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Indivisibilities and Economies of Scope in Data Envelopment Analysis

By

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Abstract

This paper attempts to examine the data envelopment analysis (DEA) model of efficiency measurement from an economic perspective. We have discussed here the use of a new DEA model to show how the presence of process indivisibilities arising from task-specific production processes exhibits economies of scope.

Key Words: Economies of scope; Indivisibilities; Multi-stage production; DEA.

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Indivisibilities and Economies of Scope in Data Envelopment Analysis

1. Introduction

Economies of scope arise from synergies in the production of similar goods. The classical notion of joint production explains instances of such synergies by the fact that some factors of production are *shared* or *indivisible* inputs. The *indivisibility* argument gives rise to the concept of minimum efficient scale, which is, in turn, used as a benchmark for defining excess capacity. The excess capacity so defined can give no indication of the extent of idleness or underutilization in plant and machinery[1]. We find literature on single product firm ignoring completely the idea of flexibility of inputs in terms of using a good deal of the same plant machinery, technical skill, etc. for producing several classes of related products[2].

Against this backdrop, following the literature on production control problems[3], this paper concentrates on a multi-stage production process where idle capacity[4] arises due to unequal length of production runs of intermediate stages, which leads to *scale effects*[5] when production is expanded. If the demand for output is downward slopping, then instead of scaling up existing output merely on basis of capacity utilization, the firm could also use the existing idle capacities together with the flexibility of inputs to diversify into other products so as to enjoy *economies of scope*. So, the requirement that the firm faces a downward slopping demand curve indicates that *scale* considerations play a role only under perfectly competitive condition whereas *scope* considerations are more relevant to the firm in planning the size of operation facilities under monopolistic competition[6].

The empirical estimation of cost structure has shed little light on the kinds of production processes that lead to *scope economies*. In this paper we have made an attempt to suggest a new

[1] On its detailed discussion, see Cassels (1937).
[2] See Marshall (1920, p.390) who has observed this phenomenon more frequently occurring in an industry.
[3] The literature on production control problems describes production by a detailed breakdown of the operations undertaken within a production unit. The process of production is defined as a set of 'tasks' to be performed by certain factors of production on raw materials in certain well-defined fashion to lead to the final product. See, for example, Aburzzi (1965) and Bakshi and Arora (1969) for details.
[4] Idle capacities in general arise due to *indivisibilities* in inputs, or a secular decline in demand for existing product, or due to demand uncertainty for the existing product. However, the task-specific idle capacities in production do not depend upon uncertainty in future demand for existing products or in a secular decline in demand conditions.
[5] For a historical discussion on the evolution of the concept of scale and its computational procedure, see, among others, Gold (1981), Sahoo *et al.* (1999) and Tone and Sahoo (2003a,b,c).
[6] There is ample evidence that no matter whatever the extent of diversification, some idle capacities may still remain. Howevber, if the production of single output is envisaged, it is in the interest of firm to have unused capacities. See Klein (1960) for details.

data envelopment analysis (DEA) model to estimate a cost frontier revealing such *scope economies* arising from task-specific idle capacities in the multistage production model.

The reminder of the paper unfolds as follows. Section 2 discusses the nature of production, and describes, using a simple example, a task-specific production process that generates scope economies due to indivisibilities. Section 3 proposes a new DEA model to estimate such scope economies, and finally Section 4 ends with some concluding remark.

2. Nature of production model

The standard neoclassical characterization of *production function*[7] contains information on the efficient input-output possibilities without specifying how production is actually organized with the firm. In this paper we look at production as a *task-specific process* in which production is broken down into its various principle stages. The idea is to bring out the inherent hidden *indivisibilities* of the activities associated with the production process by observing the task-length associated with each stage. The main observation is that production process usually consists of more than one stage of production, and the task-lengths associated with various stages need not be equal. This is because different pieces of capital equipment used at different stages of production processes serve different purpose and are designed with respect to that purpose at hand with the existing technical know-how. Now the relevant question for *economies of scope* to hold good is whether the set of tasks executed at any given point of time allows the full and continuous utilization of all factors of production or not. The answer to this question largely depends on how the tasks are arranged in the production process. To illustrate this, let us consider an example of a production process for the manufacture of doorframes[8], which has the following operations that are all exhibited in Table 1.

Table 1: Organization of production process for door manufacture

Stages	Description	Capacity per doorframe
A	Cutting the wooden parts to suitable dimensions.	Stage A takes 1 hour per door frame.
B	Assembling the door frame.	Stage B takes 2 hours per door frame.
C	Varnishing/painting and drying.	Stage C takes 3 hours per door frame.
D	Adding locks, hinges, and other accessories	Stage D takes 1 hour per door frame.

[7] Barring a few authors like Russell and Wilkinson (1979), most authors describe this concept as showing maximum output with a given quantity of input.
[8] This example is taken from Raja (1994, p.136).

