in})$ by the row-sum $\sum_{j=1}^{n} x_{ij}$ for $i = 1, \cdots, m$. The program (2) suffers no effect by this operation. Thus, using the Charnes-Cooper transformation scheme, the fractional program (2) can be expressed by a linear program as follows:

$$c(k) = \max \sum_{i=1}^{m} w_i^k x_{ik} \tag{3}$$

$$\text{subject to} \sum_{i=1}^{m} w_i^k = 1$$

$$w_i^k \geq 0. (\forall i)$$

Now, the problem is to maximize the objective (3) on the simplex $\sum_{i=1}^{m} w_i^k = 1$. Apparently the optimal solution is given by assigning 1 to $w_{i(k)}^k$ for the criterion $i(k)$ such that $x_{i(k)} = \max\{x_{ik} | i = 1, \cdots, m\}$, and assigning 0 to the weight of the remaining criteria. We denote this optimal value by $c(k)$.

$$c(k) = x_{i(k)}. \quad (k = 1, \cdots, n) \tag{4}$$

The $c(k)$ indicates the highest relative score for player k which is obtained by the optimal weight selecting behavior. The optimal weight $w_{i(k)}^k$ may differ from one player to another.

[Proposition 1]

$$\sum_{k=1}^{n} c(k) \geq 1. \tag{5}$$

Proof: Let the optimal weight for player k be $w_k^* = (w_{1k}^*, \cdots, w_{mk}^*)$, i.e., $w_{i(k)k}^* = 1$ and $w_{ik}^* = 0 \; (\forall i \neq i(k))$. Then we have

$$\sum_{k=1}^{n} c(k) = \sum_{k=1}^{n} \sum_{i=1}^{m} w_{ik}^* x_{ik} = \sum_{k=1}^{n} x_{i(k)k} \geq \sum_{k=1}^{n} x_{1k} = 1.$$

The inequality above follows from $x_{i(k)k} \geq x_{1k}$ and the last equality follows from the row-wise normalization. □

This proposition asserts that, if each player sticks to his egoistic sense of value and insists on getting the portion of the benefit as designated by $c(k)$, the sum of shares usually exceeds 1 and hence $c(k)$ cannot fulfill the role of division (imputation) of the benefit. If eventually the sum of $c(k)$ turns out to be 1, all players will agree to accept the division $c(k)$, since this is obtained by the player's most preferable weight selection. The latter case will occur when all players have the same and common optimal weight selection. More concretely, we have the following proposition.

[Proposition 2]

The equality $\sum_{k=1}^{n} c(k) = 1$ *holds if and only if the score matrix satisfies the condition*

$$x_{ik} = x_{2k} = \cdots = x_{mk} \quad \mathit{for}\ k = 1, \cdots, n.$$

I.e., each player has the same score with respect to the m criteria.

Proof: 'If' part can be seen as follows. Since $c(k) = x_{1k}$ for all k, we have $\sum_{k=1}^{n} c(k) = \sum_{k=1}^{n} x_{1k} = 1$.

The 'Only if' part can be demonstrated as follows. Suppose $x_{11} > x_{21}$, then there must be a column $h \neq 1$ such that $x_{1h} < x_{2h}$, as otherwise the second row sum can not attain 1. Thus, we have

$$c(1) \geq x_{11}, c(h) \geq x_{2h} > x_{1h} \text{ and } c(j) \geq x_{1j}\ (\forall j \neq 1, h).$$

Hence, it holds that

$$\sum_{k=1}^{n} c(k) \geq \sum_{j=1, \neq h}^{n} x_{1j} + x_{2h} > \sum_{j=1}^{n} x_{1j} = 1.$$

This leads to a contradiction. Therefore, player 1 must have the same score in all criteria. The same relation must hold for other players. □

In the above case, only one criterion is needed for describing the game and the division (imputation) proportional to this score is the fair division. However, such situations might occur only in rare instances. In the majority of cases, we have $\sum_{k=1}^{n} c(k) > 1$. We may call this the 'egoist's dilemma' and, as we see later, many societal problems (conflicts) belong to this class.

One might think that a benefit allocation proportional to $\{c(k)\}$ is a solution. However, this allocation is by no means rational if we admit coalitions among players.

2.2 Assumption on the game and fair division

In order to attain a fair division (imputation), we assume the following agreements among the players, although each 'selfish' player sticks to his most preferable weight selection behavior as expressed by the program (3).

(A1) All players agree not to break off the game.

(A2) All players are willing to negotiate with each other to attain a reasonable and fair division $z = (z_1, \cdots, z_n)$ as represented by $z = wX$ with a certain common weight $w = (w_1, \cdots, w_m)$, if it exists.

2.3 Coalition with additive property

Let a coalition S be a subset of the player set $N = (1, \cdots, n)$. The record for the coalition S is defined by

$$x_i(S) = \sum_{j \in S} x_{ij}. \quad (i = 1, \cdots, m) \tag{6}$$

This coalition aims at obtaining the maximal outcome $c(S)$:

$$c(S) = \max \sum_{i=1}^{m} w_i x_i(S) \tag{7}$$

$$\text{subject to } \sum_{i=1}^{m} w_i = 1, \quad w_i \geq 0 \ (\forall i).$$

The $c(S)$, with $c(\phi) = 0$, defines a characteristic function of the coalition S. Thus, we have a game in coalition form with transferable utility as represented by (N, c).

[Proposition 3]

The characteristic function c is sub-additive, i.e., for any $S \subset N$ and $T \subset N$ with $S \cap T = \phi$, we have

$$c(S \cup T) \leq c(S) + c(T). \tag{8}$$

Proof: By renumbering the indexes, we can assume that

$$S = \{1, \cdots, h\}, T = \{h+1, \cdots, k\} \text{ and } S \cup T = \{1, \cdots, k\}.$$

For these sets, it holds that

$$c(S \cup T) = \max_i \sum_{j=1}^{k} x_{ij} \leq \max_i \sum_{j=1}^{h} x_{ij} + \max_i \sum_{j=h+1}^{k} x_{ij} = c(S) + c(T). \qquad \square$$

We also have the following proposition.

[Proposition 4]

$$c(N)=1.$$

2.4 Another expression of the game

Let us define another game (N, v) by

$$v(S) = \sum_{j \in S} c(j) - c(S). \tag{9}$$

(We use the notation $c(j)$ instead of $c(\{j\})$.)

[Proposition 5]

(N, v) is supper additive, i.e.,

$$v(S) + v(T) \le v(S \cup T) \quad \forall S, T \subset N \text{ and } S \cap T = \phi$$

Proof: $v(S)+v(T) = \left\{\sum_{j \in S} c(j) - c(S)\right\} + \left\{\sum_{j \in T} c(j) - c(T)\right\} = \sum_{j \in S \cup T} c(j) - \{c(S) + c(T)\}$

$$\le \sum_{j \in S \cup T} c(j) - c(S \cup T) = v(S \cup T). \qquad \square$$

We have:

$$v(j) = 0\ (\forall j) \text{ and } v(N) = \sum_{j=1}^{n} c(j) - c(N) = \sum_{j=1}^{n} c(j) - 1 > 0.$$

Hence the game (N, v) is 0-normalized. Let an imputation of the game (N, v) be $y = (y_1, \cdots, y_n)$, which satisfies $y_j \ge v(j) = 0\ (\forall j)$ and $\sum_{j=1}^{n} y_j = v(N)$. Using $y = (y_1, \cdots, y_n)$, we can define a benefit allocation $z = (z_1, \cdots, z_n)$ by $z_j = c(j) - y_j\ (j = 1, \cdots, n)$. This allocation satisfies $\sum_{j=1}^{n} z_j = \sum_{j=1}^{n} c(j) - \sum_{j=1}^{n} y_j = 1$. Hence, this game has essentially the same structure as the game (N, c).

The game (N, v) starts from $v(j) = 0\ (\forall j)$ and enlarges the gains by coalitions until the grand coalition N with $v(N) = \sum_{j=1}^{n} c(j) - 1$.

2.5 A DEA minimum game

We observe here the opposite side of the egoist's game (N, c) that is defined by replacing *max* in (3) by *min* as follows:

$$\begin{aligned} & d(k) = \min \sum_{i=1}^{m} w_i^k x_{ik} \\ & \text{subject to } \sum_{i=1}^{m} w_i^k = 1 \\ & w_i^k \ge 0. (\forall i) \end{aligned} \tag{10}$$

The optimal value $d(k)$ assures the minimum division that player k can expect from the game. In this case, as a counterpart of Proposition 1, we have:

[Proposition 6]

$$\sum_{k=1}^{n} d(k) \le 1.$$

Analogously to the *max* game case, for a coalition $S \subset N$, we define

$$d(S) = \min \sum_{i=1}^{m} w_i x_i(S)$$
$$\text{subject to} \sum_{i=1}^{m} w_i = 1, \quad w_i \ge 0\ (\forall i). \tag{11}$$

Apparently, it holds that $d(N)=1$.

The DEA *min* game (N, d) is super-additive, i.e., we have

$$d(S \cup T) \ge d(S) + d(T). \qquad \forall S, T \subset N \text{ with } S \cap T = \phi \tag{12}$$

Thus, this game starts from $d(k) > 0\ (k = 1, \cdots, n)$ and enlarges the gains by coalitions until the grand coalition N with $d(N)=1$.

Between the games (N, c) and (N, d) we have the following proposition:

[Proposition 7]

$$d(S) + c(N - S) = 1. \quad \forall S \subsetneq N \tag{13}$$

Proof: By renumbering the indexes, we can assume that

$$S = \{1, \cdots, h\}, N = \{1, \cdots, n\} \text{and} N - S = \{h+1, \cdots, n\}$$

For these sets, it holds that

$$d(S) + c(N - S) = \min_i \sum_{j=1}^{h} x_{ij} + \max_i \sum_{j=h+1}^{n} x_{ij} = \min_i \left(\sum_{j=1}^{n} x_{ij} - \sum_{j=h+1}^{n} x_{ij} \right) + \max_i \sum_{j=h+1}^{n} x_{ij}$$

$$= \min_i \left(1 - \sum_{j=h+1}^{n} x_{ij} \right) + \max_i \sum_{j=h+1}^{n} x_{ij} = 1 - \max_i \sum_{j=h+1}^{n} x_{ij} + \max_i \sum_{j=h+1}^{n} x_{ij} = 1 \quad \square$$

2.6 Convex or concave game

Proposition 3 suggests that the DEA *max* game (N, c) might be a concave game, i.e., for any coalitions S and T, it holds that

$$c(S \cup T) + c(S \cap T) \le c(S) + c(T). \tag{14}$$

However, unfortunately this conjecture is not true as demonstrated by the counterexample below.

[Example 1]

Table 1 exhibits a DEA game with 4 players and 3 criteria. The scores are row-wise normalized so that the sum of row elements is equal to 1 for each row.

Table 1: Example 1

	Player A	Player B	Player C	Player D	Row-sum
Criterion 1	0.5	0.25	0.2	0.05	1
Criterion 2	0.375	0.375	0.125	0.125	1
Criterion 3	0.5	0.25	0.125	0.125	1

Let $S=\{A,B\}, T=\{B,C\}, S\cup T=\{A,B,C\}$ and $S\cap T=\{B\}$. Then we have:

$$c(S)=0.75, c(T)=0.5, c(S\cup T)=0.95, c(S\cap T)=0.375.$$

Hence it holds that $c(S)+c(T)=1.25<1.325=c(S\cup T)+c(S\cap T)$, showing the non-concavity of this game.

However, this concept, concavity, is case-sensitive and so we should check it case by case. In the case of $S\cup T=N$, we have the following proposition.

[Proposition 8]

$$1+c(S\cap T)\leq c(S)+c(T).\quad \forall S,T\subset N \; with \; S\cup T=N$$

Proof: From the super-additivity of $d(\cdot)$, we have the following inequality.

$$d(\{S-S\cap T\}+\{T-S\cap T\})\geq d(S-S\cap T)+d(T-S\cap T)$$

From Proposition 7, it holds that $d(S-S\cap T)=1-c(T)$, $d(T-S\cap T)=1-c(S)$ and $d(\{S-S\cap T\}+\{T-S\cap T\})=1-c(S\cap T)$.

Hence we have,

$$1-c(S\cap T)\geq\{1-c(T)\}+\{1-c(S)\}$$
$$1+c(S\cap T)\leq c(S)+c(T)$$

□

Similarly we have, for the game (N, d),

[Corollary 1]

$$1+d(S\cap T)\geq d(S)+d(T).\quad \forall S,T\subset N \; with \; S\cup T=N$$

From Proposition 8, we have following proposition in the case of 3 players.

[Proposition 9]

The DEA game (N, c) with 3 players is concave.

Proof: Let the three players be i, j and k. Then we have, from Proposition 8, the following inequality.

$$c(j,k)+c(i,k)\geq 1+c(k)=c(i,j,k)+c(k). \qquad \square$$

(We use the notation $c(j,k)$ instead of $c(\{j\},\{k\})$.)

Similarly we have the following proposition.

[Proposition 10]

The DEA game (N, d) with *3 players is convex.*

Furthermore, the transformed game (*N*, *v*) in the 3-player case has the convex structure.

[Proposition 11]

The DEA game (N, v) with 3 players is convex.