If the production has to be carried out in strict sequence, then during the whole work-day (assume it to be 12 hours) 12 doorframes can be cut in Stage **A**, and four doorframes will emerge as finished products from Stage **D** with six doors awaiting operation at Stage **B** and two doors awaiting operation **C**. To note that if the total number of finished doors is four, then Stage **A** has only four hours of work, Stage **B** has 8 hours, Stage **C** has full 12 hours of work, and Stage **D** has four hours of work with idle capacities of 8, 4, 0 and 8 hours existing respectively in stages **A**, **B**, **C**, and **D**. If all the stages have to be fully utilized, then here must be two groups of Stage **B** type tasks to work simultaneously to produce 12 assembled doorframes and three groups of Stage **C** types tasks arranged to work simultaneously to be able to paint and varnish all the 12 doors so that 12 finished doorframes will emerge from Stage **D**.

It is important to note that there are two alternative ways to produce 12 finished doorframes. One way is to simply replicate three times the process originally used to produce four doors. The other way is to organize production as discussed above to take maximum advantage of the existing idle capacities in various stages. In the former case the corresponding total cost will be three times that of process used to produce four units whereas in the later case the total cost will be less than three times of the original cost. What we infer from the organization of production is that tripling of output does not necessitate tripling of all inputs resulting total cost to increase less than proportionately to total output, which is an indication of *scale effects*[9].

If the demand for door is 12 units per day, then the firm will be fully engaged in producing doors. If the demand for doors falls short of 12 (or falls short of any integer multiple of 12), then the firm can possibly think of two ways for augmenting production: one by altering the rate of output with existing facilities by operating machine A more intensively and labor services in all other tasks to be intensified with more labor, and the other by acquiring another machine A to existing capacities where the production of any output greater than 12 but less than 24 (say) leaves idle capacities since the firm needs to produce 24 doors for the efficient utilization of its resources. Therefore, idle capacities generated are not due to decline in demand or uncertainty in demand for existing products, but due to indivisibilities in inputs.

The indivisibility in inputs leads to *scope effects* if the firm decides to produce related product[10], say, window frames to make use of idle capacities existing in various stages because

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[9] The indivisibility argument that leads to increasing returns scale is also shown elsewhere. See, e.g., Tone and Sahoo (2003a) where the presence of indivisibilities in all multi-stage production processes is shown to exhibit scale economies in a competitive market structure.

[10] The firm decides to diversify into those products, say, window that are having task-lengths, which match idle capacities generated in the production of any one product, i.e., door though window frames may require a slight change in the nature of tasks particularly in operation **D**. To generalize, the closer are the

the *nature of task* as well as *task-sequencing* in both the products are more or less same. Now our objective is to empirically show how indivisibility argument leads to show *scope effects*.

Given the process of production described above, if the technology were to be represented by *production function*, it would have to be in terms of all four *tasks*[11]. It is shown in Table 2 (left side) how the production of doors is expanded by taking the maximum advantage of idle capacities existing in various stages of production[12]. At the production of 12 doors, no idle capacities are found in any of these four stages.

Clearly, we observe that four production possibilities (expressed in bold letter) constitute the vertices of the *production frontier* because they are efficient in Koopmans sense. Looking at the movement of the input-output vectors along the frontier, we expect that *production function* exhibits increasing returns to scale, and as regards the expansion path, it is non-linear because doubling of output does not necessitate the doubling of all inputs during this output range even though doubling of all inputs leads to doubling of output. However, the latter production possibilities are not efficient and hence do not operate on the efficient frontier. So this observation calls into question the ability of the homogeneous characterization of production function to capture *scale effects* if *scale* arises in this fashion.

Table 2: Production Data Set for Doors and Windows

Production Data Set for Doors										Production Data Set for Windows									
	Tasks				Idle Capacity						Tasks				Idle Capacity				
Firms	A	B	C	D	A	B	C	D	Doors	Firms	A	B	C	D	A	B	C	D	Windows
1	1	1	1	1	11	10	9	11	1	13	1	1	1	1	11	10.5	10	11	1
2	1	1	1	1	10	8	6	10	2	14	1	1	1	1	10	9.0	8	10	2
3	1	1	1	1	9	6	3	9	3	15	1	1	1	1	9	7.5	6	9	3
4	**1**	**1**	**1**	**1**	**8**	**4**	**0**	**8**	**4**	16	1	1	1	1	8	6.0	4	8	4
5	1	1	2	1	7	2	9	7	5	17	1	1	1	1	7	4.5	2	7	5
6	**1**	**1**	**2**	**1**	**6**	**0**	**6**	**6**	**6**	18	**1**	**1**	**1**	**1**	**6**	**3.0**	**0**	**6**	**6**
7	1	2	2	1	5	10	3	5	7	19	1	1	2	1	5	1.5	10	5	7
8	**1**	**2**	**2**	**1**	**4**	**8**	**0**	**4**	**8**	20	**1**	**1**	**2**	**1**	**4**	**0.0**	**8**	**4**	**8**
9	1	2	3	1	3	6	9	3	9	21	1	2	2	1	3	10.5	6	3	9
10	1	2	3	1	**2**	**4**	**6**	**2**	10	22	1	2	2	1	**2**	9.0	**4**	**2**	10
11	1	2	3	1	1	2	3	1	11	23	1	2	2	1	1	7.5	2	1	11
12	**1**	**2**	**3**	**1**	**0**	**0**	**0**	**0**	**12**	24	**1**	**2**	**2**	**1**	**0**	**6.0**	**0**	**0**	**12**