Proof: Let the three players be i, j and k. Then we have,

$$\begin{aligned} v(i,k)+v(j,k) &= \{c(i)+c(k)-c(i,k)\}+\{c(j)+c(k)-c(j,k)\} \\ &= c(i)+c(j)+2c(k)-\{c(i,k)+c(j,k)\} \\ &\leq c(i)+c(j)+2c(k)-\{c(i,j,k)+c(k)\} \\ &= c(i)+c(j)+c(k)-c(i,j,k)=v(i,j,k). \end{aligned}$$

(We notice that the game (*N*, *v*) is 0-normalized.) $\square$

3. Imputations

In this section, we observe the core, Shapley value and nucleolus as the representative imputations of the cooperative game and discuss their mathematical properties associated with the DEA game. Remarkably, the Shapley value of the DEA *max* game is the same as that of the DEA *min* game. Finally we discuss the common weight problem.

3.1 Conditions for imputation

An imputation of the DEA *min* game (*N*, *d*) is a vector $z=(z_1,\cdots,z_n)$ that satisfies the following individual and grand rationalities.

Individual rationality: $z_j \geq d(j), \quad j=1,\cdots,n$

Grand rationality: $\sum_{j=1}^{n} z_j = d(N)=1.$

Let an imputation of the transformed game (N, v) be $y = (y_1, \cdots, y_n)$, which satisfies

$$y_j \geq v(j) = 0 \, (\forall j) \text{ and } \sum_{j=1}^{n} y_j = v(N).$$

A benefit allocation $z = (z_1, \cdots, z_n) = (c(1) - y_1, \cdots, c(n) - y_n)$ satisfies $z_j \leq c(j) \, (\forall j)$ and $\sum_{j=1}^{n} z_j = 1$. This is the induced imputation of the *max* game (N, c).

3.2 The core

The core of the DEA *min* game (N, d) is the set of imputations that satisfies the following collective rationality in addition to the individual and grand rationalities.

$$\text{Collective rationality: } \sum_{j \in S} x_{ij} \geq d(S), \quad \forall S \subset N. \tag{17}$$

Since the core of a convex game is not void, the core of DEA games (N, d) and (N, v) with 3 players is not void by Propositions 10 and 11. More generally, Owen (1975) has introduced linear program games associated with an economic production process and demonstrated that they have a non-empty core. Comparing our DEA game with his case, we found that the DEA game can be interpreted as the dual of his linear program. The Owen's LP game has a coalition on the right hand side vector of the constraints, whereas the coalition of the DEA game appears on the objective function vector. Since the optimal primal value that defines the characteristic function of a coalition is equal to the optimal dual value by the duality theorem, his Theorem 1 (the LP game is balanced) is valid in our case, too. Thus, the DEA games are balanced. A game has a core if and only if it is balanced (Shapley, 1967). Therefore we have the following two propositions.

[Proposition 12]

The DEA min game (N, d) is a balanced game. Hence its core is nonempty.

Proof: Although this proposition is a direct consequence of Owen (1975), we give another proof in a manner specific to the DEA game. Let an arbitrary 2^n-2 dimensional nonnegative vector be $\gamma = \{\gamma_S : S \subset N\}$ which satisfies $\sum_{S: j \in S \subset N} \gamma_S = 1 \, (\forall j \in N)$. If we can demonstrate that the following inequality holds for this vector, then the game (N, d) is a balanced game (Shapley, 1967).

$$\sum_{S \subset N} \gamma_S d(S) \leq d(N) = 1.$$

This inequality can be obtained as follows:

$$\sum_{S\subseteq N}\gamma_S d(S)=\sum_{S\subseteq N}\gamma_S\cdot\min_i\sum_{j\in S}x_{ij}$$

$$\le\min_i(\sum_{S\subseteq N}\gamma_S\sum_{j\in S}x_{ij})=\min_i\left(\sum_{j\in N}x_{ij}\sum_{S:j\in S\subseteq N}\gamma_S\right)=\min_i\sum_{j\in N}x_{ij}=1. \qquad \square$$

Similarly, we can demonstrate that the DEA game (N, v) is a balanced.

[Proposition 13]

The DEA game (N, v) is a balanced game. Hence its core is nonempty.

Furthermore, Owen (1975) developed a method for finding points in the core. Amazingly, following his method, we found that any row of the normalized score matrix X or any convex combination of rows of X is an imputation in the core. In other words, we have the following proposition.

[Proposition 14]

For any $w=(w_1,\cdots,w_m)\in R_+^m$ *in the simplex* $w_1+\cdots+w_m=1$, *the vector wX is an imputation in the core of the DEA game.*

(See Appendix A for a proof.)

However, the reverse is not always true, i.e., there are imputations in the core that cannot be expressed as *wX*. We will demonstrate this later in Example 5.

3.3 The Shapley value

The Shapley value $\phi_i(d)$ of player i for the DEA *min* game (N, d) is defined by

$$\phi_i(d)=\sum_{S:i\in S\subseteq N}\frac{(s-1)!(n-s)!}{n!}\{d(S)-d(S-\{i\})\}, \tag{18}$$

where s is the number of members of a coalition S. This value is the mathematical expectation of the marginal contribution of player i when all orders of formation of the grand coalition are equi-probable. Regarding the DEA-games, we have the following remarkable property.

[Proposition 15]

The Shapley values of the DEA max and min games (N, c) and (N, d) are the same.

Proof: For all $i\in N$, we have

$$
\begin{aligned}
\phi_i(c) &= \sum_{S:i\in S\subset N} \frac{(s-1)!(n-s)!}{n!}\{c(S)-c(S-\{i\})\} \\
&= \sum_{S:i\in S\subset N} \frac{(s-1)!(n-s)!}{n!}[\{1-d(N-S)\}-\{1-d(N-S+\{i\})\}] \\
&= \sum_{S:i\in S\subset N} \frac{(s-1)!(n-s)!}{n!}\{d(N-S+\{i\})-d(N-S)\}.
\end{aligned}
$$

By replacing $S' = N - S + \{i\}$ and defining s' as the number of members in the coalition S', the last term turns out to:

$$
\begin{aligned}
&= \sum_{S':i\in S'\subset N} \frac{(s'-1)!(n-s')!}{n!}\{d(S')-d(S'-\{i\})\} \\
&= \phi_i(d).
\end{aligned}
$$

□

[Example 2]

The Shapley values of the *max* and *min* games for the data set given in Table 1 are the same. Actually, the imputation is (0.44375, 0.30625, 0.15625, 0.09375).

It is proved that the Shapley value of a convex game belongs to the core of the game. Hence, by Proposition 12, the Shapley value of the DEA *min* game (*N*, *d*) for the 3-player case belongs to the core. Although the DEA *min* game with 4 or more players is not necessarily convex, we can demonstrate that the Shapley value of the 4-player case is included in the core. See Appendix B for a proof.

3.4 The nucleolus

Let an imputation of the DEA game (*N*, *d*) be $z = (z_1, \cdots, z_n)$. Then we define the excess of each coalition *S* as follows:

$$
e(S,z) = d(S) - \sum_{j\in S} z_j . \tag{19}
$$

In the DEA game, the excess measures the "degree of unhappiness" of coalition *S* when *z* expresses benefit, but measures the "degree of happiness" when *z* expresses cost. The nucleolus is the imputation that minimizes (lexicographically) the maximum excess.

Let $\theta(z)$ be the vector (with 2^{n-1} components) of the excesses of all coalitions $S \subset N(S \neq \phi, N)$, ordered by increasing magnitude. I.e.,

$$
\theta(z) = (e(S^1,z),\cdots,e(S^{2^{n-1}},z)), \quad e(S^1,z) \geq \cdots \geq e(S^{2^{n-1}},z).
$$

We introduce a lexicographic ordering of the vectors $\theta(z)$, i.e., $\theta(z) >_L \theta(y)$ if

$\exists k \in \{1, \cdots, 2^{n-1}\}$, such that $e(S^i, z) = e(S^i, y)$ $(i = 1, \cdots, k-1)$ and $e(S^k, z) > e(S^k, y)$. Let the entire imputation set of the DEA game (*N*, *d*) be *Z*, then the nucleolus of (*N*, *d*) is defined by

$$\mu(Z) = \{z \in Z \mid \theta(z) \leq_L \theta(y), \quad \forall y \in Z\}.$$

If the core is non-empty, then the nucleolus is included in the core.

The nucleolus of the DEA *min* game is not necessarily the same as that of the *max* game, as contrasted to the Shapley value case. This is demonstrated by the next example.

[Example 3]

The nucleolus of the data set in Table 1 is as follows. For the *min* game (*N*, *d*), it is (0.46, 0.29, 0.16, 0.09), whereas for the *max* game (*N*, *c*) it is (0.45, 0.3, 0.15, 0.1).

3.5 On the dominance relationship

We call player A *dominates* player B if it holds $x_{iA} \geq x_{iB}$ $(\forall i)$. We have the following dominance relationship on imputation.

[Proposition 16]

If player A dominates player B, then the Shapley value of A is not less than that of player B.

(See Appendix C for a proof.)

[Proposition 17]

If player A dominates player B, then the nucleolus of A is not less than that of player B.

(See Appendix D for a proof.)

3.6 On the common weight

We began this paper by introducing the most preferable weight selection behavior of players. Now we return to this subject in the knowledge of the imputation $z = (z_1, \cdots, z_n)$ induced by coalitions and allocations, e.g., the Shapley value and nucleolus. The weight $w = (w_1, \cdots, w_m) \in R^m$ associates with the imputation $z = (z_1, \cdots, z_n) \in R^n$ via $wX \in R^n$. In an effort to determine *w* in the way that *wX* approximates *z* as close as possible, we formulate the following LP with variables $w \in R^m$, $s^+ \in R^n$, $s^- \in R^n$, $p \in R$.

$$\begin{aligned}
&\min \ p \\
&\text{subject to} \quad wx_j + s_j^+ - s_j^- = z_j \quad (j = 1, \cdots, n) \\
&w_1 + \cdots + w_m = 1 \\
&s_j^+ \leq p, \quad s_j^- \leq p \quad (j = 1, \cdots, n) \\
&w_i \geq 0 \ (i = 1, \cdots, m), \quad s_j^+ \geq 0, s_j^- \geq 0 \ (j = 1, \cdots, n),
\end{aligned} \tag{20}$$

where x_j denotes the j-th column vector of X.

Let an optimal solution of this program be $(p^*, w^*, s^{+*}, s^{-*})$. Then we have two cases.

[Case 1] $p^* = 0$

In this case, it holds that $z = w^* X$ and so the imputation z is explained by the common weight w^*. All players will accept this solution since it represents the common value judgment corresponding to the cooperative game solution. We note that there still remains the uniqueness issue of w^*.

[Case2] $p^* > 0$

In this case, we have no common weight w^* which can express z as $z = w^* X$ perfectly.

[Example 4]

As Example 2 shows, the Shapley value of the DEA game for the data set in Example 1 is

$$(0.44375, 0.30625, 0.15625, 0.09375).$$

For this imputation, the optimal solution of the LP (20) satisfies $p^* = 0$ and hence this game has a common weight $w_1^* = 0.41667, w_2^* = 0.45, w_3^* = 0.13333$ that explains the game solution completely.

[Example 5]

Table 2 exhibits a DEA game with 5 players and 3 criteria.

Table 2: A 5 players and 3 criteria case

	Player A	Player B	Player C	Player D	Player E	Row-sum
Criterion 1	0.4	0.4	0.1	0.05	0.05	1
Criterion 2	0.4	0.3	0.125	0.05	0.125	1
Criterion 3	0.001	0.001	0.01	0.5	0.488	1

The imputation by the Shapley value of this game is displayed in Table 3.

Table 3: The Shapley value

	Player A	Player B	Player C	Player D	Player E	Total
Shapley	0.2064667	0.190967	0.064217	0.26755	0.2708	1

The optimal objective value of LP (20) is $p^* = 0.00264328$ with the optimal weight

$$w_1^* = 0.336314, \quad w_2^* = 0.185265, \quad w_3^* = 0.478421.$$

Hence this problem has no common weight for expressing the Shapley value. The optimal weight can approximate the Shapley value within the tolerance $p^* = 0.00264328$. Also, this example demonstrates a counterexample to the reverse of Proposition 14. We examined and confirmed that the Shapley value above belongs to the core. However, it cannot be expressed in the form wX, as evidenced by the above positive $p^* = 0.00264328$.

4. Extensions

In this section, we extend our basic model to benefit-cost game and discuss the zero weight issues.

4.1 A benefit-cost game

So far we have dealt with the DEA game in which the score matrix X represents the superiority (benefits) of players. However, there are occasions where some criteria exhibit the inferiority (costs) of players. Thus, the merit of a player is evaluated by the difference (profit) between benefits and costs.