tasks required for the new product to tasks associated with existing products, the more efficient would be the use of idle capacity. See Teece (1982) for details.

[11] The technology represented in this form can also be formally modeled in the context of production network theory developed by Färe and Whittaker (1995) and Färe and Grosskopf (1996). An empirical attempt is also made in one of our earlier paper (Tone and Sahoo, 2003, pp.186-188).

[12] While scaling up production to meet the increased demand, we have not considered various transaction cost arguments into consideration.

If the entire facilities are used to produce window frames alone, then it is shown in Table 2 (right side) how production of windows is expanded. We assume here the task-lengths to be different from those taken for doors. For window frames, task-lengths are as follows: **A**: 1 hours, **B**: 1.5 hours, **C**: 2 hours and **D**: 1 hour. Here we clearly see that three production possibilities (1A, 1B, 1C, 1D, 6Y), (1A, 1B, 2C, 1D, 8Y), and (1A, 2B, 2C, 1D, 6Y) are efficient, and hence operate on the production function for window. This production function also exhibits IRS.

However, since the main activity of the firm is in door production, it decides to produce window whenever demand for doors falls short of 12 units or falls short of any integer multiple of 12 units. Table 3 exhibits the data set showing how various combination of production of doors and windows is possible with the existing technology to exploit the idle capacities when demand falls short of 12 doors per day. We, *a priori*, expect these diversified production possibilities to exhibit economies of scope.

Let us now turn to discuss how to estimate the cost frontier that will enable us uncovering the scope effects arising from such process *indivisibilities*. Two approaches are available to estimate potential economies of scope: parametric and nonparametric approaches. We utilize here the nonparametric approach called data envelopment analysis (DEA) to reveal rather than imposing the underlying cost structure.

3. Discussion on DEA model measuring economies of scope

Baumol *et al.* (1982) define (local) economies of scope (ES) to exist between two products (y_1 and y_2) if the cost of producing two products by one firm is less than the cost of producing them separately in specialized firms, i.e.,

$$C(y_1,y_2) < C(y_1,0) + C(0,y_2) \qquad \ldots (1)$$

where $C(y_1,y_2)$ is the cost of joint production by the diversified firm, $C(y_1,0)$ and $(0,y_2)$ are the respective costs of production of y_1 and y_2 by two specialized firms. So the local degree of economies of scope (DES) for firm j is defined as

$$\text{DES}_j = \frac{C(y_1,0) + C(0,y_2) - C(y_1,y_2)}{C(y_1,y_2)}. \qquad \ldots (2)$$

DESj > 0 implies that the firm j exhibits economies of scope, DESj < 0 implies diseconomies of scope, and DESj = 0 implies that the cost function $C(y_1,y_2)$ is additive in nature.

Table 3: Production Data Set for Doors and Windows

	Tasks				Outputs		Idle Capacity					Tasks				Outputs		Idle Capacity			
Firm	A	B	C	D	Doors	Windows	A	B	C	D	Firm	A	B	C	D	Doors	Windows	A	B	C	D
D1	1	1	1	1	3	1	8	4.5	1	8	D59	1	2	3	1	4	8	0	4	8	0
D2	1	1	1	1	2	3	7	3.5	0	5	D60	1	2	3	1	3	9	0	5	9	0
D3	1	1	1	1	1	4	7	4	1	7	D61	1	2	3	1	2	10	0	5	10	0
D4	1	1	1	1	1	1	10	8.5	7	10	D62	1	2	3	1	1	11	0	6	11	0
D5	1	1	1	1	2	1	9	6.5	4	10	D63	1	2	3	1	1	1	10	21	31	10
D6	1	1	1	1	1	2	9	7	5	9	D64	1	2	3	1	2	1	9	19	28	9
D7	1	1	1	1	2	2	9	7.5	6	9	D65	1	2	3	1	3	1	8	17	25	8
D8	1	1	1	1	1	3	8	5.5	3	8	D66	1	2	3	1	4	1	7	15	22	7
D9	1	1	2	1	5	1	6	0.5	7	6	D67	1	2	3	1	5	1	6	13	19	6
D10	1	1	2	1	4	2	6	0.5	7	6	D68	1	2	3	1	6	1	5	11	16	5
D11	1	1	2	1	3	4	5	0	7	5	D69	1	2	3	1	7	1	4	9	13	4
D12	1	1	2	1	2	5	5	0.5	8	5	D70	1	2	3	1	8	1	3	7	10	3
D13	1	1	2	1	1	6	5	1	9	5	D71	1	2	3	1	9	1	2	5	7	2
D14	1	1	2	1	1	5	6	2.5	11	6	D72	1	2	3	1	10	1	1	3	4	1
D15	1	1	2	1	1	4	7	4	13	7	D73	1	2	3	1	1	2	9	19	29	9
D16	1	1	2	1	2	4	6	2	10	6	D74	1	2	3	1	2	2	8	17	26	8
D17	1	1	2	1	1	2	9	7	17	9	D75	1	2	3	1	3	2	7	15	23	7
D18	1	1	2	1	2	2	8	5	14	8	D76	1	2	3	1	4	2	6	13	20	6
D19	1	1	2	1	3	2	7	3	11	7	D77	1	2	3	1	5	2	5	11	17	5
D20	1	1	2	1	1	1	10	8.5	19	10	D78	1	2	3	1	6	2	4	9	14	4
D21	1	1	2	1	2	1	9	6.5	16	9	D79	1	2	3	1	7	2	3	7	11	3
D22	1	1	2	1	3	1	8	4.5	13	8	D80	1	2	3	1	8	2	2	5	8	2
D23	1	1	2	1	4	1	7	2.5	10	7	D81	1	2	3	1	9	2	1	3	5	1
D24	1	2	2	1	7	1	4	8.5	1	4	D82	1	2	3	1	1	3	8	18	27	8
D25	1	2	2	1	6	3	3	7.5	0	3	D83	1	2	3	1	2	3	7	16	24	7
D26	1	2	2	1	5	4	3	8	1	3	D84	1	2	3	1	3	3	6	14	21	6
D27	1	2	2	1	4	6	2	7	0	2	D85	1	2	3	1	4	3	5	12	18	5
D28	1	2	2	1	3	7	2	7.5	1	2	D86	1	2	3	1	5	3	4	10	15	4
D29	1	2	2	1	2	9	1	6.5	0	1	D87	1	2	3	1	6	3	3	8	12	3
D30	1	2	2	1	1	10	1	7	1	1	D88	1	2	3	1	7	3	2	6	9	2
D31	1	2	2	1	1	1	10	21	19	10	D89	1	2	3	1	8	3	1	4	6	1
D32	1	2	2	1	2	1	9	19	16	9	D90	1	2	3	1	1	4	7	16	25	7
D33	1	2	2	1	3	1	8	17	13	8	D91	1	2	3	1	2	4	6	14	22	6
D34	1	2	2	1	4	1	7	15	10	7	D92	1	2	3	1	3	4	5	12	19	5
D35	1	2	2	1	5	1	6	13	7	6	D93	1	2	3	1	4	4	4	10	16	4
D36	1	2	2	1	6	1	5	11	4	5	D94	1	2	3	1	5	4	3	8	13	3
D37	1	2	2	1	1	3	8	18	15	8	D95	1	2	3	1	6	4	2	6	10	2
D38	1	2	2	1	2	3	7	16	12	7	D96	1	2	3	1	7	4	1	4	7	1
D39	1	2	2	1	3	3	6	14	9	6	D97	1	2	3	1	1	5	6	15	23	6
D40	1	2	2	1	4	3	5	12	6	5	D98	1	2	3	1	2	5	5	13	20	5
D41	1	2	2	1	5	3	4	9.5	3	4	D99	1	2	3	1	3	5	4	11	17	4
D42	1	2	2	1	1	4	7	16	13	7	D100	1	2	3	1	4	5	3	9	14	3
D43	1	2	2	1	2	4	6	14	10	6	D101	1	2	3	1	5	5	2	7	11	2
D44	1	2	2	1	3	4	5	12	7	5	D102	1	2	3	1	6	5	1	5	8	1
D45	1	2	2	1	4	4	4	10	4	4	D103	1	2	3	1	1	6	5	13	21	5
D46	1	2	2	1	1	6	5	13	9	5	D104	1	2	3	1	2	6	4	11	18	4
D47	1	2	2	1	2	6	4	11	6	4	D105	1	2	3	1	3	6	3	9	15	3
D48	1	2	2	1	3	6	3	9	3	3	D106	1	2	3	1	4	6	2	7	12	2
D49	1	2	2	1	1	7	4	12	7	4	D107	1	2	3	1	5	6	1	5	9	1
D50	1	2	2	1	2	7	3	9.5	4	3	D108	1	2	3	1	1	7	4	12	19	4
D51	1	2	2	1	1	9	2	8.5	3	2	D109	1	2	3	1	2	7	3	10	16	3
D52	1	2	3	1	11	1	0	0.5	1	0	D110	1	2	3	1	3	7	2	8	13	2
D53	1	2	3	1	10	2	0	1	2	0	D111	1	2	3	1	4	7	1	6	10	1
D54	1	2	3	1	9	3	0	1.5	3	0	D112	1	2	3	1	1	8	3	10	17	3
D55	1	2	3	1	8	4	0	2	4	0	D113	1	2	3	1	2	8	2	8	14	2
D56	1	2	3	1	7	5	0	2.5	5	0	D114	1	2	3	1	3	8	1	6	11	1
D57	1	2	3	1	6	6	0	3	6	0	D115	1	2	3	1	1	9	2	9	15	2
D58	1	2	3	1	5	7	0	3.5	7	0	D116	1	2	3	1	2	9	1	7	12	1
D58	1	2	3	1	5	7	0	3.5	7	0	D117	1	2	3	1	1	10	1	7	13	1