Suppose that there are s criteria for representing benefits and m criteria for costs. Let y_{ij} $(i=1,\cdots,s)$ and x_{ij} $(i=1,\cdots,m)$ be the benefits and costs of player j $(j=1,\cdots,n)$, respectively. The merit of player j is evaluated by

$$(u_1 y_{1j} + \cdots + u_s y_{sj}) - (v_1 x_{1j} + \cdots + v_m x_{mj}),$$

where $u=(u_1,\cdots,u_s)$ and $v=(v_1,\cdots,v_m)$ are respectively the virtual weights to benefits and costs. Analogous to the expression (1), we define the relative score of player j to the total scores as follows:

$$\frac{\sum_{i=1}^{s} u_i y_{ij} - \sum_{i=1}^{m} v_i x_{ij}}{\sum_{i=1}^{s} u_i (\sum_{k=1}^{n} y_{ik}) - \sum_{i=1}^{m} v_i (\sum_{k=1}^{n} x_{ik})} \tag{21}$$

Player j wishes to maximize his score subject to the condition that the merit of all players is nonnegative, i.e.

$$\sum_{i=1}^{s} u_i y_{ik} - \sum_{i=1}^{m} v_i x_{ik} \geq 0. \quad (k=1,\cdots,n) \tag{22}$$

We can express this situation by the linear program below:

$$\begin{aligned} &\max_{v,u} \sum_{i=1}^{s} u_i y_{ij} - \sum_{i=1}^{m} v_i x_{ij} \\ &\text{subject to} \\ &\sum_{i=1}^{s} u_i (\sum_{k=1}^{n} y_{ik}) - \sum_{i=1}^{m} v_i (\sum_{k=1}^{n} x_{ik}) = 1 \\ &\sum_{i=1}^{s} u_i y_{ik} - \sum_{i=1}^{m} v_i x_{ik} \geq 0 \quad (k=1,\cdots,n) \\ &\qquad v_i \geq 0\ (\forall i), \quad u_i \geq 0\ (\forall i). \end{aligned} \tag{23}$$

Following the same scenario as the DEA game in the preceding sections, we can develop coalitions and imputations of this benefit-cost game, although the row-wise normalization is not available in this game.

4.2 Avoiding occurrence of zero weight

In Section 3.6, we presented a scheme for determining the weight w through the program (20). Eventually, it may occur that a certain weight happens to be zero for all optimal solutions. This means that the corresponding criterion is not accounted for in the solution of the game at all, even though the criterion might be taken as an important factor at the beginning. If all players agree to incorporate all criteria positively into account, i.e., to avoid zero weights, we can apply the following "assurance region method," originally developed in the DEA literature, e.g., Thompson et al. (1986) and Dyson and Thanassoulis (1988). Let the reference criterion be w_1, for example. We set constraints on the ratio w_1 vs. $w_i (i = 2,\cdots,m)$ as follows:

$$L_i \leq \frac{w_i}{w_1} \leq U_i, \quad (i = 2,\cdots,m)$$

where L_i and U_i denote the lower and upper bounds of the ratio w_i / w_1, respectively. These bounds must be set by agreement among all players. Thus, using appropriate bounds, we can avoid the occurrence of zero weights. The program (7) is now modified as:

$$\begin{aligned} & c(S) = \max \sum_{i=1}^{m} w_i^k x_i(S) \\ & \text{subject to } \sum_{i=1}^{m} w_i^k = 1 \\ & L_i \leq \frac{w_i^k}{w_1^k} \leq U_i \quad (i = 2,\cdots,m) \\ & w_i^k \geq 0. (\forall i) \end{aligned} \tag{24}$$

Refer to Allen et al. (1997) for other settings of weight restrictions.

[Example 6]

A data set with 3 players and 3 criteria is displayed in Table 4 along with its Shapley value in Table 5.

Table 4: 3 player and 3 criteria data

	Player A	Player B	Player C	Row-sum
Criterion 1	0.2	0.4	0.4	1
Criterion 2	0.5	0.2	0.3	1
Criterion 3	0.6	0.2	0.2	1

Table 5: The Shapley value of Example 6

	Player A	Player B	Player C	Sum
Shapley	0.4	0.3	0.3	1

By solving LP (20), we found $p^* = 0$ with the optimal weight ($w_1^* = 0.5, w_2^* = 0, w_3^* = 0.5$). Moreover, the optimal common weight is uniquely determined. Hence, in evaluating the Shapley value, Criterion 2 ($w_2^* = 0$) has no role at all. This is also reflected in the same Shapley score (0.3) of Players B and C, even though Player C has a higher score (0.3) in Criterion 2 than Player B (0.2). So, Criterion 2 is neglected in this imputation. In order to avoid such inconvenience, we set constraints on weights, for example, as follows.

$$0.5 \le \frac{w_2}{w_1} \le 2, \quad 0.5 \le \frac{w_3}{w_1} \le 2. \tag{25}$$

We solved the corresponding program (24) after converting the fractional terms into linear inequalities and found the Shapley value as displayed in Table 6. Now, Player C is ranked higher than B in the recognition of Criterion 2.

Table 6: The Shapley value after weight constraints

	Player A	Player B	Player C	Sum
Shapley	0.428929	0.271429	0.299642	1

The common weight for this Shapley value is obtained by solving LP (20) and we have ($p^* = 0, w_1^* = 0.357143, w_2^* = 0.282143, w_3^* = 0.360714$) which satisfies all constraints in (25).

5. Applications

We present here some of the potential applications of our DEA game. In the literature of cooperative game theory, there have been many applications to cost or benefit sharing problems. The proposed DEA game models demonstrate sharp contrast to them in that we can deal with these problems under multi-criteria environments that are common to conflicts in our society.

5.1 DEA max game

This game can be applied for the purpose of allocating benefits to players. Typical examples include research grant allocation to applicants by a foundation. The multiple criteria are, among others,

1. Novelty of subject.
2. Feasibility of research.
3. Influence on advance of science.

4. Past records of applicants.

Also, many resource distribution problems for R&D belong to this class. In this case, multiple criteria are, among others,

1. Short term profitability.
2. Contributions to the future of the company.
3. Spillover effects on the existing technologies of the company.

5.2 DEA min game

Typical potential applications of this model include cost allocation or burden sharing problems. Each player of this game wishes to minimize his share, although the participants of the game must pay a certain amount of cost in total. The U.N., NATO and many other international organizations have this kind of problem. According to a paper by Kim and Hendry (1998), in the NATO burden sharing case, they have pointed out the following items as the criteria for benefits.

1. Protection from external threat: The degree of reliance on USA (NATO) protection against an external threat.
2. Political benefits: The relative size of benefits accrued to NATO members by utilizing NATO as a policy tool in pursuing their foreign policy goals.
3. Receipt of economic and military aid: Amount of USA economic and military aid received.
4. Receipt of economic spin-offs (foreign exchange income): The number of USA troops stationed in a member nation.
5. Receipt of economic spin-offs (employment in defense industry): The number of workers employed in world's top 100 defense contractors.

Kim and Hendry (1998) analyzed this problem within the traditional DEA framework by incorporating other cost factors as outputs and benefit factors as inputs. Comparisons of both approaches will present an interesting research task.