We assume here to deal with n diversified firms, each using m inputs to produce s outputs. For each firm 'o' ($o = 1,2,\dots,n$) we denote respectively the input/output vectors by $x_o \in R^m$ and $y_o \in R^s$. The input/output matrices are defined by $X = (x_1,\dots,x_n) \in R^{m\times n}$ and $Y = (y_1,\dots,y_n) \in R^{s\times n}$. We assume that $X > O$ and $Y > O$. Given the unit input price vector $c_o \in R^m$ (> 0) for the input x_o of firm 'o', the cost efficiency[13] is defined as

$$\gamma^* = c_o x_o^* / c_o x_o = \sum_{i=1}^{m} c_{io} x_i^* \Big/ \sum_{i=1}^{m} c_{io} x_i, \qquad \dots (3)$$

where x_o^* is an optimal solution of the following linear programming problem (LP):

$$\begin{aligned}
&\text{[Cost]} \quad && C(y_o; c_o) = \min \sum_{i=1}^{m} c_{io} x_i && \dots (4)\\
&\text{subject to} && \sum_{j=1}^{n} x_{ij}\lambda_j \le x_i \ (\forall i)\\
& && \sum_{j=1}^{n} y_{rj}\lambda_j \ge y_{ro} (\forall r)\\
& && \sum_{j=1}^{n} \lambda_j = 1\\
& && \lambda_j \ge 0 \ (\forall j).
\end{aligned}$$

Now we need to compare the minimal cost of these n *diversified* firms along with their observed outputs with a frontier consisting of *additive* firms satisfying the condition: DES = 0 over the relevant range of outputs. These additive firms are hypothetical ones, which are all created from *specialized* firms. Assuming there are n_1 firms producing output y_1 alone and n_2 firms producing output y_2 alone. All possible permutations of the outputs and costs of these two sets of specialized firms are added pair wise to form the set of hypothetical additive firms. Let the number of additive firms be k *whose* output and cost of these firms are associated with superscript '+'. We then follow Evans and Heckman (1984)'s procedure to determine the admissible region where we require our hypothetical additive firms to envelop the diversified ones, i.e., each diversified firm must produce doors and windows no more than the maximal and no less than the minimal production of doors and windows of hypothetical firms. So in order to calculate economies of scope for the diversified firm 'o', we need to solve the following LP:

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[13] We call [Cost] model as the classical cost efficiency model. On this definition of cost efficiency, refer to Färe *et al.* (1985, 1994), Sueyoshi (1997,1999) and Cooper *et al.* (1999) for details.

$$[\text{Cost\_m}] \quad C^{+}(y_o;c_o) = \min \sum_{i=1}^{m} c_{io}x_i \qquad \ldots (5)$$

$$\text{subject to} \quad \sum_{j=1}^{k} x_{ij}^{+} \lambda_j \le x_i \ (\forall i)$$

$$\sum_{j=1}^{k} y_{rj}^{+}\lambda_j \ge y_{ro}(\forall r)$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0 \ (\forall j).$$

Here $C^{+}(y_o;c_o)$ represents the minimum cost of production of output vector y_o in the additive technology set when input price vector faced by firm '*o*' is c_o. The degree of economies of scope[14] (DES_o) is defined as:

$$\text{DES}_o = \frac{C^{+}(y_o;c_o)}{C(y_o;c_o)} - 1. \qquad \ldots (6)$$

As has been pointed out by Tone (2002), the cost efficiency evaluation model (4) has several shortcomings out of which the two most important ones are:

1. In case of single input, technical efficiency and cost efficiency are one and the same.
2. If two firms *A* and *B* have the same amount of inputs and outputs, i.e., $x_{iA} = x_{iB}$ ($\forall$ *i*), and $y_{rA} = y_{rB}$ ($\forall$ *r*), and the input price faced by firm A is twice that of firm B, then both firm exhibit same cost and allocative efficiencies.

These problems are due to the inherent structure of the supposed technology set P as defined by

$$P = \{(x,y) : x \ge X\lambda, y \le Y\lambda, e\lambda = 1, \lambda \ge 0\}. \qquad \ldots (7)$$

P is defined only with help of technical factors $X = (x_1,\ldots,x_n) \in R^{m\times n}$ and $Y = (y_1,\ldots,y_n) \in R^{s\times n}$, but has no concern with the unit input price vector $C = (c_1,\ldots,c_n)$.

He defined another cost-based technology set P_c as

$$P_c = \{(x,y) : \bar{c} \ge \bar{C}\lambda, y \le Y\lambda, e\lambda = 1, \lambda \ge 0\}, \qquad \ldots (8)$$

where $\bar{C} = (\bar{c}_1,\ldots,\bar{c}_n)$ with $\bar{c}_j = (c_{1j}x_{ij},\ldots,c_{mj}x_{mj})^T$ and *e* is a row unit vector. Here, the matrices *X*, *C* and $\bar{C}$ are all assumed to be positive, and the elements of bbare also assumed to be denominated in homogeneous units so that adding up the elements of $\bar{c}_{ij}$ has a meaning.

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[14] This measure of economies of scope is just an adaptation of the model by Färe (1986), who first provided a theoretical model and outlined LP for measuring ES. However, in the spirit of Färe *et al.* (1994), one can find, with the help of input and output data only, the measure of economies of scope by comparing two production frontiers for specialized firms with a production frontier for diversified firms. See Prior (1996), Kittelsen and Magnussen (2003) and Morita (2003) for the details.

Based on this new technology set P_c, the new cost efficiency is defined as

$$\bar{\gamma}^* = \overline{ec}_o^* / \overline{ec}_o, \qquad \dots (9)$$

where $\bar{c}_o^*$ is an optimal solution of the LP given below:

$$\text{[NCost]} \quad \overline{\text{ec}}^* = \min \sum_{i=1}^{m} e_i \bar{c}_i \qquad \dots (10)$$

$$\text{subject to} \quad \sum_{j=1}^{n} \bar{c}_{ij}\lambda_j \le \bar{c}_i \ (\forall i)$$

$$\sum_{j=1}^{n} y_{rj}\lambda_j \ge y_{ro} \ (\forall r)$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0 \ (\forall j).$$

Considering the objective function form $e\bar{c}$ and the input constraints in [NCost], the aggregation of these m constraints into one constraint yields a new program as follows:

Let us denote $e\bar{c}_j$ by $\bar{\bar{c}}_j$, i.e.,

$$\bar{\bar{c}}_j = \sum_{i=1}^{m} \bar{c}_{ij} = \sum_{i=1}^{m} x_{ij}c_{ij}. \ (j = 1,\dots,n) \qquad \dots (11)$$

$\bar{\bar{c}}_j$ is the total input cost of firm j for producing the output vector y_j. Using this notation, we have a new scheme as expressed by the following LP:

$$\text{[NCost-1]} \quad \bar{\bar{\text{c}}}^* = \min \ 1.\,\bar{\bar{\text{c}}} \qquad \dots (12)$$

$$\text{subject to} \quad \sum_{j=1}^{n} \bar{\bar{c}}_j\lambda_j - 1.\bar{\bar{c}} \le 0$$

$$\sum_{j=1}^{n} y_{rj}\lambda_j \ge y_{ro} \ (\forall r)$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0 \ (\forall j).$$

Alternatively, [NCost-1] can be expressed as

$$\text{[NCost-1A]} \quad \min \sum_{j=1}^{n} \bar{\bar{c}}_j\lambda_j \qquad \dots (13)$$

$$\text{subject to} \quad \sum_{j=1}^{n} y_{rj}\lambda_j \ge y_{ro} \ (\forall r)$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0 \ (\forall j).$$

Between these programmes, we have the following theorem:

Theorem 1. *The optimal objective values of* [NCost], [NCost-1] *and* [NCost-1A] *are the same.*
See Tone (2002) for the proof.

We can also express [NCost-1] in a simple input oriented BCC model framework as follows:

Since $\bar{\bar{c}}_o^*$ is the minimum cost for firm '*o*', we can express this amount as the some positive constant fractions (θ) of actual cost $\bar{\bar{c}}_o$, i.e.,

$$\bar{\bar{c}}_o^* = \theta * \bar{\bar{c}}_o, \text{ or, } \theta = \frac{\bar{\bar{c}}_o^*}{\bar{\bar{c}}_o}. \qquad \ldots (14)$$

Given the observed cost $\bar{\bar{c}}_o$, minimizing $\bar{\bar{c}}_o^*$ is equivalent to minimizing θ, which leads to the following equivalent model for [NCost-1]:

$$\text{[NCost-1E]} \quad \theta^* = \min \ \theta \qquad \ldots (15)$$

$$\text{subject to} \quad \sum_{j=1}^{n} \bar{\bar{c}}_j \lambda_j \le \theta \bar{\bar{c}}_o$$

$$\sum_{j=1}^{n} y_{rj} \lambda_j \ge y_{ro} \ (\forall r)$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \ge 0 \ (\forall j).$$

Now we have the following theorem:

Theorem 2. *At the optimum* $\theta^* \bar{\bar{c}}_o$ *in* [NCost-1E] *and* $\bar{\bar{c}}_o^*$ *in* [NCost-1] *are equal.*
See Tone (2002) for the proof.