5.3 DEA benefit-cost game

This model is applicable for the cases in which every player has both merits and demerits. As an example, we point to the trilateral relationship in security between Japan, South Korea and USA. The three countries have been playing crucial role in security in the Far-East Asia since the end of the Second World War (1945). In fact, Japan and USA have been in alliance as represented by the Japan-US Security Treaty (1951-present). A similar alliance exists between South Korea and USA, i.e., the US-ROK Mutual Defense Treaty (1954-present). Though Japan and Korea do not make a security treaty, a potential military reciprocity through USA exists, e.g., Cha (1999) called this relationship 'quasi-alliance.' Three countries are tightening the cooperative relationships in the security matter in the presence of a terrorist country (North Korea). However, the trilateral cooperation demands measurements of weights that signify the importance of each country. Although Japan and South Korea have been protected under the nuclear umbrella of USA, the both countries have a certain geopolitical advantage in the Far-East Asia. Therefore, this importance should be discussed in multi-criteria environments. This subject comes to the fore implicitly or explicitly whenever the bilateral or the trilateral cooperation demands burden sharing for the security maintenance. The burden includes governmental spending for military preparedness, basing, logistic support and others. In measuring the overall importance of each country in the bilateral or the trilateral relationship, it is absolutely essential to take into account multi-criteria regarding the relevant benefits and costs. The benefits include, among others,

1. Protection from external threat (North Korea).
2. Economic benefits achieved through the regional stability.
3. Political benefits.

The costs include, among others,

1. Defense efforts.
2. Supports for the US military presence.
3. Military operational role (risk).
4. Other duties and constraints associated with the alliance.

We can apply our DEA benefit-cost game for this purpose, which is one of our intriguing future research subjects.

As another example, we cite comparisons of cities by quality of life. The criteria for merits are represented by such factors as:

1. Living space per householder.
2. Educational expenses per student.
3. Number of hospitals per population.
4. Area of park per population.
5. Number of libraries per population.
6. Income per head.

The criteria for demerits include:

1. Pollution (emissions of CO_2 and noise).
2. Congestion (commuting time).
3. Living expenses (level of prices).
4. Crime (murder case per population).

Although many authors have analyzed this subject, e.g., the Analytic Hierarchy Process approach by Saaty (1986) and the DEA approach by Zhu (2001), the DEA game approach will add a new dimension to this field.

6. Concluding remarks

In this paper, we have introduced a societal dilemma called the 'egoist's dilemma' and studied its properties by means of data envelopment analysis (DEA) and cooperative game theory. The DEA game thus defined has two variations, the one the original selfish or bullish *max* game and the other the modest or bearish *min* game. As a special case of linear production games of Owen (1975), we also found that the DEA game has always a core. We have discussed imputations based on the cooperative game theory, e.g., the Shapley value and the nucleolus. Specifically, we found that the Shapley value of the *max* game coincides with that of the *min* game. This might be one of the remarkable characteristics of the DEA game. In Japan, a proverb says "Modesty (sympathy) is not merely for others' sake," reflecting a wisdom of living. In this sense, the Shapley solution has strong impact on consensus-making among participants of the game. Furthermore, we have studied the common weight issues that connect the game solution with the arbitrary weight selection behavior of the players. Regarding this subject, we have proposed a method for incorporating weight constraints to the game.

Future research subjects include:

1. Studies on the relationship between the core and the Shapley value of the DEA game with 5 or more players.
2. The role of other imputations, e.g., the disruptive nucleolus (Littlechild and Vaidya, 1976), the proportional nucleolus (Young et al., 1981), the bargaining set (Aumann and Maschler, 1964) and the kernel (Davis and Maschler, 1965).
3. Types of coalition, e.g., partial coalition.
4. Introduction of the concept of economies of scale to the game, especially to the benefit-cost game.

We hope this study contributes to opening a new field of research in game theory and data envelopment analysis, and provokes novel applications for resolution of social and political conflicts.

Appendix A (Proof of Proposition 14)

We prove this proposition for the DEA *min* game case. Since the data matrix X is row-wise normalized, the grand coalition N has the program (11) as

$$\begin{aligned} & d(N)=\min \sum_{i=1}^{m} w_i x_i(N)=\sum_{i=1}^{m} w_i \\ & \text{subject to } \sum_{i=1}^{m} w_i=1, \quad w_i \geq 0\ (\forall i). \end{aligned} \tag{A1}$$

Hence, $d(N)$=1 for any $w=(w_1,\cdots,w_m)\geq 0$ with $\sum_{i=1}^{m} w_i=1$, i.e., any point in the simplex is optimal. For such a w, we define the vector $z=(z_1,\cdots,z_n)$ by

$$z_j=(wX)_j=\sum_{i=1}^{m} w_i x_{ij}. \quad (j=1,\cdots,n) \tag{A2}$$

For any coalition S, we have

$$\sum_{j\in S} z_j=\sum_{j\in S}\sum_{i=1}^{m} w_i x_{ij}=\sum_{i=1}^{m} w_i \sum_{j\in S} x_{ij}=\sum_{i=1}^{m} w_i x_i(S). \tag{A3}$$

Hence, it holds that

$$\sum_{j\in N} z_j=d(N)=1. \tag{A4}$$

Since $d(S)$ is the minimum of the objective function in (11) and w is a feasible solution for (11), we have

$$d(S)\leq \sum_{i=1}^{m} w_i x_i(S) \tag{A5}$$

and, by (A3),

$$\sum_{j\in S} z_j \geq d(S). \tag{A6}$$

Thus, $z(=wX)$ is an imputation in the core. □

Appendix B

[Proposition 18]

The Shapley value of the DEA min game (N, d) with 4 players is included in the core.

Proof: Let the four players be 1, 2, 3 and 4. Then we can define the Shapley value of the game (N, d) as follows:

$$\phi_i(d)=\sum_{\pi\in\Pi}\frac{1}{4!}\{d(S_{\pi,i}\cup\{i\})-d(S_{\pi,i})\} \quad (\forall i\in\{1,2,3,4\}), \tag{B1}$$

where Π is the set of permutation π of all players and $S_{\pi,i}$ is the set of players preceding player

i in the permutation π. For a permutation π, let m_π^i be the marginal contribution of player i when the players form the grand coalition in the permutation π, i.e.,

$$m_\pi^i = d(S_{\pi,i} \cup \{i\}) - d(S_{\pi,i}) (\forall i \in \{1,2,3,4\}). \tag{B2}$$

Let m_π be an imputation of the game (N, d) such that:

$$m_\pi = (m_\pi^1, m_\pi^2, m_\pi^3, m_\pi^4). \tag{B3}$$

(m_π is an imputation because it satisfies the grand rationality and also satisfies the individual rationality from the super-additivity of $d(\cdot)$.)