To note that model (13) is simply based on a technology set that is a convex polyhedron composed of every combination of $(\bar{\bar{c}}_1, y_1), (\bar{\bar{c}}_2, y_2), \ldots, (\bar{\bar{c}}_n, y_n)$ coupled with the set of activity $(\bar{\bar{c}}, y)$ with an excess in cost $(\bar{\bar{c}} \ge \sum_{j=1}^{n} \bar{\bar{c}}_j \lambda_j)$ and shortfalls in output $(y \le Y\lambda)$. So this technology set can be interpreted as a set representing a possible correspondence between the input (cost) and output (production). Thus, we can determine for every firm the Pareto-Koopmans efficient point by employing the input oriented variable returns to scale DEA model (15).

Let us now turn to evaluate the degree of economies of scope for any diversified firm. Analogous to the procedure discussed above, to compute DES for firm '*o*', we need to solve the following LP:

$$\begin{aligned} &[\text{NCost-1E}] \quad \theta^{+*} = \min \ \theta^{+} \\ &\text{subject to} \quad \sum_{j=1}^{k} \overset{=+}{c_j} \lambda_j \le \theta^{+} \overset{=*}{c_o} \\ &\qquad \sum_{j=1}^{k} y_{rj}^{+} \lambda_j \ge y_{ro} \ (\forall r) \\ &\qquad \sum_{j=1}^{k} \lambda_j = 1 \\ &\qquad \lambda_j \ge 0 \ (\forall j). \end{aligned} \qquad \ldots (16)$$

The degree of economies of scope[15] (DES_o) is defined as θ^{+*} minus one, i.e.,

$$\text{DES}_o = \theta^{+*} - 1. \qquad \ldots (17)$$

We first computed the cost efficiency measure (θ^*) in (15), and then used this θ^* to get the minimal cost levels of the diversified firms, which are in turn used to measure economies of scope (θ^{+*}) in the additive technology in (16) for these diversified firms. Table 4 exhibits these figures.

As expected, we find here all diversified production possibilities exhibiting strong economies of scope, as the corresponding DES numbers are all positive. Also is evident that if we compare this table with Table 3 we infer that lesser the idle capacities, higher the magnitudes of economies of scope and vice versa. We need to mention here that this scope measure captures not only *scale effect* reflecting cost advantages of different scales of operations, but also *convexity effect* reflecting the cost advantage of over various output-mixes.

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[15] This measure has the natural advantage of eliminating all technical and allocative inefficiencies before calculating potential economies of scope (but not realized gains from diversifications), which is defined on the boundary of the cost-based technology set. This measure has been found applications in studies by Grosskopf *et al.* (1987), Grosskopf and Yaiswarng (1990) and Fried *et al.* (1998).

Table 4: Efficiency and Scope Measures of Diversified Firms							
Firms	θ^*	θ^{+*}	Scope	Firms	θ^*	θ^{+*}	Scope
D1	1	2.00000	1.00000	D60	1	1.38636	0.38636
D2	1	2.00000	1.00000	D61	1	1.42424	0.42424
D3	1	2.00000	1.00000	D62	1	1.46212	0.46212
D4	1	2.00000	1.00000	D63	0.636364	2.00000	1.00000
D5	1	2.00000	1.00000	D64	0.636364	2.00000	1.00000
D6	1	2.00000	1.00000	D65	0.636364	2.00000	1.00000
D7	1	2.00000	1.00000	D66	0.681818	1.86667	0.86667
D8	1	2.00000	1.00000	D67	0.727273	1.81250	0.81250
D9	0.941176	1.81250	0.81250	D68	0.772727	1.76471	0.76471
D10	0.911765	1.80645	0.80645	D69	0.818182	1.72222	0.72222
D11	0.93573	1.76019	0.76019	D70	0.863636	1.68421	0.68421
D12	0.921569	1.78724	0.78724	D71	0.909091	1.65000	0.65000
D13	0.921569	1.78724	0.78724	D72	0.954545	1.61905	0.61905
D14	0.872549	1.88764	0.88764	D73	0.636364	2.00000	1.00000
D15	0.823529	2.00000	1.00000	D74	0.636364	2.00000	1.00000
D16	0.872549	1.88764	0.88764	D75	0.659091	1.93103	0.93103
D17	0.823529	2.00000	1.00000	D76	0.704545	1.80645	0.80645
D18	0.823529	2.00000	1.00000	D77	0.75	1.75758	0.75758
D19	0.852941	1.93103	0.93103	D78	0.795455	1.71429	0.71429
D20	0.823529	2.00000	1.00000	D79	0.842593	1.67233	0.67233
D21	0.823529	2.00000	1.00000	D80	0.891414	1.63173	0.63173
D22	0.823529	2.00000	1.00000	D81	0.940236	1.59534	0.59534
D23	0.882353	1.86667	0.86667	D82	0.636364	2.00000	1.00000
D24	0.947368	1.72222	0.72222	D83	0.636364	2.00000	1.00000
D25	0.962963	1.63968	0.63968	D84	0.685185	1.85749	0.85749
D26	0.950292	1.60615	0.60615	D85	0.734007	1.73394	0.73394
D27	0.981481	1.50149	0.50149	D86	0.782828	1.68387	0.68387
D28	0.968811	1.56640	0.56640	D87	0.83165	1.63968	0.63968
D29	1	1.60526	0.60526	D88	0.880471	1.60038	0.60038
D30	1	1.64912	0.64912	D89	0.929293	1.56522	0.56522
D31	0.736842	2.00000	1.00000	D90	0.636364	2.00000	1.00000
D32	0.736842	2.00000	1.00000	D91	0.674242	1.88764	0.88764
D33	0.736842	2.00000	1.00000	D92	0.723064	1.76019	0.76019
D34	0.789474	1.86667	0.86667	D93	0.771886	1.64885	0.64885
D35	0.842105	1.81250	0.81250	D94	0.820707	1.60615	0.60615
D36	0.894737	1.76471	0.76471	D95	0.869529	1.56825	0.56825
D37	0.736842	2.00000	1.00000	D96	0.91835	1.53437	0.53437
D38	0.736842	2.00000	1.00000	D97	0.674242	1.88764	0.88764
D39	0.793372	1.85750	0.85750	D98	0.712121	1.78724	0.78724
D40	0.849903	1.73394	0.73394	D99	0.760943	1.67257	0.67257
D41	0.906433	1.68387	0.68387	D100	0.809764	1.57173	0.57173
D42	0.736842	2.00000	1.00000	D101	0.858586	1.53529	0.53529
D43	0.780702	1.88764	0.88764	D102	0.907407	1.50278	0.50278
D44	0.837232	1.76019	0.76019	D103	0.712121	1.78724	0.78724
D45	0.893762	1.64885	0.64885	D104	0.75	1.69697	0.69697
D46	0.824561	1.78724	0.78724	D105	0.798822	1.59326	0.59326
D47	0.868421	1.69697	0.69697	D106	0.847643	1.50149	0.50149
D48	0.924951	1.59326	0.59326	D107	0.896465	1.47042	0.47042
D49	0.868421	1.74747	0.74747	D108	0.75	1.74747	0.74747
D50	0.912281	1.66346	0.66346	D109	0.787879	1.66346	0.66346
D51	0.95614	1.67890	0.67890	D110	0.8367	1.56640	0.56640
D52	1	1.59091	0.59091	D111	0.885522	1.48004	0.48004
D53	1	1.54545	0.54545	D112	0.787879	1.71154	0.71154
D54	1	1.50000	0.50000	D113	0.825758	1.63303	0.63303
D55	1	1.45455	0.45455	D114	0.874579	1.54187	0.54187
D56	1	1.40909	0.40909	D115	0.825758	1.67890	0.67890
D57	1	1.36364	0.36364	D116	0.863636	1.60526	0.60526
D58	1	1.35606	0.35606	D117	0.863636	1.64912	0.64912

4. Concluding remark

In any multi-stage production process idle capacities arise due to unequal length of production runs of intermediate stages, which leads to scale effects when production is expanded. If the final output can be scaled to the nearest integer value of that production run which has the largest idle capacity, then scale economies are realized since total cost do not increase proportionately to the volume of output. Such a characteristic is called process indivisibility in the multi-stage production literature. However, when demand for output falls short of its optimal scale or any integer multiple of its optimal scale, then scope rather than scale considerations play a meaningful role for the firm's planning the size of the operations. Here, instead of scaling up the existing product, the firm needs to diversify into several classes of related products by exploiting the maximum advantage of idle capacities available in various stages of production so as to enjoy the economies of scope.

Now, the natural question is how to reveal the underlying cost frontier exhibiting such scope economies arising from idle capacities needed for diversification. We have discussed in this paper the use of our new DEA model help revealing this frontier by capturing indivisibilities in the various task-specific processes that go into the production process, and have highlighted its advantage over the old cost DEA model.

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