Then we can define the Shapley value of the game (N, d) as follows:

$$\phi(d) = (\phi_1(d), \phi_2(d), \phi_3(d), \phi_4(d)) = \sum_{\pi \in \Pi} \frac{1}{4!} m_\pi. \tag{B4}$$

Let p be a mapping such that:

$$p : \Pi \mapsto \Pi, \quad p(\pi) = p((1,2,3,4)) = (2,1,4,3). \tag{B5}$$

Then obviously $\{\pi\}=\{p(\pi)\}=\Pi$. Hence we can define the Shapley value of the game (N, d) as follows:

$$\phi(d) = \frac{1}{2} \sum_{\pi \in \Pi} \frac{1}{4!} (m_\pi + m_{p(\pi)}). \tag{B6}$$

If $\frac{1}{2}(m_\pi + m_{p(\pi)}) (\forall \pi \in \Pi)$ is included in the core, the Shapley value, which is a convex combination of $\frac{1}{2}(m_\pi + m_{p(\pi)})$, is included in the core because the core is also a convex set. The imputation $\frac{1}{2}(m_\pi + m_{p(\pi)}) (\forall \pi \in \Pi)$ satisfies the individual and grand rationalities because both m_π and $m_{p(\pi)}$ satisfy them. For a permutation $\pi = (1,2,3,4)$, imputations m_π and $m_{p(\pi)}$ are defined respectively by (B2) and (B5) as follows:

$$\begin{aligned} m_\pi &= (m_\pi^1, m_\pi^2, m_\pi^3, m_\pi^4) \\ &= (d(1), \quad d(1,2) - d(1), \quad d(1,2,3) - d(1,2), \quad 1 - d(1,2,3)), \end{aligned} \tag{B7}$$

and

$$\begin{aligned} m_{p(\pi)} &= (m_{p(\pi)}^1, m_{p(\pi)}^2, m_{p(\pi)}^3, m_{p(\pi)}^4) \\ &= (d(1,2) - d(2), \quad d(2), \quad 1 - d(1,2,4), \quad d(1,2,4) - d(1,2)). \end{aligned} \tag{B8}$$

We can show that the imputation $\frac{1}{2}(m_\pi + m_{p(\pi)})\ (\pi = (1,2,3,4))$ satisfies the collective rationality for each coalition as follows:

[Case 1] The coalition $\{1,2\}$

$$
\begin{aligned}
&[m_\pi^1 + m_{p(\pi)}^1] + [m_\pi^2 + m_{p(\pi)}^2] \\
&= [d(1) + d(1,2) - d(2)] + [d(1,2) - d(1) + d(2)] \\
&= 2d(1,2).
\end{aligned}
$$

[Case 2] The coalition $\{1,3\}$

$$
\begin{aligned}
&[m_\pi^1 + m_{p(\pi)}^1] + [m_\pi^3 + m_{p(\pi)}^3] \\
&= [d(1) + d(1,2) - d(2)] + [d(1,2,3) - d(1,2) + 1 - d(1,2,4)] \\
&= d(1) - d(1,2,4) + d(1,2,3) - d(2) + 1.
\end{aligned}
$$

From Corollary 1 and super-additivity of $d(\cdot)$, it holds that $d(1) - d(1,2,4) \geq d(1,3) - 1$ and $d(1,2,3) - d(2) \geq d(1,3)$, respectively. Hence we have

$$[m_\pi^1 + m_{p(\pi)}^1] + [m_\pi^3 + m_{p(\pi)}^3] \geq 2d(1,3).$$

For the three cases of the coalition $\{1,4\}$, $\{2,3\}$ and $\{2,4\}$, we can similarly confirm the collective rationality.

[Case 3] The coalition $\{3,4\}$

$$
\begin{aligned}
&[m_\pi^3 + m_{p(\pi)}^3] + [m_\pi^4 + m_{p(\pi)}^4] \\
&= [d(1,2,3) - d(1,2) + 1 - d(1,2,4)] + [1 - d(1,2,3) + d(1,2,4) - d(1,2)] \\
&= 2 - 2d(1,2).
\end{aligned}
$$

From Proposition 7, it holds that $2 - 2d(1,2) = 2c(3,4)$. Hence we have

$$[m_\pi^3 + m_{p(\pi)}^3] + [m_\pi^4 + m_{p(\pi)}^4] = 2c(3,4) \geq 2d(3,4).$$

[Case 4] The coalition $\{1,2,3\}$

$$
\begin{aligned}
&[m_\pi^1 + m_{p(\pi)}^1] + [m_\pi^2 + m_{p(\pi)}^2] + [m_\pi^3 + m_{p(\pi)}^3] \\
&= [d(1) + d(1,2) - d(2)] + [d(1,2) - d(1) + d(2)] + [d(1,2,3) - d(1,2) + 1 - d(1,2,4)] \\
&= d(1,2,3) + d(1,2) - d(1,2,4) + 1.
\end{aligned}
$$

From Corollary 1, it holds that $d(1,2) - d(1,2,4) \geq d(1,2,3) - 1$. Hence we have

$$[m_\pi^1 + m_{p(\pi)}^1] + [m_\pi^2 + m_{p(\pi)}^2] + [m_\pi^3 + m_{p(\pi)}^3] \geq 2d(1,2,3).$$

For the case of the coalition {1,2,4}, we can similarly confirm the collective rationality.

[Case 5] The coalition {1,3,4}

$$\begin{aligned}
&[m_\pi^1 + m_{p(\pi)}^1] + [m_\pi^3 + m_{p(\pi)}^3] + [m_\pi^4 + m_{p(\pi)}^4] \\
&= [d(1) + d(1,2) - d(2)] + [d(1,2,3) - d(1,2) + 1 - d(1,2,4)] \\
&\quad + [1 - d(1,2,3) + d(1,2,4) - d(1,2)] \\
&= d(1) - d(1,2) - d(2) + 2.
\end{aligned}$$

From Corollary 1 and Proposition 7, it holds that $d(1) - d(1,2) \geq d(1,3,4) - 1$ and $d(2) = 1 - c(1,3,4)$, respectively. Hence we have

$$[m_\pi^1 + m_{p(\pi)}^1] + [m_\pi^3 + m_{p(\pi)}^3] + [m_\pi^4 + m_{p(\pi)}^4] \geq d(1,3,4) + c(1,3,4) \geq 2d(1,3,4).$$

For the case of the coalition {2,3,4}, we can similarly confirm the collective rationality.

Thus, summing up, the imputation $\frac{1}{2}(m_\pi + m_{p(\pi)})(\forall \pi \in \Pi)$ is included in the core because it satisfies the collective rationality in addition to the individual and grand rationalities. Hence the Shapley value of the game (*N*, *d*) with 4 players is also included in the core. □

[Appendix C] (Proof of Proposition 16)

We can decompose (18) into two terms as follows:

$$\phi_A(d) = \sum_{S: A \in S, B \in S \subset N} \frac{(s-1)!(n-s)!}{n!} \{d(S) - d(S - \{A\})\} \tag{C1}$$

$$+ \sum_{S: A \in S, B \notin S \subset N} \frac{(s-1)!(n-s)!}{n!} \{d(S) - d(S - \{A\})\} \tag{C2}$$

Similarly, for player B, we have the decomposition:

$$\phi_B(d) = \sum_{T: A \in T, B \in T \subset N} \frac{(t-1)!(n-t)!}{n!} \{d(T) - d(T - \{B\})\} \tag{C3}$$

$$+ \sum_{T: A \notin T, B \in T \subset N} \frac{(t-1)!(n-t)!}{n!} \{d(T) - d(T - \{B\})\} \tag{C4}$$

where t is the number of members of a coalition T.

For each coalition S in (C1) there is a coalition T in (C3) that has the same membership with S, and *vice versa*. For these S and T, we have, since A dominates B,

$$d(S) = d(T) \text{ and } d(S - \{A\}) \leq d(T - \{B\}).$$

Hence, we have an inequality between (C1) and (C3):

$$d(S) - d(S - \{A\}) \geq d(T) - d(T - \{B\}).$$

Similarly, for each coalition S in (C2), there is a coalition T in (C4) that has the same membership with S except A and B, and *vice versa*. For these sets S and T, we have, since A dominates B,

$$d(S) \geq d(T) \quad \text{and} \quad d(S - \{A\}) = d(T - \{B\}).$$

Hence, we have an inequality between (C2) and (C4):

$$d(S) - d(S - \{A\}) \geq d(T) - d(T - \{B\}).$$

Therefore it holds that

$$\phi_A(d) \geq \phi_B(d). \qquad \square$$

[Appendix D] (Proof of Proposition 17)

Suppose that player A dominates player B and the nucleolus of player A is less than that of player B, i.e., $\mu_A < \mu_B$ where $\mu = (\mu_A, \mu_B, \cdots, \mu_i, \cdots)(i \in N)$ is the nucleolus of the game (N, d). For any coalition $T \subset N$ with $A, B \notin T$, we define the excesses of the coalition $T + \{A\}$ and $T + \{B\}$ as follows:

$$e(T + \{A\}, \mu) = d(T + \{A\}) - \sum_{i \in T} \mu_i - \mu_A,$$

$$e(T + \{B\}, \mu) = d(T + \{B\}) - \sum_{i \in T} \mu_i - \mu_B.$$

Since A dominates B, we have $d(T + \{A\}) \geq d(T + \{B\})$. Hence we have

$$e(T + \{A\}, \mu) > e(T + \{B\}, \mu). \tag{D1}$$

We define a coalition T^* and a real number E such that:

$$E = e(T^* + \{A\}, \mu) = \max_{A,B \notin T \subset N} e(T + \{A\}, \mu). \tag{D2}$$

Then it holds, from (D1), that

$$E > e(T + \{B\}, \mu) \, (\forall T). \tag{D3}$$

Let $\theta(\mu)$ be the vector of the excesses of all coalitions $S \subset N \, (S \neq \phi, N)$, ordered by increasing magnitude. I.e.,

$$\theta(\mu) = (e(S^1, \mu), \cdots, e(S^{2^n - 1}, \mu)), \quad e(S^1, \mu) \geq \cdots \geq e(S^{2^n - 1}, \mu). \tag{D4}$$

We divide coalitions $\{S^i\}$ into $\{S^1, \cdots, S^{k-1}\}$, $\{S^k, \cdots, S^{k+h}\}$ and $\{S^{k+h+1}, \cdots, S^{2^n - 1}\}$ in such a way that:

$$e(S^{k-1}, \mu) > e(S^k, \mu) = \cdots = e(S^{k+h}, \mu) = E > e(S^{k+h+1}, \mu). \tag{D5}$$

Then the coalition $T^* + \{A\}$ is included in $\{S^k, \cdots, S^{k+h}\}$. For $\forall S^i \in \{S^1, \cdots, S^{k-1}\}$, it holds, from (D2) - (D3), that

$$A,B \in S^i \text{ or } A,B \notin S^i. \tag{D6}$$

We define $\mu'_A = \mu'_B$ as $\dfrac{\mu_A + \mu_B}{2}$. Then we have $\mu_A < \mu'_A = \mu'_B < \mu_B$. Using μ'_A and μ'_B, we define an imputation μ' as follows:

$$\mu' = (\mu'_A, \mu'_B, \mu'_C, \cdots) = (\frac{\mu_A + \mu_B}{2}, \frac{\mu_A + \mu_B}{2}, \mu_C, \cdots) \tag{D7}$$

Then, for $\forall T \subset N$ with $A,B \notin T$, we have

$$e(T, \mu') = e(T, \mu), \tag{D8}$$

$$\begin{aligned} e(T + \{A,B\}, \mu') &= d(T + \{A,B\}) - \sum_{i \in T} \mu'_i - \mu'_A - \mu'_B \\ &= d(T + \{A,B\}) - \sum_{i \in T} \mu_i - \mu_A - \mu_B = e(T + \{A,B\}, \mu), \end{aligned} \tag{D9}$$

and

$$\begin{aligned} e(T + \{B\}, \mu') &= d(T + \{B\}) - \sum_{i \in T} \mu'_i - \mu'_B \\ &\le d(T + \{A\}) - \sum_{i \in T} \mu'_i - \mu'_A = e(T + \{A\}, \mu') \\ &< d(T + \{A\}) - \sum_{i \in T} \mu_i - \mu_A = e(T + \{A\}, \mu) \\ &\le E. \end{aligned} \tag{D10}$$

For $\forall S^i \in \{S^1, \cdots, S^{k-1}\}$, we have, from (D6), (D8) and (D9),

$$e(S^i, \mu') = e(S^i, \mu) > E. \tag{D11}$$

For $\forall S^i \in \{S^{k+h+1}, \cdots, S^{2^n-1}\}$, it holds, from (D8) - (D10), that

$$e(S^i, \mu') < E. \tag{D12}$$

For $\forall S^i \in \{S^k, \cdots, S^{k+h}\}$, we have, from (D8) - (D10),

$$e(S^i, \mu') \le E = e(S^i, \mu). \tag{D13}$$

Furthermore, $\exists S^i = T^* + \{A\} \in \{S^k, \cdots, S^{k+h}\}$ and it holds, from (D10), that

$$e(S^i, \mu') < e(S^i, \mu) = E. \tag{D14}$$

From (D11)-(D14), we have $\theta(\mu) >_L \theta(\mu')$, i.e.,

$\exists t \in \{k, \cdots, k+h\}, \quad e(S^i, \mu) = e(S^i, \mu')(i = 1, \cdots, t-1)$ and $e(S^t, \mu) > e(S^t, \mu')$.

This leads to a contradiction with the definition of the nucleolus. Therefore, if player A dominates player B, then the nucleolus of player A is not less than that of player B. □

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